Global Journal of Mechanical Engineering, Volume 3, Number 2, 2002
ISSN 1595 – 7578

FINITE ELEMENT ANALYSIS OF A TRANSIENT 2-DIMENSIONAL HEAT CONDUCTION PROBLEM

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ABSTRACT

The analysis of a transient 2-Dimensional heat conduction problem by the Finite Element Method is hereby presented The solution approach was that of partial discretisation : 4-node isoparametric elements were used in the discretisation of the problem domain in the spatial coordinate, while linear temporal elements are used in the discretisation of the time domain. The Galerkin's Weighted Residual Method was used in the development of the system equation in the space domain, and their transformation into the time domain. The resulting system of equation, which is a two -point recursive relation, was solved using the Gauss - Cholesky Method .The developed algorithm was used on a linear, transient 2-Dimensional problem, and the results obtained were an improvement on those obtained by Bruch, J.C . and Zyvoloski , G .

Keywords: ***Finite element, 2-Dimensional, Transient, Energy, Heat, Conduction, Jacobian***

INTRODUCTION

Design calculations and safety considerations have made it important for the engineer to have a precise knowledge of the temperature field .particularly non-stationary temperature field, to be encountered in service by components and /or systems. The attempt in this project is to develop, using the Finite Element Method, an algorithm which can be used in determining the non-stationary field of components, referred to as solution domain. The focus here will be on problems which are best analyzed or can be approximated as 2-Dimensional. Examples of such class of problems where the transient conduction of heat is of particular interest are the cooling or heating of furnaces, heat treatment of metals, thermo-structural analysis of space deployable Vehicles etc. Many engineering problems in real life, including the examples given above preclude analytical or closed form solutions. This may arise from any or combination of the following: irregular and varied geometry, mixed boundary conditions, non-linear material behaviour, e.t.c. When such situations arise, resort has to be made to numerical or approximate methods of solution. Amongst the numerical methods are to be found the finite difference method (FDM) and the finite element method (FEM). These two being the most popular ones. However, the FEM offers several advantages over the FDM. For example, the accuracy of the results can be improved, without complicating the boundary conditions, by using higher order elements the finite element analysis. While the same attempt in the FDM usually poses some difficulty .For this and other reasons, the FEM has been used in this work.

MATHEMATICAL FORMULATION OF THE TRANSIENT HEAT CONDUCTION EQUATION FOR A 2-D CASE

In order to obtain the governing differential equation, certain assumption have to be made:

(a) The material particles of the body are at rest, that is, convective heat transfer within the solution domain would not be considered.

(b) The thermophysical properties are not temperature dependent.

(c) No phase change or latent heat effect takes place.

(d) The analysis for heat conduction is decoupled from the stress condition.

FORMULATION

From the basic equations of heat transfer: (i) Energy balance equation, (ii) Rate equations, we have for a rectangular domain:

Heat inflow in time dt + Heat generated in time dt = Heat outflow in time dt + change in internal energy in time dt.

$$q_x dt + q_y dt + \dot{q} dx dy dt = q_{x+dx} dt + \rho c dT dx dy \qquad \text{------------------} (1.0)$$

Expressing q_{x+dx} and q_{y+dy} as a two-term Taylors' Series and substituting in equation (1.0)

$$= (qx + qy)dx + \dot{q} dx dy dt = \left(qx + \frac{\partial q_x dx}{\partial x} + qy + \frac{\partial q_y dy}{\partial y} \right) dt + \rho c dT dx dy \qquad \text{------------------} (2.0)$$

$$= -(\frac{\partial q_x}{\partial x})dx - \frac{\partial q_y dy}{\partial y})dt + \dot{q} dx dy dt = \rho c dT dx dy \qquad \text{------------------} (3.0)$$

$$\Rightarrow But \Rightarrow q_n = -k_n A_n \frac{\partial T}{\partial n}, n = x, y \text{ ---------------------------- } (4.0)$$

$$-\frac{\partial}{\partial x}(-k_x A_x \frac{\partial T}{\partial x})dx - \frac{\partial}{\partial y}(-k_y A_y \frac{\partial T}{\partial y})dydt + q\,dxdydt = \rho c dT dxdy \text{ ---------------- } (5.0)$$

Equation (7.0) is the equation that governs the transient 2-D conduction of heat within the confines of the assumptions made

$$= \frac{\partial}{\partial x}(k_x \frac{\partial T}{\partial x})dxdy + \frac{\partial}{\partial y}(k_y \frac{\partial T}{\partial y})dxdydt + \dot{q}\,dxdydt = \rho c dT dxdy \text{ ---------------- } (6.0)$$

It would be seen that the equation is a second order partial differential equation, and as such would require, in addition to the initial condition, two boundary conditions.

Initial condition: $T\,(t = 0) = T_0$

$$= \frac{\partial}{\partial x}(k_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial T}{\partial y}) + \dot{q} = \rho c \frac{\partial T}{\partial t} \text{ ---------------------------- } (7.0)$$

Boundary condition:

$$k_n \frac{\partial T}{\partial n} + q = 0 \rightarrow on \Rightarrow \Gamma_1$$

$$T - \hat{T} = 0 \Rightarrow on \Rightarrow \Gamma_2$$

FINITE ELEMENT ANALYSIS OF A TRANSIENT 2-D HEAT CONDUCTION PROBLEM

The application of the Finite Element Method is now made to the analysis of a transient 2-D heat conduction problem. The concepts and procedure normally adopted when using FEM are incorporated, particularly as they relate to this problem.

DISCRETIZATION

The solution domain is divided into sub-regions, using 4-node isoparametric elements. See figure (3.0). The isoperimetric element discretization admits the simulation of solution regions with curved boundaries, to a high degree of accuracy. Also they are well suited to non-structural problems. The present problem under consideration is an example of non-structural problem.

INTERPOLATION FUNCTIONS

One of the characteristics of isoparametric element formulation is that the interpolation of element coordinates and the element nodal field variable (temperature) are done via the same interpolation functions. These are usually defined in natural coordinate system. For the 4-node isoparametric element used it takes the form:

$N_1 = ¼(1+r)(1-s)$

$N_2 = ¼(1+r)(1+s)$ See figure (4.0) .... (1)

$N_3 = ¼(1-r)(1+s)$

$N_4 = ¼(1-r)(1-s)$

Where the element coordinates and field variable are interpolated as:

$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$ .... (2)

$T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4$

However, since the nodal temperature are a function of the global or Cartesian coordinates, and the interpolation functions a function of the local natural coordinates, a mapping is needed. This mapping is from the Cartesian coordinate and vice versa. For an isoparametric element formulation, this is achieved via the Jacobian. Which for a two dimensional problem, is given as:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$$

ELEMENT CHARACTERISTICS MATRICES

Applying Galerkin's method to equation (6.0), with q = 0, we have,

$$\int_{\Omega^{(e)}} [N]^T \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) - \rho c \frac{\partial T}{\partial t} \right] d\Omega^e = 0 \quad \text{-- (7.0)}$$

Applying the Green Gauss theorem to the derivatives of second order gives:

$$\int \left[N^T\right] k_x \frac{\partial T}{\partial x} n_x d\Gamma^{(e)} - \int \frac{\partial}{\partial x}[N]^T k_x \frac{\partial T}{\partial x} d\Omega^{(e)} - \int \frac{\partial}{\partial y}[N]^T k_y \frac{\partial T}{\partial y} d\Omega^{(e)}$$

$$+ \int [N]^T k \frac{\partial T}{\partial y} n_y d\Gamma^{(e)} - \int [N]^T \rho c \frac{\partial T}{\partial t} d\Omega^{(e)} \quad \text{------------------------------------(8)}$$

Rearranging,

$$\int_{\Omega^{(e)}} \frac{\partial[N]^T}{\partial x} k_x \frac{\partial T}{\partial x} d\Omega^{(e)} + \int \frac{\partial[N]^T}{\partial y} k_x \frac{\partial T}{\partial y} d\Omega^{(e)} + \int \frac{[N]^T}{1} \rho c \frac{\partial T}{\partial t} d\Omega^{(e)} =$$

$$\int_{\Gamma^{(e)}} [N]^T k \frac{\partial T}{\partial x} n_x d\Gamma^{(e)} + \int_{\Gamma^{(e)}} [N]^T k \frac{\partial T}{\partial y} n_y d\Gamma^{(e)} \quad \text{------(8.1)}$$

Substituting

$$T = \hat{T} = [N]\{T\}^{(e)}$$

$$\Rightarrow \left[\int_{\Omega^{(e)}} \frac{\partial[N]^T}{\partial x} k_x \frac{\partial[N]}{\partial x} d\Omega^{(e)} + \int_{\Omega^{(e)}} \frac{\partial[N]^T}{\partial y} k_y \frac{\partial[N]}{\partial y} d\Omega^{(e)} \right] \{T\}^{(e)} + \int_{\Omega(e)} [N]^T \rho c [N] \left(\dot{T} \right)^{(e)} =$$

$$\int_{\Gamma^{(e)}} [N]^T ky \frac{\partial T}{\partial x} ny d\Gamma^{(e)} \quad \text{------(8.2)}$$

In matrix form,

$$[C]^{(e)}\{\dot{T}\}^{(e)} + [K]^{(e)}\{T\}^{(e)} = \{F\}^{(e)} \quad \text{------(9.0)}$$

Where,

$$[K]^{(e)} = \int_{\Omega^{(e)}} \frac{\partial[N]^T}{\partial x} k_x \frac{\partial[N]}{\partial x} d\Omega^{(e)} + \int_{\Omega^{(e)}} \frac{\partial[N]^T}{\partial y} k_x \frac{\partial[N]}{\partial y} d\Omega^{(e)} \quad \text{------(9.1)}$$

$$[C]^{(e)} = \int_{\Omega^{(e)}} [N]^T \rho c [N] d\Omega^{(e)} \quad \text{------(9.2)}$$

$$\{F\}^{(e)} = \int_{\Gamma^{(e)}} [N]^T k_x \frac{\partial T}{\partial x} n_x d\Gamma^{(e)} + \int_{\Gamma^{(e)}} [N]^T k_y \frac{\partial T}{\partial y} n_y d\Gamma^{(e)} \quad \text{------(9.3)}$$

In evaluating these integrals, numerical integration is required.

ASSEMBLAGE OF THE GLOBAL MATRICES

The element characteristics matrices are now assembled to give the global matrices [C], [K], and {F}. That is:

$$[C] = \sum_{e=1}^{m} [C]^{(e)}$$

$$[K]=\sum_{e=1}^{m}[K]^{(e)}$$

$$\{F\}=\sum_{e=1}^{m}\{F\}^{(e)}, m = no....of.....elements.$$

These then gives the system equation

$$[C]\{\dot{T}\}+[K]\{T\}=\{F\} \quad \text{--(10)}$$

INCORPORATION OF THE TIME DOMAIN

This incorporation can either be done via the finite element method or the finite difference method. Nevertheless, the finite element approach would be used here as this lead to an unconditionally stable scheme.

Using a linear temporal element, an approximate solution would be:

$\{T\} = N_i \{T_i\} + N_{i+1} \{T_{i+1}\}$ ....(10.1)

$N_i = 1-\mu$

Interpolation functions ....(10.2)

$N_{i+1} = \mu$

$$\mu = \frac{t-t_i}{\Delta t} \quad \text{--(10.3)}$$

$$\dot{T} = \frac{dT}{dt} = \frac{dT}{d\mu}\frac{d\mu}{dt} \quad \text{--(10.4)}$$

$$\frac{d\mu}{dt} = \frac{1}{\Delta t} \quad \text{--(10.5)}$$

Substituting equations (10.2), (10.3), (10.5) into (10.4)

$$\Rightarrow \dot{T} = \frac{d}{d\mu}\left[(1-\mu)T_i + \mu T_{i+1}\right]\frac{1}{\Delta t} = -\frac{1}{\Delta t}T_i + \frac{1}{\Delta t}T_{i+1} \quad \text{------------------------------(10.6)}$$

$F = N_i f_i + N_{i+1} f_{i+1}$ ....(10.7)

Using the Galerkin's method (10)

$$\Rightarrow \int_{t_i}^{t_{i+1}} W\left[(C)\{\dot{T}\}+[K]\{T\}-\{F\}\right]dt = 0. \quad \text{------------------------------(10.8)}$$

Substituting (10.1), (10.6), and (10.7)

$$\Rightarrow \int^1 W\left[\left[[C](-\frac{1}{\Delta t}\{T_i\}+\frac{1}{\Delta t}\{T_{i+1}\})+[K]((1-\mu)\{T_i\}+\mu\{T_{i+1}\}-((1-\mu)\{f_i\}+\mu\{f_{i+1}\})\right]\right]\Delta t d\mu = 0 \quad ...(10.9)$$

$$\left(\Delta t\int_0^1 W(1-\mu)d\mu\right)\{f_i\}+\left(\Delta t\int_0^1 W\mu d\mu\right)\{f_{i+1}\} \quad \text{------------------------}(10.10)$$

dividing $\Rightarrow$ through $\Rightarrow$ by $\Rightarrow$ $\int_0^1 W d\mu$

$$\left(\Delta t\int^1 W(1-\mu)d\mu\right)\{f_i\}+\left(\Delta t\int^1 W\mu d\mu\right)\{f_{i+1}\} \quad \text{------------------------}(10.10)$$

$$\Rightarrow [[C]+\theta[K]\Delta t]\{T_{i+1}\} = [[C]-(1-\theta)[K]\Delta t]\{T_i\}+[(1-\theta)\{f_i\}+\theta\{f_{i+1}\}]\Delta t \quad \text{------------------------}(10.11)$$

$$Where \Rightarrow \theta = \frac{\int_0^1 W\mu d\mu}{\int_0^1 W d\mu}$$

For Galerkin's method W = N = shape functions

Choosing N = μ , which gives a more stable and accurate result

$$\Rightarrow \theta = \frac{\int_0^1(\mu)\mu d\mu}{\int_0^1(\mu)d\mu} = \frac{\left[\frac{\mu^3}{3}\right]}{\left[\frac{\mu^2}{2}\right]} = \frac{2}{3}$$

Hence,the final equation to be implemented on the computer is :

$$\left[[C]+\frac{2}{3}[K]\Delta t\right]\{T_{i+1}\}=\left[[C]-\left(1-\frac{2}{3}\right)[K]\Delta t\right]\{T_i\}+\left[\left(1-\tfrac{2}{3}\right)\{f_i\}+\tfrac{2}{3}\{f_{i+1}\}\right]\Delta t \quad \text{------------}(10.12)$$

Which is a two- point recursive scheme.

RESULTS AND CONCLUSIONS

The program was applied to a linear, 2-D transient heat conduction problem for which both analytical and finite element solutions exist. The problem was taken from that solved by Zyvoloski et al.

SAMPLE PROBLEM

The domain of the problem is a square one with no heat generation, and the boundary conditions are specified. Only a quarter of the problem was considered because of the bi-symmetric nature of the problem.

Other specifications:

Thermal conductivity = 1.25 (BTU/mhr °F)

Density x specific heat = 1.0 (BTU/m^2 °F)

Length of domain in the x-direction = 3.0 m

Length of domain in the y-direction = 3.0 m

Initial temperature = 30.0 °F

The domain was divided into 100 elements with 121 nodes. In considering only one-quarter of the domain, only twenty five isoparametric elements with thirty six nodes were considered.

It can be concluded that the Finite Element Analysis, using partial discretization, has been applied successfully to the analysis of linear, transient 2-D heat conduction problem. This is amplified by the results obtained which compare favourably with both analytical and existing finite element solution.

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MAIN PROGRAM (THEFCAP)

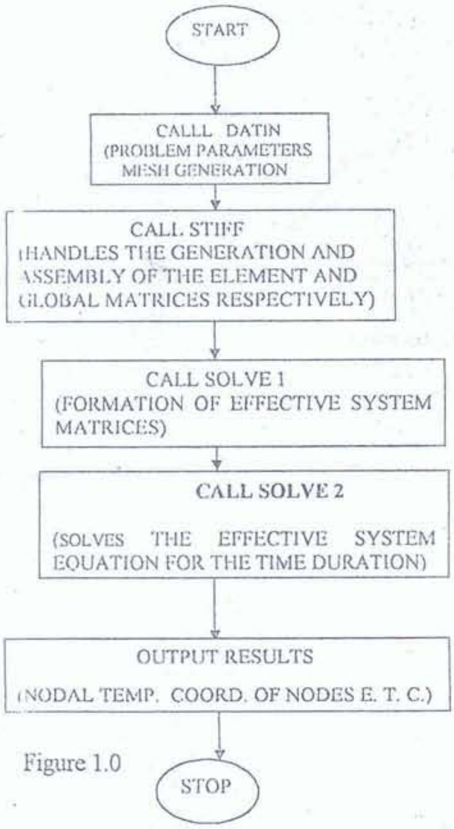


Figure 1.0

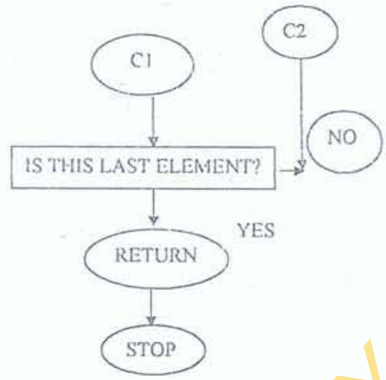
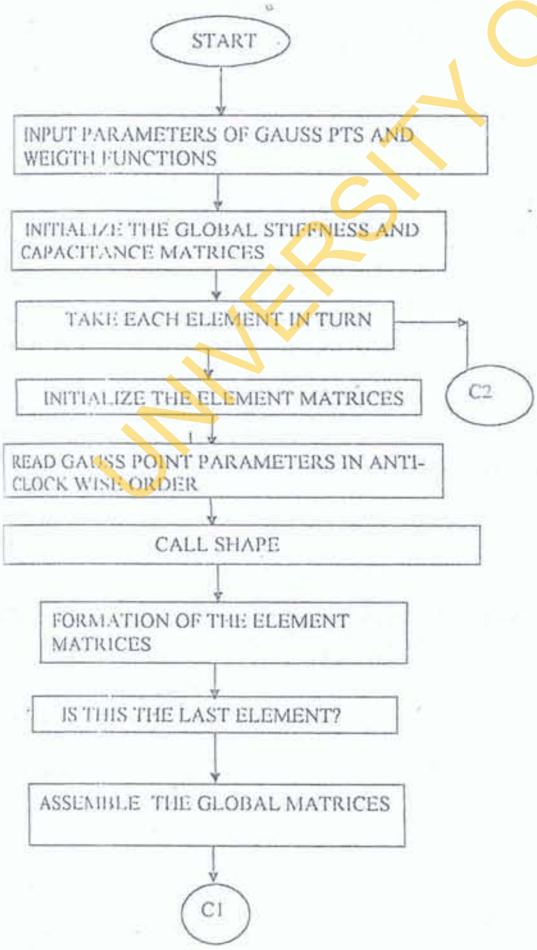


Figure 2.0

SUBROUTINE STIFF



SUBROUTINE SOLVE 1

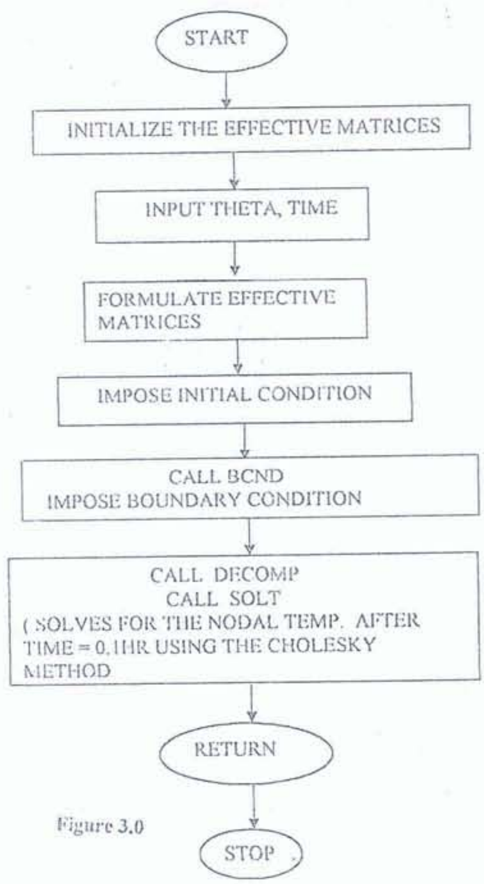


Figure 3.0

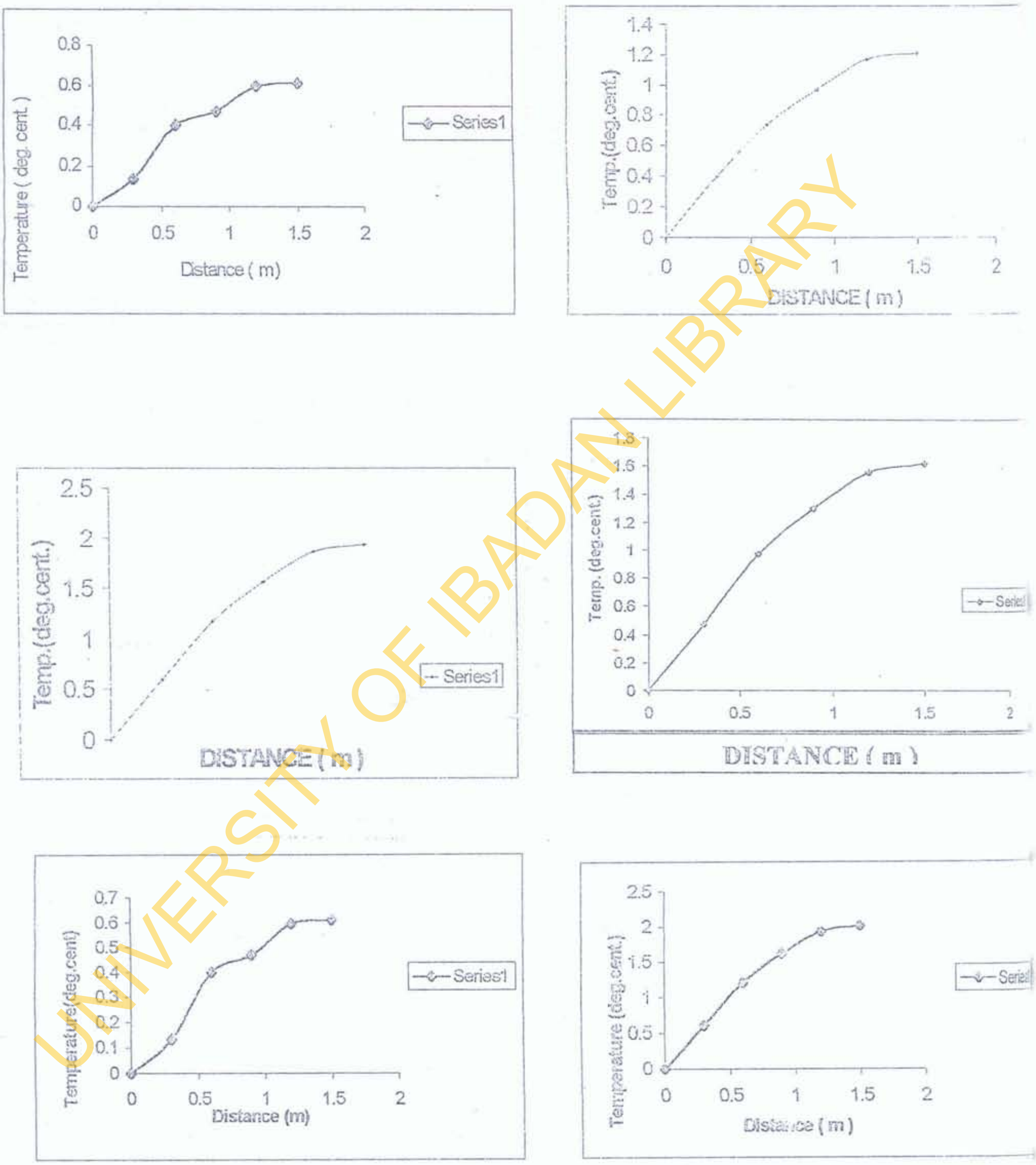


Figure 4.0: NODAL TEMPERATURE AT THE END STATE (= 1.2HR IN STEPS OF .1000HR)

ISSN 1595 – 7578

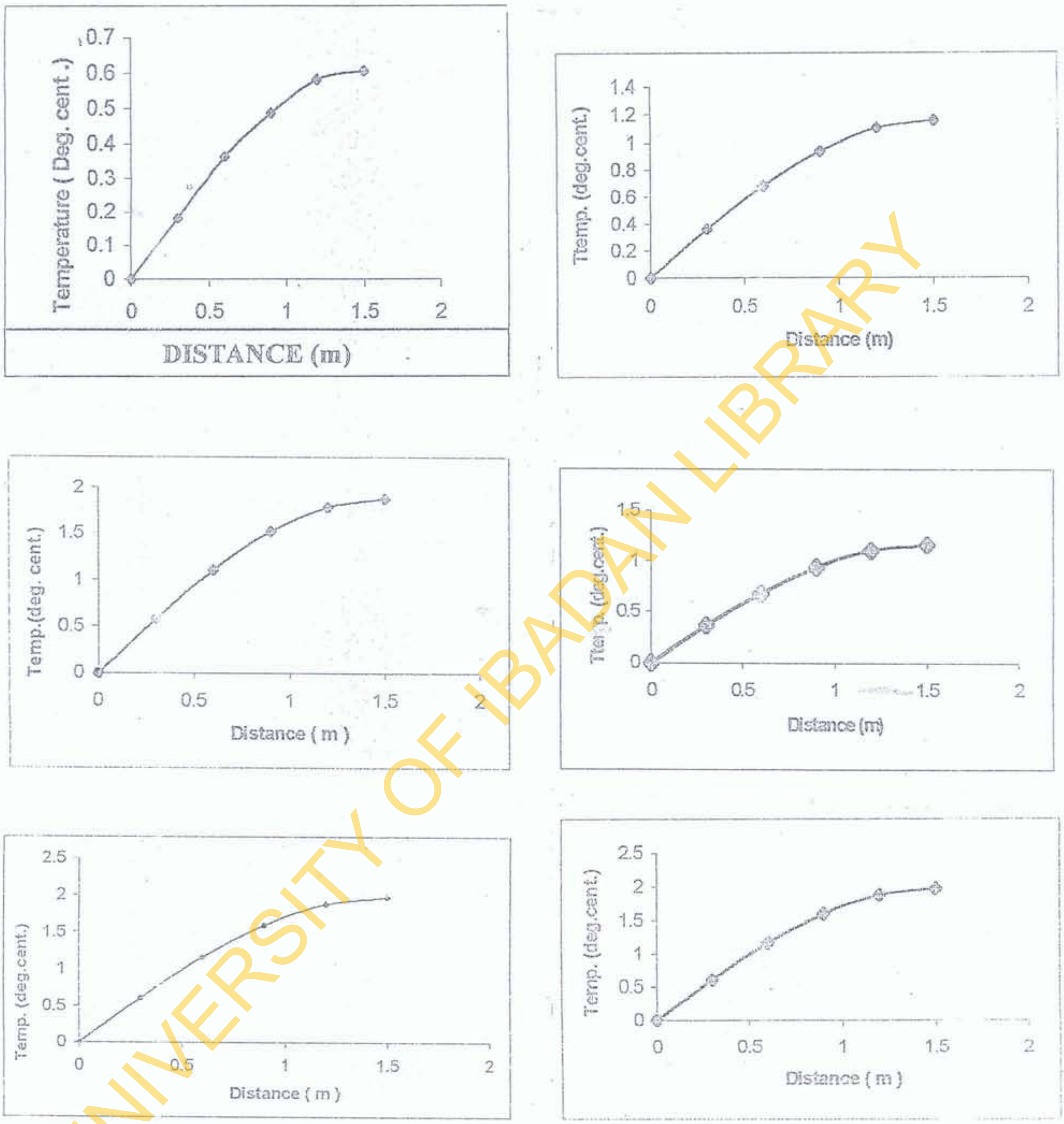


Figure 5.0: NODAL TEMP. AT THE END STATE (1.2HR) IN STEPS OF 0.075

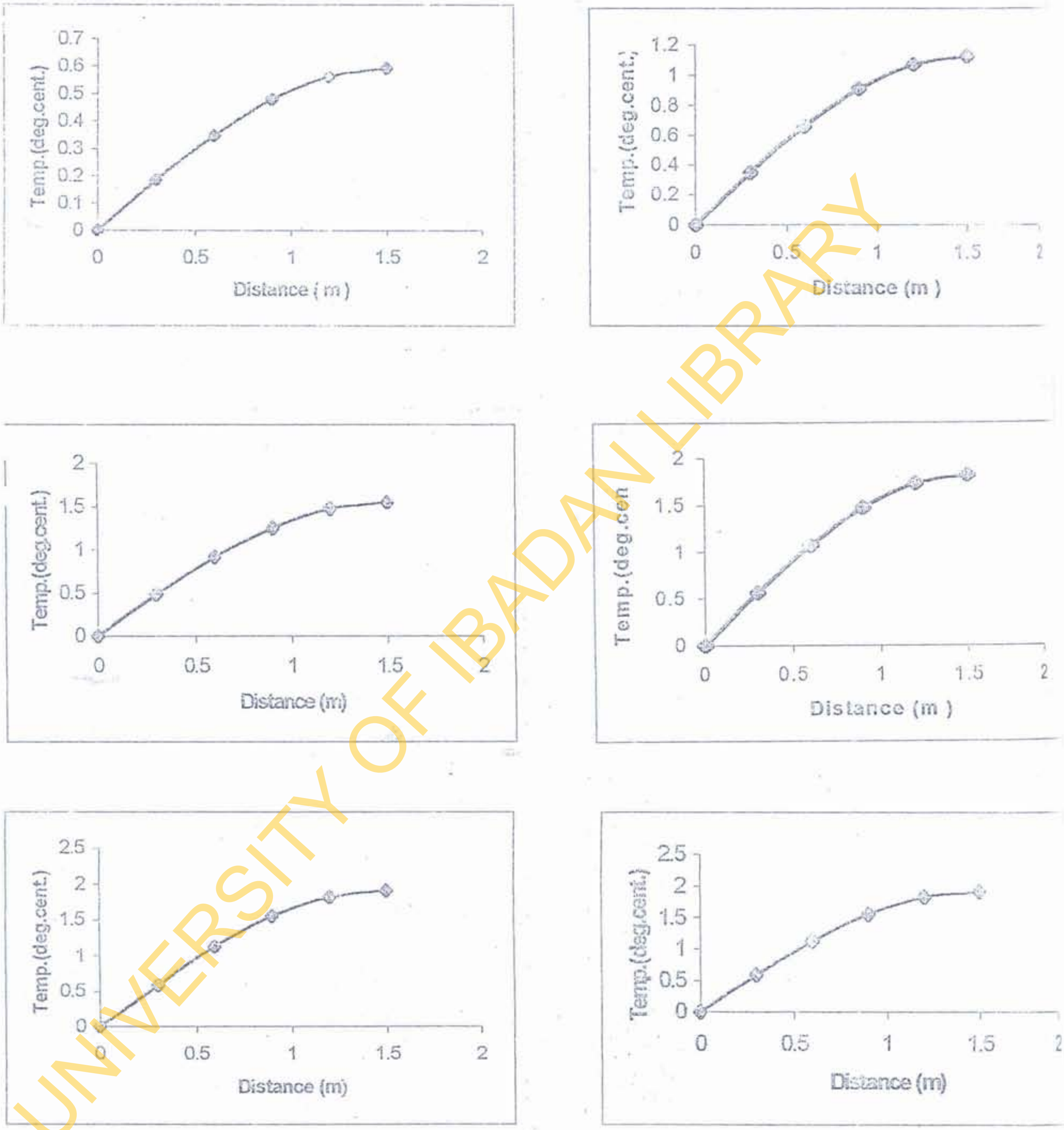


Figure 6.0: NODAL TEMPERATURES AT THE END STATE (= 1.2HR) IN STEPS OF 0.0500HR

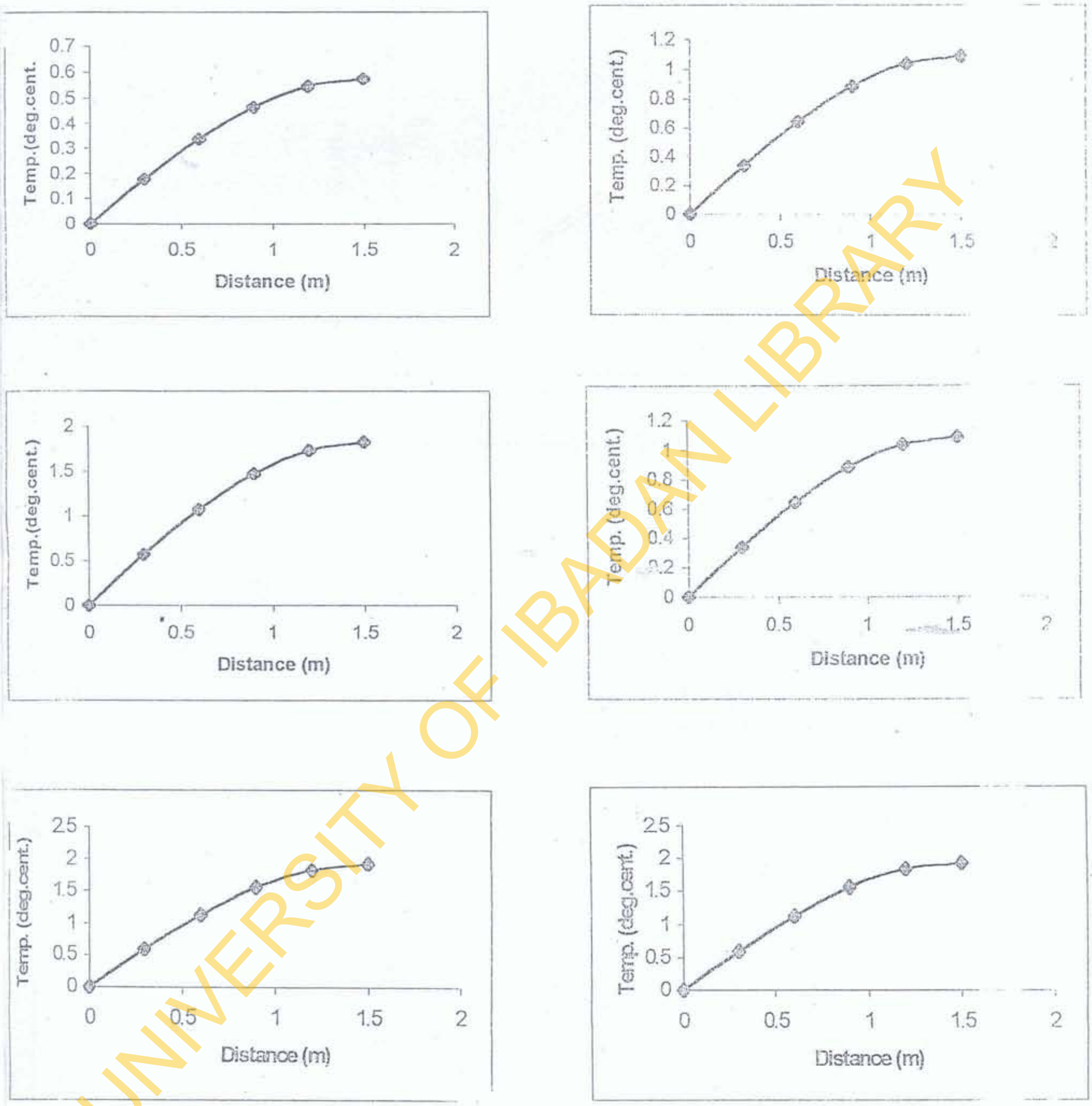


Figure 7.0: NODAL TEMPERATURES AT THE END STATE (=1.2 HR) IN STEPS OF .025HR