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Abstract - This study utilised correlation and distribution analyses to investigate the acceptability of application of two fractal dimension estimators to characterise the waveforms originating from excited nonlinear pendulum. Parameters selection sensitive simulation of the excited nonlinear pendulum waveforms was performed with the constant time step fourth order Runge-Kutta algorithm with codes developed in FORTRAN90. However, the waveforms validated by Gregory and Jerry (1990) and treated as time series were characterised using developed codes of Carlos (1998) and Hurst fractal dimension estimation procedures. The validation results compare qualitatively well and the correlation coefficients between Carlos (1998)-based and Hurst's exponent based dimension estimate for the angular displacement and velocity are respectively $R^2 = 0.68$ and $R^2 = 0.66$. A higher correlation coefficient ($R^2 = 0.84$) existed between the estimated Hurst's exponent of the angular displacement and velocity. The Hurst distribution exhibited both full spectrum and peak values range 0.04 to 1.00 and percentage probability range 2 to 12. The sum of this study results is the interchange possibility and utility of the two fractal dimension estimators as waveforms characterising tool.

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I. Introduction

Oldrich *et al* in 2001 describes fractals as a rough shape that can easily be subdivided in parts of which are at least approximately a reduced copy of the whole (self-similar). The use of a concept known as fractal dimension has made it easy for researcher in this field to measure the extent to which a one-dimensional thread fills up a three dimensional space (Scott,1990). The use of fractal dimension as an estimating tool for characterising nonlinear systems is becoming more appreciated in the recent times. The aim of Clark *et al* (1995) paper was to distinguish chaotic and non-chaotic (regular or random) behaviour using fractal dimension. The authors employed an electronic circuit modelling of a ball bouncing on an oscillating table in demonstrating these different dynamic characteristics. Findings show that the fractal dimension of 1.07 and 1.7 respectively implies regular data and chaotic data. It was also shown that the fractal dimension which is infinite refers to data that are random. It is concluded from the paper that the bouncing ball circuit system dynamics has a chaotic attractor of low fractal dimension. This paper has shown that fractal dimension can be utilised in characterising nonlinear system. Salau and Ajide (2012) asserted that the study of images play a significant role in engineering and several fields of study. The authors employed fractal dimension as the estimator for sectional images characterisation of selected dynamic systems. Findings obtained from the paper showed the high potentiality of fractal disk dimension as characterising tool for images.

The chaotic driven impact of two important parameters of excited Duffing oscillator has been studied using fractal disk dimension (Salau and Ajide, 2013). The outcome of the study has provided a very robust platform for the relevance of the use of fractal dimension as a reliable estimator for characterising nonlinear dynamic systems.

Hurst exponent can easily be computed from fractal dimension. Wikipedia (2013) describes Hurst exponent as what can be utilized as a measure of long term memory of time series. It relates to the autocorrelations of the time series and the rate at which these decrease as the lag between pairs of values increases. It was further understood here that Hurst exponent were originally developed in hydrology for the practical matter of determining optimum dam sizing for the Nile River's volatile rain and drought conditions that have been observed over a long period of time. The name "Hurst Exponent" was coined from Harold Edwin Hurst who was the lead researcher in this field. An extract from Ian (2013) article shows that Hurst exponent has a very wide application. It occurs in several areas of mathematics (Fractals and Chaos theory, Spectral Analysis, e.t.c.), biophysics, computer networking, hydrology and just to mention but a few. Hurst exponent has been found to be very useful in marketing, medicine, engineering and host of other fields. In business, the simulation model which describes the

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multi-level supply chain has been done using Hurst exponent (Chien-Yuan and Jinsheng, 2013). The exponent satisfactorily analyzes the dynamic behaviour of inventory under various factors that include lead time, demand pattern, information sharing and RFID (Radio Frequency Identification). The outcome of the study showed that the lead time and RFID utilization effectualness greatly influence the inventory dynamics under the stock and supply line discrepancies of specified parameters. This paper has also shown the utility of Hurst exponent as a relevant estimator of supply chain dynamics. The trend in prices of gold has been analyzed by Priyadarshini and Babu (2010). The author utilised fractal dimension index (FDI) that was computed from Hurst exponents. The monthly gold rates required for study was collected from January 1971 to April 2010 (40 years). The results of FDI calculated from a constant Hurst exponent (H=1.047) revealed that it is useful tool for determining the amount of market growth. The author affirmed that this study has opened-up an immeasurable advantage for a businessman to venture into markets that have the most opportunity. Yu-Zhi *et al* (2011) paper dwell on the possibility of Hurst exponent being utilised in ecology. A careful application of this exponent to rodent populations revealed that it is very convenient and effective exponent for detecting nonlinear systems in natural populations. In medicine, Kamalanand and Jawahar (2012) paper studied the behaviour of Human Immuno Virus (HIV) system using the three dimensional HIV model and a chaotic measure popularly referred to as Hurst exponent. Results of the study showed that Hurst exponents of cells and viral load vary nonlinearly in the selected parameters range. It was further shown that the three dimension HIV model can accommodate both persistent and anti-persistent dynamics of HIV states. The research output of this paper has clearly shown the high clinical relevance of Hurst exponent. This is because the analysis of the complexity of the HIV model is helpful for choosing appropriate parameter estimation methods. It has great benefits for identifying suitable treatment strategies.

Despite the richness of fractal dimensions and Hurst exponent as estimating tools in nonlinear dynamics, they are yet to be extensively explored in many nonlinear mechanical engineering system dynamics as characterising exponents. Extensive literature study shows that Duffing oscillator and excited nonlinear pendulum are just two of the so many mechanical systems that Hurst exponent has not been significantly used to characterize. The main objective of this paper is to investigate the correlation and distribution analyses of estimated Fractal dimensions and Hurst's exponent from Waveforms of Excited Nonlinear Pendulum. The quest of filling this research gaps is a strong motivation for this paper.

II. Methodology

a) Equation of Motion

According to Gregory and Jerry (1990), the dimensionless representation of the damped, sinusoidally excited pendulum with fixed lumped-mass and length is described by equation (1) called equation of motion. This equation expresses Newton's second law with the various terms on the left representing respectively variation with time (t) of the acceleration, damping and gravitation effect. The variation with time of the angular displacement, velocity and acceleration are $\theta(t)$, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$. The angular velocity of the forcing is ω_D, g is the forcing amplitude (not gravitational acceleration) and q is the damping parameter.

$$\frac{d^2\theta}{dt} + \frac{1}{q}\frac{d\theta}{dt} + \sin(\theta) = g\cos(\omega_D t) \quad (1)$$

The transformation of equation (1) under the assumptions ($\theta_1 = angular\ displacement = AD$ and $\theta_2 = angular\ velocity = AV$) to a pair of first order differential equation leads to equations (2) and (3).

$$\dot{\theta}_1 = \theta_2 \quad (2)$$

$$\dot{\theta}_2 = g\cos(\omega_D t) - \frac{1}{q}\theta_2 - \sin(\theta_1) \quad (3)$$

The transient and steady angular displacements and velocities were obtained by simultaneous simulation of equations (3) and (4) using constant time step Runge-Kutta fourth order algorithms over large number of excitation periods. The results of the periodic time history of the steady angular displacement and velocity are the waveforms objects for the present investigation.

b) Fractal Dimension of Waveforms

This study utilised Carlos (1998) procedure to evaluate the fractal dimension of the waveforms. The method is of Hausdorff dimension (where for a curve: $1 \le D_h \le 2$) origin and in the simplified form equal to equation (4) for a curve of length (L) covered by N-open balls of radius (ε). To achieve equal axes Carlos (1998) employed two linear transformations (assuming topology invariance under the transformation) that map an original waveform into another such that the transformed waveform is embedded in an equivalent metric space. The equivalent transformation equations for the present study are given respectively for increasing simulation periods (T_i), consecutive steady angular displacements ($\theta_{1,i}$) and consecutive steady angular velocities ($\theta_{2,i}$) by equations (5) to (7). The transformation maps N-points of angular displacement and velocity waveforms to another that belongs to a unit square respectively. In the resulting unit square, Carlos (1998) justified that each of the waveforms can be visualised as covered by a grid of $N \times N$ cells where N of them containing one point of the transformed waveforms. Substituting in equation (4) the length (L) of the transformed waveform and $\varepsilon = 1/(2N')$ where $(N' = N-1)$ results in a modified expression for dimension given by equation (8).

$$D_h = D = \lim_{\varepsilon \to 0}\left[1 - \frac{\ln(L)}{\ln(\varepsilon)}\right] \tag{4}$$

$$T_i^* = \frac{T_i - T_{\min}}{T_{\max} - T_{\min}} \tag{5}$$

$$\theta_{1,i}^* = \frac{\theta_{1,i} - \theta_{1,\min}}{\theta_{1,\max} - \theta_{1,\min}} \tag{6}$$

$$\theta_{2,i}^* = \frac{\theta_{2,i} - \theta_{2,\min}}{\theta_{2,\max} - \theta_{2,\min}} \tag{7}$$

$$D_h = D = \phi \approx \left[1 + \frac{\ln(L)}{\ln(2N')}\right] \tag{8}$$

It is to be noted that the approximation to ϕ expressed in equation (8), improves as $N' \to \infty$.

c) Hurst Exponent (H) of Waveforms

According to Mandelbrot (1983), Hurst (1951), Daniel and Benjamin (2005) the Hurst exponent $(0 \le H \le 1)$ can be obtained from the rescaled range (R/S) statistic which is the range (R) of partial sums of deviation of times series from its mean, rescaled by its standard deviation (S). In the present study, the rescaled range (R/S) and the standard deviation (S) for the periodic time history of the steady displacement and velocity waveforms are given correspondingly by equations (9 & 11) and (10 &12) for the time span (τ). Literatures recommended that time span ($\tau \ge 4$) to ensure the reliability of the estimated Hurst exponent, which is the slope of best line to the log-log plot of (τ) versus (R/S).

$$\left[\frac{R}{S}\right]_{1,\tau} = \frac{1}{S_{1,\tau}}\left[\max_{1\le t\le\tau}\sum_{k=1}^{\tau}(\theta_{1,k} - \bar{\theta}_{1,\tau}) - \min_{1\le t\le\tau}\sum_{k=1}^{\tau}(\theta_{1,k} - \bar{\theta}_{1,\tau})\right] \tag{9}$$

$$\left[\frac{R}{S}\right]_{2,\tau} = \frac{1}{S_{2,\tau}}\left[\max_{1\le t\le\tau}\sum_{k=1}^{\tau}(\theta_{2,k} - \bar{\theta}_{2,\tau}) - \min_{1\le t\le\tau}\sum_{k=1}^{\tau}(\theta_{2,k} - \bar{\theta}_{2,\tau})\right] \tag{10}$$

$$S_{1,\tau} = \left[\frac{1}{\tau}\sum_{k=1}^{\tau}(\theta_{1,k} - \bar{\theta}_{1,\tau})^2\right] \tag{11}$$

$$S_{2,\tau} = \left[\frac{1}{\tau}\sum_{k=1}^{\tau}(\theta_{2,k} - \bar{\theta}_{2,\tau})^2\right] \tag{12}$$

In a related study on fractal dimensions of time sequences by Sy-Sang and Feng-Yuan (2009) equation (13) give the expression for the relationship between the dimension and the Hurst exponent.

$$D = \phi = 2 - H \tag{13}$$

It is important to note that in the present study the estimated fractal dimension using equations (8) and (13) are respectively represented by DCS-Carlos (1998)-based and DHE- Hurst's exponent based.

d) Studied Cases and Simulation Parameters

The present investigation focuses three cases at fixed excitation frequency ($\omega_D = 2/3$), the details are provided as follow:

Case-I : $(q, g = 2, 0.9)$. The case enables the investigation of the variation of estimated fractal dimension with increasing number of time series to verify existence of dimension convergence.

Case-II : $(q, g = 4, 1.5)$. The original and the transformed Poincare sections obtained from this case was compared visually with its equivalent from Gregory and Jerry (1990) to validate topological invariance under linear transformation and FORTRAN programme developed for this present study. As in Case-I, convergence of estimated fractal dimension demonstrated with increasing number of time series.

Case-III : This case focuses the parameter plane defined by $2.0 \le q \le 4.0$ and $0.9 \le g \le 1.5$. First part of the case dealt with a total of 2601 nodal points selected on the plane including 51coordinates each along the axes (q and g) at constant step size to enable correlation analysis. However the second part involves a total of 10201 nodal points selected on the parameters plane including 101coordinates each along the axes (q and g) at constant step size. Runge-Kutta simulation with constant time step, estimation of fractal dimension and Hurst exponent were performed for each node. Thereafter correlation and Hurst exponent distribution analyses were performed for the entire nodes of the corresponding part.

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Simulation of equations (2) and (3) for each set of parameters selected with fourth order Runge-Kutta algorithms were effected from initial conditions (0, 0) using constant time step ($\Delta t = T_p/500$) over 3010 excitation periods ($T_p = 2\pi/\omega_D$) in which the results from the first ten periods are regarded unsteady. The fractal dimension was estimated using the steady angular displacement and velocity solutions at 3000 consecutive excitation periods. However the Hurst exponent estimate was based on the first 2048 consecutive steady solutions over ten different time spans ($\tau = 4$, 8, 16, 32, 64, 128, 256, 512, 1024 and 2048).

Comparison of the distribution of the estimated Hurst exponents was investigated in 100-equal intervals between the limits-values (Hurst exponents) for the angular displacement and velocity.

III. Results and Discussion

Figure 1 refers. The visual quality of the untransformed and the transformed Poincare sections are same and compare very well with the corresponding result reported by Gregory and Jerry (1990). This demonstrated topological invariance under linear transformation as required by Carlos (1998) approach of estimating fractal dimension. Likewise the invariance of Poincare section (periodic phase plots) suggested the invariance of the associated angular displacement and velocity waveforms that are the objects of analyses in this study. However, quantitative comparison of the limits-values along the angular displacement (AD) and angular velocity (AV) axes are different.

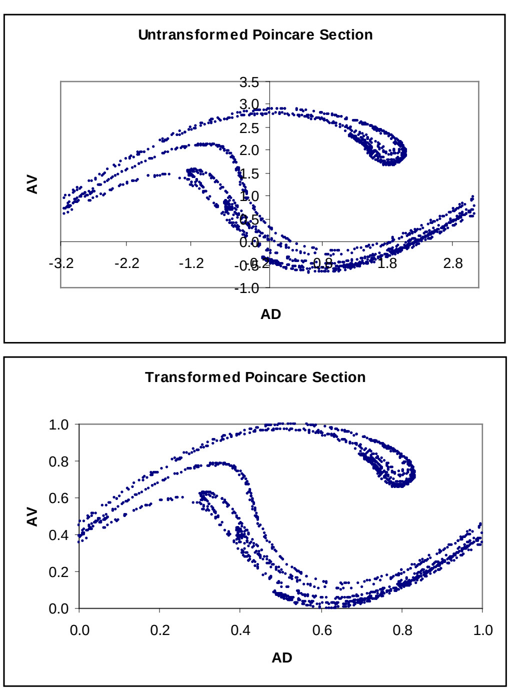

Figure 1 : The untransformed and the transformed Poincare sections for Case-II

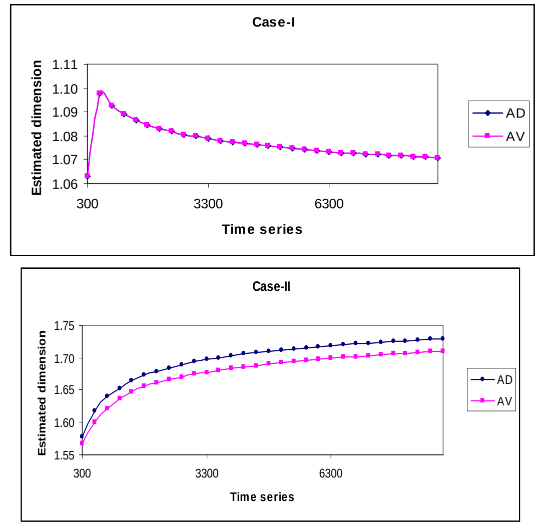

Figure 2 : Variation of estimated fractal dimension with increasing number of time series

Figure 2 refers. The variation of the estimated fractal dimension with increasing number of time series shows evidence of convergence for both Case-I and Case-II. In Case-I the observed variation of estimated fractal dimension remain the same for the angular displacement and velocity. However, a consistent relative higher estimated fractal dimension variation for the angular displacement is observed for Case-II. That is angular velocity sustained lower estimated fractal dimension relative to angular displacement in Case-II.

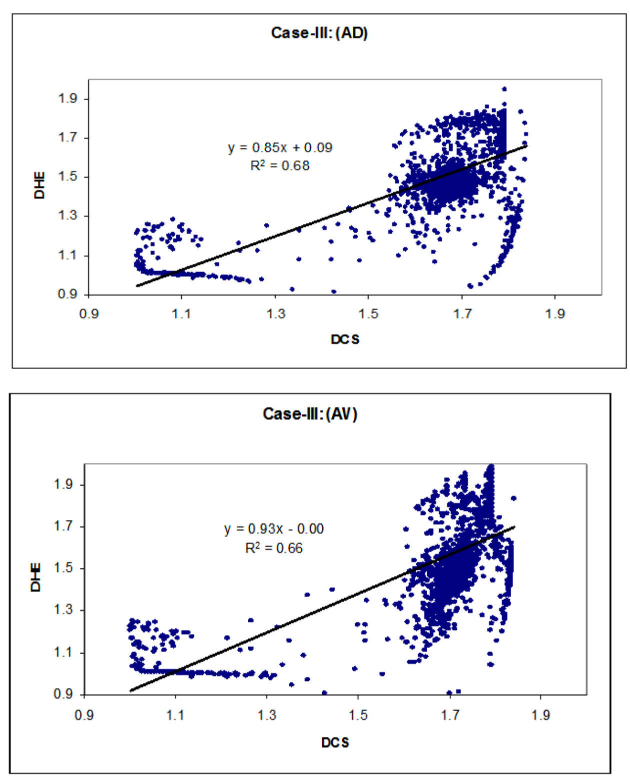

Figure 3 : Correlation of estimated fractal dimensions in Case-III

Figure 3 refers. The correlation coefficients between Carlos (1998)-based and Hurst's exponent based dimension estimated for the angular displacement (AD) and angular velocity (AV) are respectively $R^2 = 0.68$ and $R^2 = 0.66$. Higher correlation coefficients may be achieved by increasing the number of time series beyond its present value i.e. 3000-for dimension estimate and 2048-for Hurst exponent estimate. The quality of the visual pattern created by the scatter plots of the dimensions correlation of angular displacement (AD) and angular velocity (AV) are the same. It is important to note that the small and negligible quantitative difference between the obtained correlation coefficients is an indication of common source of time series (i.e. the source is periodic sampling of solutions to harmonically excited nonlinear pendulum).

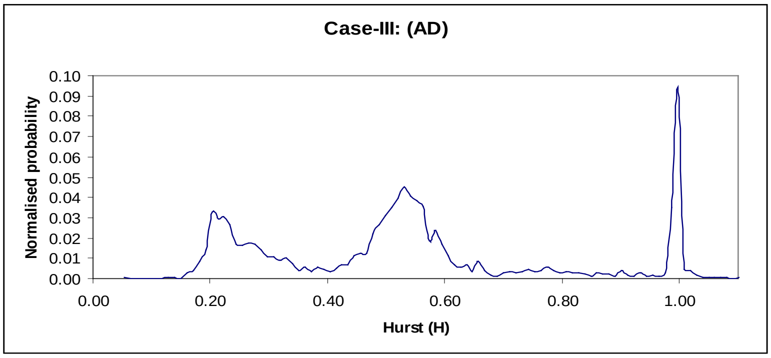

Figure 4 : Correlation of estimated Hurst's exponent between angular displacement and velocity

Figure 4 refers. There is higher correlation coefficient between the estimated Hurst's exponent of the angular displacement and velocity with value being $R^2 = 0.84$. Higher coefficient value indicate common source of the angular displacement and velocity waveforms.

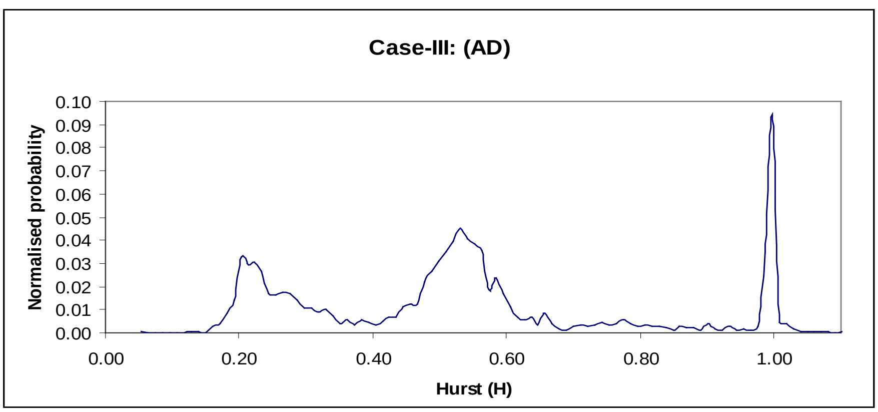

Figure 5 : Normalised probability distribution of estimated Hurst's exponents of angular displacement waveforms obtained from 10201 nodal parameter points

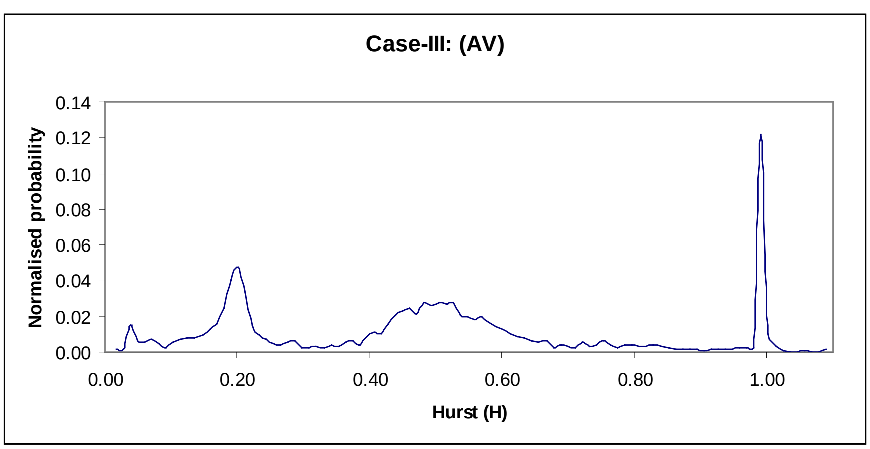

Figure 6 : Normalised probability distribution of estimated Hurst's exponents of angular velocity waveforms obtained from 10201 nodal parameter points

Figures 5 and 6 refer. In figure 5, the normalised probability distribution of the estimated Hurst's exponents of the angular displacement waveforms peaked at three different Hurst-values: 1.00, 0.53 and 0.20. The corresponding percentage probability is 9, 5 and 3. Likewise the distinct distribution peak points for the angular velocity waveforms are four at Hurst- values: 0.99, 0.50, 0.20 and 0.04. The corresponding percentage probability is 9, 3, 5 and 2. The full spectrum of Hurst-values ($0 \leq H \leq 1$) coupled with the multiple peak points observed in these distributions is an indication of the richness of the pendulum dynamics when driven by arbitrary parameters selection from this parameter plane while holding constant the drive frequency. That is the pendulum can behave periodically, quasi-periodically or worst still chaotically depending on the choices of parameters from this plane.

IV. Conclusions

This study established good correlation coefficients for two differently estimated fractal dimensions while the distribution of Hurst exponents exhibited full spectrum for the waveforms from excited nonlinear pendulum. The Hurst exponents' distributions apart from having full spectrum show multiple distinct peak values as a possible indicator of pendulum multiple behaviours such as periodic, quasi-periodic, random or chaotic. The study therefore show that the pendulum drive parameters plane consist of large selection combinations with associated rich dynamics. Thus the present study has established the possible interchange and utility of the two fractal dimension estimators as waveforms characterising tool.

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