MATHEMATICS:

A FORMIDABLE VEHICLE FOR DEVELOPMENT

An Inaugural Lecture delivered at the University of Ibadan

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The Vice-Chancellor, Deputy Vice-Chancellor (Administration), Deputy Vice-Chancellor (Academic), Registrar, Librarian, Provost of the College of Medicine, Dean of the Faculty of Science, Deans of other Faculties and Postgraduate School, Dean of Students, Distinguished Ladies and Gentlemen.

1. PREAMBLES

It is my desire to start this lecture by thanking and praising God Almighty through Jesus Christ our Lord, for making this occasion a reality. The Omnipotent, Omnipresent and Omniscience God has been faithful and merciful to me and my family in many ways. I thank the Creator of Heaven and the Earth for keeping me alive to see this day. Glory be to His holy name.

I pay tribute to all my teachers who had delivered previous inaugural lectures from the Department of Mathematics in this university. I must mention in particular, Professor H.O. Tejumola, who delivered his lecture, *Mathematics: A Tool for All Human Sciences*, in 1978, late Professor C.O.A. Sowunmi who delivered his lecture, *A Mathematical Essay on Polygamy and Mass Action* in 1986; Professor A.O. Kuku who delivered his lecture, *Mathematics as a Universal Language* in 1987; Professor S.A. Ilori delivered his inaugural lecture, *Mathematics for Recognition and Relevance* in the year 2003 and the most recent inaugural lecture from the Department of Mathematics, titled *Proofs and Paradigms: A Mathematical Journey into the Real World* was delivered by my former doctoral research supervisor, Professor G.O.S. Ekhaguere on April 8, 2010. My inaugural lecture today is the sixth from the Department of Mathematics, since its establishment in 1948. I deliver this lecture as the 13th person to occupy a Chair in mathematics of this University and I thank God and the present University Administration for fast tracking the process that enabled me to give the lecture before the first year anniversary of the announcement of my professorship.

Professor S.A. Ilori in his inaugural lecture, *Mathematics for Recognition and Relevance* talked briefly on two of the traditions to inaugural lectures. First tradition is to give an opportunity to the newly appointed or newly promoted Professor to justify his or her elevation to a Chair and the other tradition is to address a particular problem of public interest in the Professor's discipline and offer possible solutions.

As a newly appointed Professor by promotion, I shall blend the two traditions but heavily lean on the first tradition in order to justify my appointment and announce my research accomplishments. Since mathematics learning, research and applications have not found the much needed favour and interest, especially in most developing nations, including Nigeria, I shall give accounts of some of the existing non-trivial applications of the subject, with a view of stimulating interests of our people in the applications of mathematical results and techniques to diverse areas of research activities in our university and elsewhere in this country.

2. THE LIFE AND SURVIVAL OF MATHEMATICAL IDEAS

Michael F. Barnsley, a Professor of mathematics at the Mathematical Sciences Institute, Australian National University recently declared (Barnsley, 2010) that *nature and evolution provide the notion of a creative system: a core stable form, a fertile environment, a determination to survive, and a random stimuli. In the same vein, the mind of a mathematician provides a locus for creative systems, a place where mathematical structures live and evolve.*

In the first chapter of the book of Genesis, God created the heaven and the earth after which He created man in His own image, blessed them and ordered them to *Be fruitful and multiply; fill the earth and subdue it; have dominion over the fish of the sea, over the birds* *of the air and over every living thing that moves on the earth.* (Genesis, 1; 28) NKJV.

We learn from some mysterious events in the scriptures that everything in our universe is a *living* thing. According to the divine injunction, mathematicians and other professionals are faced with the extraordinary richness and complexity of the physical observable universe, which we are part of. What on earth can a mathematician create? The answer is: a vast landscape of lovely constructions, born for the first time, to live on the realm of ideas. For the realm of ideas belongs to sentient beings like us. The divine order enjoins us to dominate the universe, albeit, within our natural human capability because there is a spiritual world above the comprehension of the human brain but which is below God in the supernatural. God questioned Job during the time of his affliction in the book of Job Chapter 38, verses 4 - 11 and I quote:

Where were you when I laid the foundations of the earth? Tell me if you have understanding. Who determined its measurements? Surely you know! Or who stretched the line upon it? To what were its foundations fastened? Or who laid its cornerstone, when the morning stars sang together, and all the sons of God shouted for joy? Or who shut in the sea with doors, when it burst forth and issued from the womb; when I made the clouds its garment, and thick darkness its swaddling band; when I fixed my limit for it , and set bars and doors, when I said " This far you may come, but no farther, and here your proud waves must stop.
Verses 17- 21: *Have the gates of death been revealed to you? Or have you seen the doors of the shadow of death?. Have you comprehended the breadth of the earth? Tell me if you know all this. Where is the way to the dwelling of light? And darkness where is its place, that you may know the paths to its home? Do you know it because you were born then, or because the number of your days is great?*

The above scriptures tell us about the limitation of human knowl-

edge and ideas in spite of God's divine order for us to have dominion over the earth. Nevertheless, it is clear that the system of which we are part is by its very nature creative. Our genes are creative and allow creative mutations but they must be stable in their creativity. Their creativity is our wellspring. Not only must our genes, through the mechanisms of biology and random mutation invent new viable forms, they must also be prone to do so.
Our mathematical creativity may actually be initiated by random events at the deepest level, after deductive reasoning, consistencies, experience, and even intuition are factored out of the process. But the creative mind contains something much more important than a random idea generator; it provides an environment in which the wild seed of a new idea is given a chance to survive. It is a fertile place. It has its refugia and extinctions. In giving credit for creativity, we really praise not random generation but the determination to give life to new forms.

But when does a newly thought-up mathematical concept, C survive? According to Barnsley 2010, C must be consistent with the existing mathematics, true, correct, non-trivial and interesting. An important condition for C to survive in the minds and words of mathematicians is that it is, itself, a creative system and adaptable in some mathematical sense. *Stochastic analysis and applications*, which is my major research area has been found to be a creative system. This shall become clear later when I dwell on my research experience in what follows.

3. THE NATURE AND STRUCTURE OF MATHEMATICS

Mathematics has been in the world for a long time. It emerged in Babylonia in the 3rd millennium BC and has ever since permeated science and civilisation. It crosses both national boundaries and subject boundaries and in the words of *R. W. Hamming* (The unreasonable effectiveness of mathematics, American Math Monthly

87, 1980)-

Our main tool for carrying out the long chains of tight reasoning required by science is mathematics.

All over the advanced and front-line developing worlds, mathematics is given the prominence its deserves with respect to all round developments. The subject is everywhere in applications although it may be hidden, not obvious on the surface. Mathematics develops the power to reason. It shows better than any other subject, how reason can lead to truth. In most cases in other sciences, there are some intermediaries between the problem and the reasoning, there are no such standing things in mathematics.

Economists reason as well, but sometimes two economists reason to two different conclusions. Philosophers reason, but never come to any conclusion. In mathematics, problems can be solved, using reason; and solutions can be checked and shown to be correct. Reasoning needs to be learned, and mathematics is the best way to learn it.

In the year 1906, J. D. Fitch wrote to his fellow Americans: *Our future lawyers, clergy, and statesmen are expected at the University to learn a good deal about curves, angles, number and proportions; not because these subjects have the smallest relation to the needs of their lives, but because in the very act of learning them they are likely to acquire that habit of steadfast and accurate thinking, which is indispensable in all the pursuits of life.*

We do not know who J.D Fitch was, but his statements remain correct till this day. In the same vein, Thomas Jefferson (1743-1826), the US third President said *Mathematics and natural philosophy are so peculiarly engaging and delightful as would induce everyone to wish an acquaintance with them. Besides this, the faculties of the mind, like the members of a body, are strengthened and improved by exercise. Mathematical reasoning and deductions are, therefore,*

a fine preparation for investigating the abstruse speculations of the law.
Also in 1834, the US Congressional Committee on Military Affairs reported: *Mathematics is the study which forms the foundation of the course at West Point. This is necessary, both to impart to the mind that combined strength and versality, the peculiar vigour and rapidity of comparison necessary for military action, and to pave the way for progress in the higher military sciences.*
According to Underwood Dudley, a retired mathematician at DePauw University, (Notices of the AMS, Vol 57 (5), 2010, 608-613) a contemporary student gave the following testimony: *The summer after my freshman year I decided to teach myself algebra. At school nest year my grades improved from a 2.6 gpa to a 3.5 gpa. Tests were easier and I was much more efficient when taking them and this held true in all other facets of my life. To sum this up: algebra is not only mathematical principles, it is a philosophy or way of thinking, it trains your mind and makes otherwise complex and overwhelming tests seem much easier both in school and in life.*

The above quotations are foods for thought especially to us as a nation. It is therefore, important for us as a nation to deeply appreciate the roles and connections of mathematical sciences to the technological, scientific and socio-economic development of all nations. Our recent experience of diminishing success of students in West African Examination Council (WAEC) and National Examination Council (NECO) examinations in mathematics and the unpopularity of the subject as a course of study at the tertiary levels further underscores the need for intensive efforts in the development and awareness of mathematical sciences in Nigeria. This lecture is aimed at shedding lights on the various connections of mathematics to development, thereby encouraging our government and stake holders in education to undertake profitable policy formulations and actions.

Professor A.O. Kuku (2004), about eight years ago, made detailed explanations concerning the phenomenal growth of mathematics in the last one hundred years, This had made contemporary developments in the subject inaccessible to most literate people the world over. Various cultures like the Egyptians, Babylonians, Greeks, Indians, Mayans, Chinese, Europeans, etc. had made significant contributions to the developments of mathematics at various periods of history. The historical development of the subject, as well as the evolution of modern mathematics, are replete with lessons in symbiosis between pure and applied mathematics, with pure mathematics, once created, helping to solve applied problems ,while applied mathematics motivates the creation of new areas of pure Mathematics. However, the subject has come a long way since Archimedes used to struggle to find the volume of a sphere through weighing of infinitesimals (into which such a sphere was divided). Methods currently applied have become very profound, sophisticated, rather technical and diversified, and we rightly see mathematics today in its various ramifications. These are:

The developments in mathematics in the last one hundred years in particular, have been so extraordinary and phenomenal that it is believed that more mathematics has been created in the last fifty years than in all previous age together. The subject is now so large with so many sub-disciplines that it is absolutely impossible for any one to grasp it all. The fact that the subject continues to grow phenomenally in various directions also creates a crying need to continue to find efficient; economical ways of coordinating and unifying ideas. At the moment, we have 97 sections in Mathematics according to the 2010 edition of the classifications by the American Mathematical Society (www.ams.org).

Virtually every area of human activities now requires sophisticated mathematics for their in-depth study, while problems in mathematics that have confounded mathematicians over the years have been solvable only through highly sophisticated and abstract techniques. Furthermore, as we find ways of solving other still outstanding problems, there will be more theories developed, and earlier results generalised to deepen our insight. All these considerations; therefore, raise serious pedagogical issues as to how to minimise the current global illiteracy in contemporary mathematics that has resulted in hostility towards the subject from parents, funding institutions and the general public. A lot of ideas that should have filtered down to the industry, business, government and the entire educational system are still circulating among relatively few specialists.

4. CLASSIFICATION OF MATHEMATICS

History of the development of Mathematics shows that by the beginning of the twentieth century, the central core of Mathematics - Algebra, Analysis and Topology had emerged; the knowledge of which is sine-que-non for any modern practicing Mathematician. Indeed, efficiency in the understanding and usage of the language of these three areas is imperative for frontier research excellence in contemporary mathematics. This central core is usually referred to as Pure Mathematics, while each of these areas has also been broken into various subdivisions constituting part of the core of Mathematics, e.g. Algebra has such subdivisions as algebraic number theory, linear algebra, commutative algebra, group representations, category theory, k-theory etc. Analysis has such subdivisions as Real analysis, Complex analysis, Functional analysis, Harmonic analysis, Stochastic analysis, Fourier analysis, etc. Topology has such ramification as Differential topology, Algebraic topology, Analytic topology, etc. A modern Mathematician soon realises the cross fertilisation of ideas to the extent that some areas are difficult to classify under the rubrics of algebra, analysis or topology since they could be a combination of all. Under the rubrics of Applied Mathematics, also extending into Mathematical Sciences, we have such area as mechanics, electromagnetic theory, quantum mechanics, various types of Engineering, Operations research, Computer science, Statistics, Mathematical modelling, Mathematical physics, Mathematical economics, Mathematical biology, Mathematical linguistics, even Mathematical history otherwise known as Cliometrics, each of which invariably makes use of ideas from the central core. Many areas of natural, technological and social sciences now require profound and sophisticated Mathematics for their in-depth study resulting even in new Mathematics and further extension of Mathematical frontiers.

5. IMPACT OF MATHEMATICS ON SCIENCE AND TECHNOLOGY

Most of the reasons for the well known underdevelopment of our nation could be traced to low level of science and technology development, which in turn could be reduced to low level of development in the mathematical sciences. The present developing and least developed countries are those that were left behind during the radical mathematical, scientific and technological revolutions of the 19-20th century. Many developed countries in realisation of this fact, are devoting a sizeable part of their budget on education to training in the mathematical sciences. Recent statistics showed that in the U.S.A. for instance, Mathematical education which involves at least 40 million students and two million teachers accounts for about ten percent of the Nation's educational annual expenditure (Kuku, 2004). The picture is similar in most European and Asian countries.

Indeed, some industries and technologies would not exist as they do today without mathematical sciences. For instance:

(a). Unification theory (of electricity and magnetism) of Faraday has led to electrical generation technology.

(b). Maxwell mathematical equations are the fundamental foundation of modern communication systems - radio, television, etc;

(c) Computer Revolution is Mathematics revolution.
There would have been no computers today without the work of Alan Turing, an English Mathematician who a few years before the Second World War, gave a cogent and complete analysis of the mo-

tion of computation leading to the conclusion that it should be possible to construct universal computers which could be programmed to carry out any possible computation. Turing's logical proof of the existence of universal computers resulted in the modern all purpose digital computers and influenced the thinking of other pioneers in the development of modern computer like John Von Neuman. Computers have not only been useful in improving the quality of life of people, but also recorded success in its use to solve outstanding Mathematical problems (e.g. four colour problem, classification of simple groups). Computer graphics or computer vision, and shape analysis in particular have recently benefited from the intervention of the Gromov-Hausdorff distance results of the famous Russian Mathematician and the recipient of the 2009 Abel prize of the Norwegian Academic of Science and Letters, Mikhail Gromov (Notices of the AMS, Volume 57 (3), 391-403). Moreover, new areas of Mathematics have been created through the computer (e.g. complexity theory, proof theory, theory of algorithms), while computers are currently being used to teach Mathematics (e.g. calculus, matrices, probability, statistics, and some geometry). Computers have also opened the way to new technologies for the betterment of the society as well as proved useful in solving various problems arising in technology, business, commerce, economics, etc. Thus, computer is a creation and a creator of mathematics, and hence a creative system.

(d) The abstract area of Mathematics known as Fourier Analysis has since become a subject important in studying electromagnetic waves such as X-ray, visible light, microwaves, radio waves, and their harmonic components. Many electrical and electronic devices such as nuclear magnetic resonance and X-ray crystallographic spectrometers are based on Fourier Analysis. Also, Fourier Analysis has provided a basis for understanding quantum theory, and hence, modern chemistry and physics. It also led to the discovery of the time-series analyses used in oil exploration for interpreting seismic rocks suspected of bearing petroleum. An ability to decompose sound into its harmonic components using Fourier analysis has allowed computers to generate and recognise human speech.

(e) Like Fourier analysis, wavelet analysis deals with expansion of functions, but in term of wavelets i.e. given fixed function with mean zero. Variations of this theory have found applications in image processing. Acoustics, coding (in form of quadratic mirror filters and pyramid algorithms) and in oil exploration, analysis of rapidly changing transient signals, electric currents in the brain, impulse underwater sounds and monitoring of power plants. It has also served as a scientific tool for sorting out complicated structures occurring in turbulence, atmospheric flows, and study of stellar structures.

(f) It is noteworthy that the 1985 Nobel Prize in Chemistry was awarded to the Mathematicians, H.H. Hauptmann and J. Karle - for their development of the models for the determination of crystalline structure based on Fourier Analysis and probability. Similarly, the 1997 Nobel Prize in Economics was jointly awarded to two mathematicians, Scholes and Merton for their development of option price models based on the dynamics of the underlying asset price that follows a geometric Brownian motion Black and Scholes, 1973; Merton, 1973).

(g) Group theory, discovered (in the abstract) by Galois while studying mathematical symmetries associated with the solution of polynomial equation, has subsequently been applied to significant advantage in the study of subatomic particles, in crystallography, information theory, photochemistry and even in the elucidation of certain complicated marriage systems studied by anthropologists. Graph theory, the mathematical study of abstract networks, was considered an esoteric kind of pure mathematics until it was applied to problems of transportation, communication, urban planning, electrical networks, neurophysiology and sociology.

(h) Partial differential equations (PDE's) have been used to model shocks in non-linear waves, vortices in fluid flows with various applications including accurate tracing of hurricanes, and to study blood flow through the heart, the efficient mixing of fuel in the internal combustion engine, aircraft flight, and the way in which radio telescopes sense distant galaxies. They have also been used to model icebergs melting in the sea, crystal growth and the flow of oil and water through a reservoir.

(i) Linear PDE's govern small functions of small disturbances from equilibrium while non-linear PDE's govern large disturbances. The real world is generally non-linear. PDE's are being used to solve problems in geometry, physics engineering etc. For example, non-linear elliptic equations are PDE's arising in geometry - especially in the construction of surfaces with prescribe curvature with diverse applications in engineering and physics.

(j) Physiological fluid dynamics has various applications including computational models of the heart, the kidney, the pancreas, the ear and many other organs. Blood flow in the heart is governed by coupled equations of motion - of the muscular heart walls, elastic heart valve leaflets and the blood flowing in the cardiac chambers. This will be explained a bit more below. Computational Fluid Dynamic (CFD) is a primary aerodynamical design tool for any problem and the wind tunnel is treated more of an evaluation and confirmation tool.

(k) Computer solutions of PDE's arising in physiological fluid allow the study of flow of suspensions, blood clotting, wave propagation of the inner ear blood flow in the arteries and veins and airflow in the lungs. Computer graphics have been particularly useful in many studies including the theory of surface inspired originally by the study of soap films. Computer graphics enhance the understanding of global and stability problems in the calculus of variations.

(l) Greenhouse effect is the warming of the surface of the earth due to re-radiated energy by the Greenhouse gases - e.g. ozone methane etc. from the atmosphere to the surface. The Greenhouse theory says that recent modification of the atmospheric gaseous composition will result in the gradual warming of the earth's surface as well as a cooling of the upper atmosphere, leading to an unprecedented modification of the earth's climate. A three-dimensional circulation model, involving numerical solutions of PDE's are used to compute differences between a climate forced by increases in Greenhouse gases and a controlled climate.

(m) At the turn of the 20th century, Poincare realised that the behaviour of trajectories of celestial bodies could display a chaotic motion - a motion forever oscillating yet irregular and aperiodic. In 1963, a numerical examination of some specific systems of Differential equations from meteorology revealed the presence of chaotic trajectories in specific non-Hamilatonian system but also suggested new directions of research in the theory of dynamical systems. Copious applications of these ideas exist in ecology, economics, physics, chemistry, engineering, fluid dynamics and meteorology. Dynamical systems (especially those representing chaotic behaviour) involves topology, number theory, measure and ergodic theory, combinatorics, etc.

(o) The Nineteenth Century witnessed the study of Hamiltonian mechanics i.e. study of many particles moving without friction and governed by equations which take standard form when the Hamiltonian - total energy of the system is taken as the standard point. Modern Hamiltonian mechanics is the study of symplectic manifolds - sympletic geometry. A sympletic manifold is a higher dimensional surface on which the Hamiltonian procedure of passing from Hamiltonian to differential equations can be implemented.

(p) A lot of social and economic problem can be modelled mathematically using notably the theory of games or combinatorial scheduling theory. Problems connected with inventory control, industrial production and efficiency in the allocation of resources can be solved by various methods in operations research including simplex method in linear programming.

(q) Population biology has to do with counting, estimating and predicting population sizes. The problems involved range from determining the mechanisms that cause and maintain biological rhythms to problems posed by the management of exhaustible resources like timber and fish or even to geographical distribution of genes, age distribution of populations, the spread of forest disease, and genetic engineering. It is noteworthy that population problems have led to the development of many theories and methods that are central to the core of mathematics e.g. theories of probability, dynamical system and wave propagation.

6. RECENT APPLICATIONS IN MEDICINE, BUSINESS, INDUSTRY AND GOVERNMENT

Applications of Mathematics to development projects in medicine, business, industry and government all over the world, offer a wealth of exciting problems for Mathematicians. In a variety of settings, Mathematics is a key component in important projects. Recent applications and breakthrough in mathematics were published in the Mathematical Moments section of the American Mathematical Society website (www.ams.org/mathmoments). Some extracts from the site are contained in what follows:

(a) **Forecasting Weather**: Forecasting the weather requires enormous amounts of data and computation. In order to have an accurate model of the weather, one must know the temperature, humidity, air pressure and wind speed (among other things) at different points and elevations. Although incorrect forecasts may be more memorable, current three- to seven-day forecasts are better than 36-hour forecasts were just 20 years ago. Increases in computing power have helped improve weather forecasts, but it's the mathematics behind the models that has led to the great increase in accuracy. Collected information is the basis for numerical calculations that output approximate solutions to the relevant nonlinear partial differential equations. Weather models take into account the rotation of the Earth and the perpetual interaction among land, sea, and air. While more data and better computers are obvious sources of improved forecasting, the not-so-obvious sources of better sampling techniques and better use of data have helped as well. Mathematics is also used in models for forecasting atmospheric conditions, such as ozone levels, in major metropolitan areas.

(b) **National Security, Defence Industry and Criminology:** Mathematics has helped investigators in several major cases of human rights abuses, election fraud, and in national security.

(i) **Use of Mathematics for Business Information Security**: Professor M. O. Ajetunmobi (2006) talked extensively on the use of mathematics for business information security. These are physical security, operational security and management and policy securities. A mathematician's major concern is therefore, on the operational security, which deals with how business environment does things and includes the use of computers, telecommunication systems, computers and management information. Operational issues include access control, authentication and security topologies. The goals of business information security are: prevention of security breaches, detection of security breaches and response to develop strategies and techniques to deal with an attack or loss of data. Cryptography, the mathematical science of keeping oral and written forms of communication secret has found applications in all kind of security issues. Cryptographic algorithms span the mathematical world of number theory, complexity theory, elliptic curves, vector calculus, tensors and set theory.

Mathematics is a key to the promotion of national and world peace. According to information provided at the website of the US National Security Agency/Central Security Service
(NSA/CSS) (www.nsa.gov), the agency of the Defence Department responsible for solving cryptographic problems for the US federal

government, is the largest employer of mathematicians in the US. NSA mathematicians use tools from diverse areas of mathematics including number theory, Fourier analysis, statistics, to solve cryptographic problems. The core missions are to protect U.S. national security systems and to produce foreign signals intelligence information. The National Security Agency/Central Security Service (NSA/CSS) is home to America's code-makers and code-breakers. The National Security Agency has provided timely information to U.S. decision makers and military leaders for more than half a century. The Central Security Service was established in 1972 to promote a full partnership between NSA and the cryptologic elements of the armed forces. NSA/CSS provides products and services to the Department of Defense, the Intelligence Community, government agencies, industry partners, and select allies and coalition partners. In addition, the agency delivers critical strategic and tactical information to war planners and war fighters.

The Signals Intelligence mission collects, processes, and disseminates classified intelligence information from foreign signals for intelligence and counterintelligence purposes and to support military operations. This Agency also enables Network Warfare operations to defeat terrorists and their organizations at home and abroad, consistent with U.S. laws and the protection of privacy and civil liberties. According to the Director NSA, Lt. General Keith Alexander :*The majority of our nation's intelligence for counterterrorism, hard targets and support to military operations comes from the National Security Agency /Central Security Service. For the good of the nation, it is imperative that NSA/CSS maintain its cryptologic superiority.*

(ii) **Mathematics Techniques in the Design and analysis of Future Military Systems:** The operational Requirements Analysis has two primary roles: (1) analysis of tactical systems to determine their operational requirements and (2) the performance of simulation analysis of the advanced weapons systems. Mathematics is used at every step of the design and analysis phases to assess the mission effectiveness of proposed and existing systems. Data on mission, system, and crew performance, such as survivability, system effectiveness, crew workload, enemy encounters, etc is collected during simulator tests. The raw data is processed using several mathematical and statistical methods and computing and the results are compared across different system versions or tactical concepts. Relevant mathematical disciplines include modelling, probability, statistics, differential equations and applied mathematics. Other scientists and engineers together with computer programmers work as a team with mathematicians in the defence industry (Society for Industrial and Applied Mathematics: Career in Mathematics).

(iii) **The 2009 Election in Iran:** A mathematical result known as Benford's Law states that the leading digits of truly random numbers aren't distributed uniformly, as might be expected. Instead, smaller digits, such as 1's, appear much more frequently than larger digits, such as 9's. Benford's Law and other statistical tests have been applied to the 2009 election in Iran and suggest strongly that the final totals are suspicious.

(iv) **Guatemalan Disappearances:** Here, Mathematics and Statistics are being used to extract information from over 80 million National Police archive pages related to about 200,000 deaths and disappearances. Sampling techniques give investigators an accurate representation of the records without them having to read every page. Families are getting long-sought after proof of what happened to their relatives, and investigators are uncovering patterns and motives behind the abductions and murders. Tragically, the people have disappeared. But because of this analysis, the facts won't. Graph theory, combinatorics and statistics are just some of the Mathematical fields being used today by real life investigators in advanced countries to solve actual crimes.

(c) **Revealing Secrets of Nature:** Mathematical ecology deals with the study of Mathematical modelling of complex biosystem.

Mathematical ecology researchers are faced with the task of simulating several interconnected networks of organisms across different scales of time, size and space. New areas of Mathematics such as nonlinear dynamical systems and spatial statistics have become applicable.

(d) **Medical Applications of Mathematics**

(i) **Mathematical Models in Population Biology and Epidemiology:** These are concerned with analysing the spread of disease using a system of evolutionary equations that reflect the dynamics among three distinct categories of population, those susceptible S to a disease, those infected I with the disease, and those who recovered R from the disease. This SIR model is applicable to a range of diseases from small pox to HIV/AIDS. Armed with reliable models, mathematicians help public health officials in battling the complex, rapidly changing world of modern diseases.

(ii) **Health Care Delivery**: Mathematical fields of probability and statistics play a role. For example, years of data proved that the annual chest x-ray and some diabetes drugs actually did more harm than good. Those practices have now been abandoned, saving lives and money. An area of medicine that requires more analysis is chemotherapy. Many dosing strategies are prescribed based on a patient's ability to tolerate side effects not necessarily on demonstrated efficacy. At times, untreated tumours shrink, while the treated tumours grow, and differential equations and numerical analysis have been used to solve this puzzle by modelling the interactions of tumour cells, immune cells, host cells, and drugs in patients. This allows for more complex combinations of chemotherapy and immunotherapy so as to maximise the treatments benefits while minimising their side effects.

(iii) **Defeating Disease:** From modelling microscopic genes and proteins to tracing the progression of an epidemic through a country, mathematics plays an important role in combating disease. For example, the basic model used to analyse the dynamics of infectious disease is a system of differential equations. A new field called data mining', involving statistics and pattern recognition, helps to locate significant information in the vast amounts of data collected from studies of diseases in populations. Mathematics also plays a key role in connecting changes in the human genome to specific diseases. Mathematics has helped recent fights against foot-and-mouth disease in the United Kingdom and against Chagas disease, a disease affecting millions of people in Latin America. Epidemiologists studying the mouth and foot epidemic also use mathematical models to conclude that early efforts were insufficient to stop what would become a calamitous spread of the disease. The government accepted the conclusions and took a course of action that, although drastic, did indeed arrest the outbreak. In Latin America, mathematicians computationally tested several courses of action against Chagas disease and found a surprisingly simple yet highly effective step (keeping dogs out of the bedroom) to greatly reduce the infection rate. These examples share three important characteristics: a mathematical model of the disease, modern computers to do calculations required by the model and researchers with the insight to design the former so as to take advantage of the power of the latter.

(iv) **Genetic Information about Diseases:** The state-of-the-art technology used by researchers to identify active (expressed) genes in cells is the microarray: a gene chip imprinted, not with circuits, but with DNA. Active genes or fluorescently tagged cell samples placed on the chip reveal themselves when they bind with their DNA complements on the chip. The amount of data generated by this microscopic activity is enormous: just one row in an array can have 15,000 points. Pattern recognition and image analysis are two fields which use mathematics to help extract important genetic information about several diseases, including Alzhiemer's and Parkinson's from microarray data. In the future, microarrays may enable an individualised approach to medicine, in which your doctor could use these chips to diagnose disease and determine the best treatment for your unique genetic profile. In one particular area of medicine, cancer research, the points in each column of an array can be thought of as genetic coordinates of samples from tumours. Yet there are so many coordinates that it is difficult to determine which tumours are similar. Algorithms employ statistics and different measures of distance in higher dimensions to group genetically similar tumours into clusters so that experiments can be done on treatments corresponding to the clusters. In one case, microarray technology not only distinguished between two different types of leukaemia (verifying in the time it took to hit "Return" what had taken 35 years to discover) but also found different clusters within tumours that had been thought to be similar - resulting in clinical trials to confirm the distinction.

(v) **Source Localisation in the Human Brain:** Surgical therapy has become an important therapeutic alternative for patients with medically intractable epilepsy. Based on the EEG signal, a mathematical algorithm with the appropriate software tools have been developed to obtain a correct and anatomically precise localisation of the epileptic focus.

(vi) **Toxicology:** Mathematicians together with other scientists have been involved in the development of mathematical models to simulate the flow of air in the respiratory tract. These models are used to determine where chemicals end up in the respiratory tract as a result of air flow. The simulation of the airflow within the respiratory tract has provided needed information in the study of the evolution of cancer tumours
(www.chalmers.se/Math/EN/research/gmmc/solved-problems).

(e) **Applications in Politics: Making Votes Count:** Mathematicians study voting methods in the hope of finding equitable procedures, so that no one is unfairly left out in the cold. The outcome of elections that offer more than two alternatives with no preference by a majority is determined more by the voting procedure used than by the votes themselves. Mathematicians have shown that in such elections, illogical results are more likely than not. For example, the majority of this group wants to go to a warm place but the South Pole is the group's vacation destination, in the same way most elections are conducted; they will all go to the South Pole and six people will be disappointed, if not frostbitten. Elections in which only the top preference of each voter is counted are equivalent to a school choosing its best student based only on the number of A's earned. The inequity of such a situation has led to the development of other voting methods. In one method, points are assigned to choices, just as they are assigned to grades. Using this procedure, these people will vacation in a warm place, a more desirable conclusion for the group.

(f) **Resolving Traffic Problems:** The mathematical study of traffic is relatively new, but a US Federal report concluded that the information revolution, i.e., the combination of more powerful computers, telecommunications and better numerical models will affect transportation as much as the invention of the automobile and jet engine. Analysing traffic (like predicting weather) requires many variables (driver speed, length of trip, time of day and origination point) and involves chaos theory (a small change down the road can drastically change travel conditions). Unlike weather; however, traffic can change in response to a forecast as alternative routes are chosen today by drivers and in the future, perhaps by the cars themselves.

(g) **Finding Oils:** As high as the prices of petroleum products are, they would be much higher without modern oil exploration techniques, which make operations more efficient (and cleaner). Drilling a well can cost 20 million dollars, so drillers now rely on mathematical models of reservoirs rather than hunches, to choose sites. The models approximate a reservoir's characteristics from data collected using sound waves beamed underground, and from the resulting systems of nonlinear equations. In fact, one company estimates that it solves over 250,000 systems a day. The reservoir simulations are derived from partial differential equations describing fluid flow and from terabytes of data, but they still contain a good deal of uncertainty. Researchers are using statistics to quantify the uncertainty involved, thus giving planners models that are more descriptive of subsurface properties such as permeability. One thing is certain, however: Finding new sources of energy to meet future energy demand will continue to depend on advances in the mathematical sciences.

(h) **Revolutionising Computing:** In about 20 years, computer chips will be so small that the effects of quantum mechanics will replace the physical laws we take for granted. While today's computing is based on bits that are either 0 or 1, the basic unit in quantum computing is the quantum bit - the qubit - which can be 0 and 1 simultaneously (with a probability associated with each). In the strange world of quantum computing, complicated procedures such as factoring large numbers are done much faster because the many steps involved can be done concurrently. The ultimate goal of mathematicians, physicists, computer scientists and engineers in the field is to create a quantum computer that could solve in seconds some problems that would take today's most powerful computers billions of years to solve. Among the capabilities of a quantum computer would be the ability to do the calculations necessary to break today's electronic encryption methods. This is not as alarming as it may sound, because cryptographers have already designed algorithms to take advantage of the quantum mechanics principle that observing a system's state changes it. Thus, users of a quantum communications network could detect any attempt to intercept their communication. It is somehow ironic that the laws that govern the barrier to the miniaturisation of today's computers may provide a boom to future computing.

(i) **Preserving the Past:** Structures that have stood for thousands of years are now crumbling because of air pollution. Mathematicians are using models that incorporate factors such as humidity, temperature and the level of pollution to better understand the degradation process (which occurs when pollutants reacting with water vapour transform the outer surface of stone into a vulnerable layer of porous gypsum). The models, based on differential equations, can point to better strategies for restoring ancient monuments, perhaps preventing their destruction. One difficulty in modelling the deterioration is that the process depends heavily on constantly changing conditions among many variables, such as humidity. Due to this large number of relevant features, simplifying assumptions are made - for example, that the air temperature equals that of the gypsum layer along the air-gypsum boundary, to make the models manageable. The resulting non-linear equations are then solved numerically, and despite the simplifications, the predictions are accurate. Researchers who developed these models have recently discovered the following: that there is a humidity threshold below which stone isn't converted to gypsum, that removing existing gypsum can be counterproductive, and that the size of the advancing decaying front varies with the square root of both time and the concentration of pollutants.

(j) **Power Distribution in Parliament:** Votes are cast by full membership in each house of Congress, but much of the important manoeuvring occurs in committees. Graph theory and linear algebra are two mathematics subjects that have revealed a level of organisation in Congress, groups of committees, above the known levels of subcommittees and committees. The result is based on strong connections between certain committees that can be detected by examining their memberships but which were virtually unknown until uncovered by mathematical analysis. Mathematics has also been applied to individual congressional voting records. Each legislator's record is represented in a matrix whose larger dimension is the number of votes cast (which in a House term is approximately 1000). Using mathematical notion of eigen-values and eigenvectors, researchers have shown that the entire collection of votes for a particular Congress can be approximated very well by a two-dimensional space. Thus, for example, in almost all cases the success or failure of a bill can be predicted from information derived from two coordinates. Consequently it turns out that eigenvalues are some of the important values.

(k) **Recognising Speech:** Current speech recognition systems perform fairly well in non-conversational settings such as dictation or requests for directory assistance. Applications like this may not appear impressive, but because of accents, inflections, and pauses, even simple situations require sophisticated techniques to transform speech waveforms into words accurately. One of the most common techniques is a mathematical tool known as a hidden Markov model, involving conditional probabilities, which trains on candidate sounds so as to locate the best match for a given input. Dictating directions to machines, a luxury now, may become a necessity as input devices become too small. Researchers are looking for new mathematical models and algorithms (which will probably use subjects like statistics and machine learning) that can filter out noises, understand casual speech, and adjust to different speakers. Those are difficult problems, but once solved, it won't be long before your voice replaces your keyboard, mouse, and - best of all - your many remotes.

(l) **Tracing Travelling Routes:** The Travelling Salesman Problem entails finding the shortest route that passes through each assigned town exactly once. The problem is noteworthy for its complexity, which grows exponentially with the number of towns, and for its applications, which range from wiring a chip to scheduling airline crews. Mathematics researchers use graph theory and linear programming to solve the problem when feasible and to find near-optimal solutions in other instances, saving industry time and money. There may never be a workable general solution to the Travelling Salesman Problem. Yet even without knowing the best answer, mathematicians still can estimate how close to optimal a given route is. Perhaps even more surprising: Operating on a map of 25,000 towns, current algorithms design paths whose lengths are within 0.01% of that of a shortest path.

(m) **Predicting Storm Surges**: Storm surge is often the most devastating part of a hurricane. Mathematical models used to predict surge must incorporate the effects of winds, atmospheric pressure, tides, waves and river flows, as well as the geometry and topography of the coastal ocean and the adjacent floodplain. Equations from fluid dynamics describe the movement of water but most often such huge systems of equations need to be solved by numerical analysis in order to better forecast where potential flooding will occur. Much of the detailed geometry and topography on or near a coast require very fine precision to model, while other regions such as large open expanses of deep water can typically be solved with much coarser resolution. So, using one scale throughout either has too much data to be feasible or is not very predictive in the area of greatest concern, the coastal floodplain. Researchers solve this problem by using an unstructured grid size that adapts to the relevant regions and allows for coupling of the information from the ocean to the coast and inland. The model was very accurate in tests of historical storms in Southern Louisiana and is being used to design better and safer leeves in the region and to evaluate the safety of all coastal regions.

(o) **Determining Rogues:** It doesn't take a perfect storm to generate a rogue wave, an open-ocean wave that is much steeper and more massive than its neighbours, and which appears with little or no warning. Sometimes, winds and currents collide causing waves to combine non-linearly and produce towering walls of water. Mathematicians and other researchers are collecting data from rogue waves and modelling them with partial differential equations to understand how and why they are formed. A deeper understanding of both their origins and their frequency will result in safer shipping and offshore platform operations. Since rogue waves are rare and short lived (fortunately), studying them is not easy; so some researchers are experimenting with light to create rogue waves in a different medium. Results of these experiments are consistent with sailors' claims that rogues, like other unusual events, are more frequent than what is predicted by standard models. The standard models had assumed a bell-shaped distribution for wave heights, and anticipated a rogue wave about once every 10,000 years. This purported extreme unlikelihood led designers and builders to not account for their potential catastrophic effects. Today's recognition of rogues as rare, but realistic, possibilities could save the shipping industry billions of dollars and hundreds of lives.

(p) **Environmental Engineering**: There is a formidable interplay between mathematical modelling, technology and stochastic control in solving environmental engineering problems. Environmental engineering is invariably aimed at controlling the evolution of systems that contains inherent uncertainty. Existing theory in this field has applied mathematical results in state space models, Markov decision processes, dynamic programming, control of linear systems, Kalman filtering, system identification and adaptive control. Environmental engineering has challenged mathematicians in managing uncertainty. There is a fast exploding research field aimed at developing mathematical models and techniques that adequately deal with the problems of uncertainty encountered in environmental decision making. There are many fast developing branches of mathematics that contain concepts, techniques, theorems, and algorithms that have much to offer in this area. The latter include the theories of signal processing, stochastic and robust programming, large deviation theory and the theory of singular perturbations of operators and viability theory [Notices of the AMS, 57 (10): 2010].

(q) **Combustion Engine Optimisation:** It is nowadays possible to utilise computational fluid dynamics (CFD) methods to simulate the physical and chemical processes inside combustion engines. By varying the design parameters of the engine, different configurations can be simulated and their performances compared over a set of representative engine load conditions. The goals when designing a new engine are typically in conflict: for highly efficient engines, we have high burning temperatures which may result in the formation of a lot of nitrogen oxides, in addition to high fuel consumption. Multi-criteria optimisation techniques have been employed to get good designs that have Pareto optimal points taking into account constraints on the combustion process, like a maximum pressure inside the engine during the combustion stroke. Even the best CFD codes can take days to run one simulation, while the still will produce numerical errors giving a serious challenge for optimisation. Recently, a team of researchers consisting of engineers, mathematicians, physicists and computer scientists at Chalmers University of Technology, Goteborg, Sweden, Volvo Car Corporation and Volvo Powertrain Corporation have devised and tested a new algorithm for the design of a new diesel engine (www.chalmers.se/Math/EN/research/gmmc/solved-problems). For a given set of evaluated designs, an explicit approximation model is constructed, based on radial basis functions. Its approximate Pareto optimal set guides the search for one or more new designs, with the aim to reduce the uncertainty of the approximation model near the Pareto front. The new method avoids potential pitfalls stemming from simulation errors, and further produces a final approximation model that can be used for post-processing of the proposed designs.

(r) **Mathematical Cardiology/Physiology:** The exposition here is a short extract from a very recent article published by Professor John Cain of Virginia Commonwealth University (Cain, 2011). There are many important research problems in cardiology that have been jointly attacked by mathematicians, clinicians and biomedical engineers, and which have contributed to improved understanding of cardiac abnormalities. Since some of the most exciting research problems in mathematical cardiology involve electrical wave propagation in heart tissue, research in cardiac electrophysiology has attracted deep attention of the above named group of researchers. The quantitative study of electrophysiology was pioneered by George Ralph Mines (1886-1914), and his contribution has continued to influence the mathematical study of reentrant arrhythmia. Nearly half a century after Mines' death, in a stunningly elegant blend of mathematics and experimentation, British physiologists Alan Hodgkin and Andrew Huxley introduced a model of electrical propagation in the squid axon. The first of many adaptation of their mathematical model to cardiac tissue soon after led

Hodgkin and Huxley to share the 1963 Nobel Prize in Medicine or physiology for their efforts. Examining the electrical activity in a person's body can reveal a great deal of physiological information.

Electrical activities in the brain and the heart are respectively recorded by someone undergoing electroencephalogram (EEG) and electrocardiogram (ECG). Bodily fluids contain positively charged ions that transverse cell membranes, the resulting electrical currents elicit changes in the voltage V across the cell membrane. In the absence of electrical stimulation, V stays at rest at a constant value. Electrical stimuli can cause a resting cardiac cell to respond in a rather dramatic fashion resulting in an abnormality called an action potential. Mathematically modelling the cardiac action potential is an attractive research topic partly because such models tend to be rooted in the Nobel Prize winning work of Hodgkin and Huxley. The key idea is to model the cardiac cell membrane as an electrical circuit. In building a realistic model, the tricky part is to determine the specific form of the ionic current. A major modelling challenge is to simultaneously keep the model minimally complicated so that it is amenable to mathematical analysis but make the model sufficiently detailed that it can reproduce as much clinically relevant data as possible. There is a large repository of ionic models available in the literature. Although the model span a wide range of complexity, most are based upon the original Hodgkin Huxley formalism and are presented as systems of ordinary differential equations (Cain, 2011).

Adapting this single cell model to the tissue level, the resulting system of partial differential equations has been well studied, revealing (i) the existence of travelling pulse solutions (solitary action potentials) (ii) existence of periodic wave train solutions (iii) stability of these solutions and how they involved from initial data (iv) asymptotic estimates of action potential duration and velocity in terms of the parameters (v) existence of spiral (2-D tissue) and scroll (3-D tissue) wave solutions (vi) asymptotic estimates of the rotation frequencies of spiral and scroll waves which are important in the genesis of certain arrhythmias. Action potentials can propagate through cardiac tissue because individual cells are electrically coupled. In the domain Ω formed by the heart tissue, transmembrane voltage has both spatial and temporal dependence: $V = V(x, y, z, t)$ where (x, y, z) belongs to Ω. A model of the electrical wave propagation in cardiac tissue has emerged in the form of a boundary value problem for partial differential equations involving V, its gradients, ionic current with Neumann boundary conditions (Keeners and Sneyd, 1998). The resulting equation presents a nice challenge for numerical analysts. Many diverse related cardia-vascular problems are similarly receiving mathematical treatments. Modelling groups around the world including those led by Peter Hunter of Auckland Bioengineering Institute, University of Auckland and Rob Macleod (Scientific Computing and Imaging Institute, University of Utah) have made considerable progress in this field.

(s) **Mathematical Biology:** Avner Friedman, a Distinguished Professor of Mathematics and former Director of the Mathematical Biosciences Institute at Ohio State University explained some recent achievements and direction of research in Mathematical Biology (Friedman, 2010). What follows is a short summary from the article.

Recent years have witnessed unprecedented progress in the biosciences. Perhaps the most visible event is the completion of the Human Genome Project - the first step toward a molecular genetic understanding of the human organisms. Subsequent discovery of non-coding genes and deeper understanding of the genomic/proteomic machinery continue to advance biology at a revolutionary pace. Advances are reported continually in the fights against cancer and degenerative diseases of the brain, such as Alzheimer's, Parkinson's, and ALS, and in the management of health threats such as AIDS, insect disease vectors, and antibiotic resistance. Society is eager to see basic research quickly translated into longer and better quality of life through deeper understanding of disease mechanisms and better medical treatment. Accordingly, many topics from bioscience have been given high priority on the US national agenda.

Behind the headlines lie astonishing advances in basic science and technology, including medical imaging, nanoscale bioengineering, and gene expression arrays. These technologies have rapidly generated massive sets of loosely structured data and enabled researchers to elucidate basic biomedical mechanisms and pathways. This explosion of experimental results has challenged researchers' abilities to synthesise the data and draw knowledge from them. Thus the emergence of models and the existence of large data sets that require quantitative analysis, coupled with strong public support for accelerated progress in the biosciences, present a great opportunity for the mathematical sciences.

To successfully exploit this opportunity will require mathematical scientists to learn the bioscientists' language so that they can understand the underlying biology clearly enough before they bring the power of mathematics to bear. While we can expect that established methods in mathematical sciences will be of immediate use, the quantitative analysis of fundamental problems in bioscience will undoubtedly require new ideas and new techniques. Indeed, when viewed over a long time scale, biological applications launched new fields within mathematics, for example, pattern formation in reaction-diffusion equations and combinatorial problems arising in sequence alignment. There already exist several mathematical bioscience research groups in departments of mathematics, statistics, computer science and biology, as well as biostatistics centres in medical research facilities around the country. In addition, individual topics from mathematical biosciences have been featured in the programs of some of the existing mathematical institutes in the United States.

Nevertheless, the current size of the mathematical biosciences community is relatively small compared with the demands of the biosciences. Therefore, there is a need to encourage an influx of mathematicians and statisticians into mathematical biosciences and to nurture a new generation of researchers more systematically than before. Friedman revealed further that these challenges have motivated the founding of the Mathematical Biosciences Institute at Ohio State University as one of the National Science Foundation/ Division of Mathematical Sciences (NSF/DMS) -sponsored Mathematical Institutes. Since 2002 when the Institute became operational, there have been influx of thousands of enthusiastic, both mathematicians and biologists to the Institute.

What is Mathematical Biology?: If the unit of physics is an atom, then the unit of life is a cell; but a cell is infinitely more complex. A cell in mammals typically contains 300 million molecules. Some are very large, such as the DNA molecules, which consist of many millions of atoms. But a cell is not just a huge collection of molecules. The cell maintains control and order among its molecules as exemplified, for instance in the DNA-RNA-protein machinery. A cell absorbs nutrients and generates biomass to perform specific functions, such as secreting chemicals or engulfing pathogens; it adapts to its microenvironment by moving toward sources of nutrients or by remaining quiescent when resources are scarce, and a cell replicates when conditions are favourable. Consequently, mathematical modelling of cellular processes is quite challenging. Furthermore, since the human body has 1013 cells of different types and functions continuously talking to each other, it is quite clear that mathematical models of biological processes are extremely challenging.

Work in mathematical biology is typically a collaboration between a mathematician and a biologist. The latter will pose the biological questions or describe a set of experiments, while the former will develop a model and simulate it. In order to develop a model, for instance in terms of a system of differential equations, the mathematician needs to determine a diagram of relationships among the biological variables and specify rate parameters. Typically some of these parameters are not found in the literature and need to be estimated. They are determined in an iterative process of simulations aimed at achieving good fit with the experimental data. This process may take much iteration. Hence, it is crucial that each simulation does not take too much computational time. When the model simulations finally agree with experimental results, the model may be considered useful for suggesting new hypotheses that are biologically testable. It may suggest, for example, a particular therapy that is represented, in the model, in the form of an increase in one or several rate parameters.

How useful is mathematical biology? This is a question with two parts: (1) does mathematics advance biology and (2) does biology inspire new mathematics? What follow are examples of research conducted at the Mathematical Biosciences Institute, which illustrate how both disciplines, mathematics and biology, benefit from each other.

Ischemic Wounds: Chronic wounds represent a major public health problem worldwide, affecting 6.5 million individuals annually in the United States alone. Vascular complications commonly associated with problematic wounds are primarily responsible for wound ischemia (shortage of blood flow), which severely impairs healing response. Recent experiments with a porcine model to study healing in a preclinical approach were conducted by Roy et. al. in (Friedman, 2010). In those experiments, a full-thickness bipedicle dermal flap was developed first, such that blood supply was isolated from underneath the flap and from two long edges. One circular wound was then developed in the centre of the flap (ischemic wound) and another on the normal skin (non-ischemic wound) of the same animal as a pair-matched control.

In order to determine therapeutic strategies that may help heal ischemic wounds, Xue et. al. (Friedman, 2010 and references therein) developed a mathematical model that incorporates the main variables involved in the wound closure phase of the healing process, namely, several types of blood and tissue cells, chemical signals, and tissue density. The model was formulated in terms of a system of partial differential equations in a viscoelastic, partially healed domain where a portion of the boundary, namely the open wound's surface, is a free boundary unknown in advance. However, each simulation of the free boundary problem in the 3-dimensional geometry takes too much time. The challenge then was how to simplify the geometry while still imposing conditions of ischemia. Xue et. al. assumed that the wound is circular with radius r but that many small incisions of size δ are made at $r = L$ with adjacent incisions separated by distance ϵ. Taking $\delta, \epsilon \to 0$ in appropriate proportions and applying homogenisation theory, they deduced that each boundary condition $u = u_s$ (for a solution of $\Delta u = f$) before the incisions changed into a boundary condition

$$(1-\alpha)(u-u_s)+\alpha\frac{\partial u}{\partial r}=0 \text{ at } r=L$$

after the incisions were made, where α is a measure of ischemia; α near 1 means extreme ischemia. The results are in tight agreement with the experimental results of Roy et al. The model is now going to be used as a tool to suggest biologically testable hypotheses for improved healing, thereby reducing the need for guesswork and time-consuming animal testing.

Cancer-Inspired Free Boundary Problems: The mathematical theory of free boundary problems has developed extensively over the last forty years but the range of new applications has remained modest. Recently, histological changes in biology offered new mathematical models and inspired new theories and examples occurred in tumour growth, wound healing, and developmental biology, to name a few. Consideration here is on tumour models involving a new class of free boundary problems related to symmetry-breaking bifurcations of a spherical tumour and its stability. Consider a tumour that occupies a region $\Omega(t)$ at time t and assume that all the cells in $\Omega(t)$ are identical tumour cells and are uniformly distributed; due to proliferation, the region $\Omega(t)$ will expand but only as long as there is sufficient supply of nutrients σ assuming that the concentration satisfy a diffusion system:

$$\sigma_t - \Delta\sigma + \sigma = 0 \text{ in } \Omega(t), \sigma = 1 \text{ on } \partial\Omega(t),$$

and the proliferation rate S is assumed to depend linearly on σ:

$$S = \mu(\sigma - \tilde{\sigma}) \ (\mu > 0, \ 0 < \tilde{\sigma} < 1);$$

roughly speaking, if $\sigma > \tilde{\sigma}$, the tumour expands, and if $\sigma < \tilde{\sigma}$, the tumor shrinks. Tumour models with several types of cells (proliferating, quiescent and dead) have been analysed mathematically but the existence of spherical stationary solutions and their bifurcation and stability remain mostly open problems.

Mr Vice Chancellor Sir, it is clear from the foregoing discussion that mathematics is indeed a formidable vehicle for development, as its use permeates all fields of human activities and promotes frontier research. I will now proceed to give an account of my major contributions to mathematical research and applications.

7. ACCOUNT OF MY CONTRIBUTIONS

7.1. Stochastic Differential Equations (SDE)

Let me start by giving a brief appraisal of what this field is all about. I shall give a short definition and motivation for studying Stochastic Differential Equations (SDE) and where this field has found applications. An SDE is a generalisation of a deterministic ordinary differential equation incorporating random phenomenon into its formulation and parameters. This makes SDEs more useful for modelling real life problems much more realistically than their deterministic counterpart. It also involves much more complex analysis using specially developed calculus called stochastic calculus (e.g. Ito Calculus pioneered in 1949 by K. Ito (1915-2008), a Japanese mathematician) as distinct from the Newtonian calculus which is applicable to deterministic equations.

Systems in many branches of science, engineering, industry and government are often perturbed by various types of environmental noise arising from some uncertainties and random errors in measurements. If we consider the simple population growth model:

$$\frac{dN}{dt}(t) = a(t)N(t) \tag{7.11}$$

with initial condition $N(0) = N_0$, where $N(t)$ is the size of the population at time t and $a(t)$ is the relative growth. It might happen that $a(t)$ is not completely known or subject to some random fluctuation so that it may be written in the form:

$$a(t) = r(t) + \sigma(t)\xi.$$

Here ξ stands for the noise process. Equation (7.11) then becomes:

$$\frac{dN}{dt}(t) = r(t)N(t) + \sigma(t)N(t)\xi,$$

which is given in integral form by:

$$N(t) = N_0 + \int_0^t r(s)N(s)ds + \int_0^t \sigma(s)N(s)\xi(s)ds. \tag{7.12}$$

The question is: What is the mathematical interpretation for the noise term $\xi(t)$ and what is the integration

$$\int_0^t \sigma(s)N(s)\xi(s)ds? \qquad (7.13).$$

It turns out that a reasonable mathematical interpretation for the 'noise' term $\xi(t)$ is the so called white noise which is formally regarded as the derivative of a Brownian motion (a Wiener process) $B(t)$. Hence we have $\xi(t) = \frac{dB}{dt}(t)$ or $\xi(t)dt = dB(t)$ and therefore, we have the integral that appears in Equation (7.12) given by

$$\int_0^t \sigma(s)N(s)\xi(s)ds = \int_0^t \sigma(s)N(s)dB(s). \tag{7.14}$$

If the Brownian motion $B(t)$ were differentiable, then the integral would have no problem at all. Unfortunately, we know that $B(t)$ is nowhere differentiable hence the integral cannot be defined in the ordinary way. The stochastic nature of the Brownian motion was used by K. Ito in 1949 to establish the integral now known as Ito stochastic integral. This led to the study of a class of stochastic differential equations driven by a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ of the form:

$$dX(t) = E(t, X(t))dt + F(t, X(t))dB(t), \ X(0) = X_0, \ t \in [0, T], \tag{7.15}$$

where the coefficients E, F belong to the appropriate spaces of real or vector valued stochastic processes.

Let it be known that many generalisations and extensions of Ito integral have appeared in the literature. These include generalisation to stochastic integrals driven by semimartingales and several formulations of the non commutative quantum stochastic integrals driven by operator valued processes on certain topological spaces.

7.2. Some Examples of Applications of SDE to Real Life Problems

In order to motivate my audience and stimulate multidisciplinary research collaborations, I will proceed to list some of the recent areas where SDE (7.15) has been applied for modelling and solving real life problems. Individuals interested in the details are welcome to the Department of Mathematics, University of Ibadan and the newly established Adegoke Olubummo International Mathematical Centre (AOIMC), in our Department, for collaborations. Some of these areas are:

(a) Population Dynamics, Protein Kinetics and Genetics
The simplest deterministic model of population growth is the exponential equation

$$\frac{dN}{dt}(t) = AN(t), \ \ A > 0. \tag{7.21}$$

Allowing the vagaries of environment, A can be modelled to vary randomly as $A + \sigma\xi(t)$, for some zero mean process $\xi(t)$. Incorporating a finite supportable carrying capacity K, Equation (7.21) becomes

$$\begin{aligned} \frac{dN}{dt}(t) &= A(K - N(t))N(t) \\ &= \lambda N(t) - N^2(t) \end{aligned} \tag{7.22}$$

where $AK = \lambda$.
On randomizing the parameter λ in Equation (7.22) to $\lambda + \sigma\xi(t)$, we obtain an SDE of the form

$$dN(t) = \left[\lambda N(t) - N^2(t)\right] dt + \sigma N(t) dW(t) \tag{7.23}$$

which is explicitly solvable.
A frequently studied deterministic model of multi-species interaction is the Voltera-Lotka system:

$$\frac{dN_i}{dt}(t) = N_i(t)\left(A_i + \sum_{j=1}^{d} b^{i,j} N_j(t)\right), \; i = 1, 2 \cdots d$$

in the case of d different species. Randomising the growth parameters A_i as $A_i + \sigma_i\xi_i(t)$ leads to a system of SDE with independent Wiener processes given by:

$$dN_i(t) = N_i(t)\left(A^i + \sum_{j=1}^{d} b^{i,j} N_j(t)\right) dt + \sigma_i N_i(t) dW_i(t). \tag{7.24}$$

Explicit solutions are not known for equation (7.24), so approximate solutions are usually obtained numerically.

Protein Kinetics
Stochastic counterparts of many ordinary differential equations modelling chemical kinetics such as the Brusselor equations can be derived by randomizing coefficients. For example, the kinetics of the proportion X of one of two possible forms of certain proteins can be modelled by an ODE of the form

$$\frac{dX}{dt}(t) = \alpha - X - \lambda X(1 - X) \tag{7.25}$$

where $0 \leq X \leq 1$ and the other form has proportion $Y = 1 - X$. For random fluctuations of the interaction coefficients λ of the form $\lambda + \sigma\xi(t)$, with white noise $\xi(t)$, we have the stochastic version of (7.25) given by the Stratonovich SDE:

$$dX(t) = (\alpha - X(t) + \lambda X(t)(1 - X(t))dt + \sigma X(t)(1 - X(t))dW(t). \tag{7.26}$$

The solutions of (7.26) are not known explicitly but remain in the interval $[0, 1]$.
The Ito equation equivalent to (7.26) given by

$$dX(t) = AX(t)dt + \sigma X(t)(1 - X(t))dW(t) \tag{7.27}$$

has been applied to genetics with $X(t)$ representing the proportion at time t of one of the two possible alleles of a certain gene. A discrete time Markov process can be constructed to model the changes from generation to generation in the alleles proportion due to natural selection.

(b) Experimental Psychology and Neuronal Activity
The coordination of human movement particularly of periodically repeated movement has been extensively investigated by experimental psychologists with the objective of gaining deeper understanding into neurological control mechanism. The neurological system is extremely complicated, yet in some situations, a single characteristic appears to dominate and a satisfactory phenomenological SDE model can be constructed to describe its dynamics.

Neuronal Activity: Many stochastic models have been proposed to describe the spontaneous firing activity of a single neutron. These are usually based on jump processes and allow arbitrary large hyperpolarisation values for the membrane potential. A model incorporating several features of neutronal activities has been derived in the form of SDE.

(c) Investment Finance and Option Pricing
Investment Finance: SDE have been used to model share price

dynamics in models of investment finance. Merton [47] considered an investor who chooses between two different types of investment, one risky and the other safe (riskless). At each instant of time, the investor must select the fraction f of his wealth that he put into risky asset with the remaining fraction $1 - f$ going into the safe one. If his current consumption rate $c \geq 0$, then his wealth $X(t)$ satisfies the SDE

$$dX(t) = (\{(1-f)a + fb\}X(t) - c)dt + f\beta X(t)dW(t). \quad (7.28)$$

If the investor has perfect information about his current wealth, Markovian feed back controls of the form:
$U(t, X(t)) = (f(t, X(t)), C(t, X(t))$ provides a natural way for choosing his current investment and consumption rate.

Option Pricing: Suppose that the price $X(t)$ of a risky asset evolves according to the ItO SDE in integral form given by:

$$X(t) = X_0 + \int_0^t b(s, X(s))dW(s), \quad t \in [0, T]$$

an European call option with strike price K gives the right to buy the stock at time T at a fixed price K. The resulting payoff is then given by

$$f(X(T)) = (X(T) - K)^+.$$

Suppose that we apply a dynamical portfolio strategy or hedging strategy (c_t, η_t) where η_t is the amount of riskless asset of constant value 1 say, and the amount c_t is the risky asset. Then the value $V(t)$ of the portfolio at time t is

$$V(t) = c_t X(t) + \eta_t$$

An important problem is to determine the fair price of the option. By the Nobel prize winning Blacks-Scholes formula, we have

$$V(0) = E(f(X(T)).$$

The corresponding self financing hedging strategy in quite general situation leads to a perfect replication of the claim

$$V(T) = f(X(T)).$$

Please note that the Black - Scholes equations have been applied to real option valuation of assets and valuation of flexibility or opportunity for real investments in the energy sector such as oil and gas, electricity as well as in the real sector.

(d) Turbulent Diffusion and Radio - Astronomy

SDE have long been used to model turbulent diffusion and related issues, dating back to Langevin's equations for Brownian motion. If $X(t) \in \mathbb{R}^3$ represents the position of a fluid particle at time t and $V(t)$ its velocity, a simple model for the Lagrangian dynamics of such a particle is of the form:

$$\begin{aligned} dX(t) &= V(t)dt \\ dV(t) &= -\frac{1}{T}V(t)dt + \sigma dW(t) \end{aligned} \tag{7.29}$$

where T is a large relaxation time for the process $V(t)$. Variation of the last equations have been considered with coloured noise. In some instances, Poisson processes have been used as the driving processes. Other areas of research activities where SDE have found applications are:

(e) Helicopter Rotor and Satellite Orbit Stability,

(f) Biological Waste Treatment, Hydrology and Indoor Air Quality,

(g) Seismology and Structural Mechanics,

(h) Fatigue Cracking, Optical Bistability and Nemantic Liquid Crystals,

(i) Blood Clotting Dynamics and Cellular Energetics,

(j) Josephson Junctions, Communications and Stochastic Annealing

7.3. My Contribution to Classical Stochastic Differential Equations

Mr. Vice Chancellor Sir, my contributions in this field concern some qualitative and numerical aspects of classical stochastic differential equations driven by Brownian motions.

Oscillatory Behaviour of Solutions of Stochastic Delay Differential Equations

1. My former doctoral student, Dr. A.O. Atonuje and I have published some results (Atonuje and Ayoola, 2007) concerning the non-contribution to the oscillatory behaviour of solutions of stochastic delay differential equations (SDDE) of the form:

$$\begin{aligned} dX(t) &= -\sum_{j=1}^{n} a_i(t)X(t-r_i)dt + \mu X(t)dB(t), \; t \geq 0 \\ X(t) &= \nu(t), \; t \in [\tilde{t}-\rho, \, 0]. \end{aligned} \tag{7.31}$$

We were able to prove that even when non-oscillatory solutions exist in the corresponding deterministic delay differential equation, the presence of noise perturbation stimulates an oscillation subject to certain conditions on the delay terms

.

2. We have also shown that in the absence of the noise term, non -oscillatory solutions can occur for the deterministic case but with the presence of noise, all solutions of SDDE oscillate almost certainly whenever the feedback intensity is negative (Atonuje and Ayoola, 2007, 2008a, b). Delay and noise play complementary roles in the oscillatory behaviour of the solution of the SDDE (7.31).

Finite Element Solutions of Stochastic Partial Differential Equations

3. Another former doctoral student of mine, Dr. I. N. Njoseh and myself published some results (Njoseh and Ayoola, 2008a) on the finite element method for a strongly damped stochastic wave equation driven by a space - time noise of the form:

$$\begin{aligned} U_{tt} + \alpha A U_t + AU &= dW \text{ in } \Omega,\ t > 0, \\ U(\cdot, t) &= 0, \text{ on } \delta\Omega,\ t > 0 \\ U(0) = \phi,\ U_t(0) &= \nu, \text{ in } \Omega \end{aligned} \tag{7.32}$$

where Ω is a bounded domain in $I\!R^d$, $d \geq 2$, with smooth boundary $\delta\Omega$ and $A = -\Delta$ self adjoint operator, W a Wiener process.
We provided some error estimates of optimal order for semi-discrete and fully discrete finite elements schemes by using L_2 - projections of the initial data as starting values.

4. We also carried out a finite element analysis of the stochastic Cahn- Hilliard Equation (Njoseh and Ayoola, 2008b) of the form:

$$U_t - \Delta(-\Delta U + f(U) = a(U)W_{tx} \text{ on } \Omega$$

with initial condition

$$U(0, \cdot) = u_0, \text{ in } \Omega \text{ and } \frac{\partial U}{\partial n} = \frac{\partial}{\partial n}\Delta U = 0, \text{on } \partial\Omega, \tag{7.33}$$

where Ω is a smooth bounded domain in $I\!R^d$.
The equation is semi-linear parabolic of fourth order. It has been used to model phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably.

7.4 My Contribution to Quantum Stochastic Differential Equations and Inclusions

Quantum stochastic calculus is a differential calculus incorporating various noises in quantum world. The first quantum stochastic calculus was introduced by **R. L. Hudson and K.R Parthasarathy (Hudson and K.R Parthasarathy, 1984)**, for Boson noises. This is roughly speaking, a sort of Ito calculus for the most fundamental noises in quantum theory. The study of stochastic calculi for several types of noises such as Boson, Fermi, free, Boolean, monotonic, etc is still a hot topic.

Quantum stochastic differential equations (QSDE) are stochastic differential equations for operator processes driven by quantum noises. In addition to reducing to classical SDE in special cases, they are applied in the study of quantum information, quantum open systems, quantum measurement, in the study of quantum Markov processes and dilations of quantum Markov semi-groups.

Quantum information theory treats any problem related to transmission of information through quantum systems, to storing, encoding, decoding information in quantum systems. Foundations of quantum mechanics are relevant and measurement theory is involved in modelling decoding and error procedures, at least.

Quantum Markov semi-groups are the natural mathematical objects for modelling the irreversible evolution of open quantum systems. This is governed by the so-called master equation whose solutions are given by a Markov semi-group. These are mathematically viewed as non-commutative generalisation of classical Markov semi-groups acting on a commutative algebra (a function space).

Quantum measurement theory is a very important topic in quantum probability. It deals with the issues of measurements of observables inside quantum mechanics. It has applications in open system theory, quantum optics, operator theory, quantum probability and quantum and classical stochastic processes.

Quantum probability is a transversal subject which finds its fundamental axioms in quantum physics and has deep connections with domains such as quantum mechanics, quantum field theory, quantum optics and scattering theory (Attal, 1998). Sometimes, quantum probability is regarded as part of functional analysis (C^*-algebra,, von Neumann algebra theory, non-commutative geometry, quantum groups, etc)

Building upon the fundamental foundation laid by G.O.S. Ekhaguere in his pioneering paper (Ekhaguere, 1992) concerning the existence and some qualitative properties of Lipschitzian quantum stochastic differential inclusions of the form (7.63) below, my research activities in this field concern both the theoretical and numerical aspects of quantum stochastic differential equations (QSDE) and inclusions (QSDI) driven by quantum martingales in the weak and strong sense within the framework of the Hudson-Parthasarathy (H-P) [37] formulations of quantum stochastic calculus (QSC). It is well known that the H-P formulation of QSC sufficiently generalise the Ito calculus.
Quantum stochastic calculus employs the principles of quantum probability which is a non-commutative generalisation of classical theory of probability. Random variables (or observables in the language of physics) are represented by self adjoint operators in a complex Hilbert space and probabilities, or states are represented by unit vectors in the Hilbert spaces. Thus, problems in classical stochastic calculus can be reformulated in quantum forms based on the fact that given any family $S = (S_t,\ t \in T)$ of commuting self adjoint operators in a complex Hilbert space H, which are collectively cyclic, (i.e there exists a unit vector $\phi \in H$ for which the set $\{e^{ixS_t}\phi : x \in I\!R\}$ is total in H), it can be shown that there exists a probability space $(\Omega,\ \mathcal{F}\ P)$, a family of real valued measurable functions (classical stochastic processes) $(X_t : t \in T)$ on Ω and a Hilbert space isomorphism

$$D_S : H \to L^2(\Omega, \mathcal{F},\ P)$$

such that the vacuum vector (unit vector in H) is mapped to the function identically 1 and the operator S_t becomes the operator of multiplication by X_t (Hudson, 2001).

A quantum stochastic process consisting of commuting self - adjoint operators is completely equivalent to a classical stochastic process (Attal, 1998). In general, quantum stochastic calculus concerns non-commuting self adjoint operators thereby containing the classical theory as a special case. It should therefore, be noted that several benefits have been achieved by interpreting classical probability in non-commutative quantum form. Such benefits include a better understanding of classical stochastic flows and some parts of Wiener space analysis and Wiener chaos expansions where a fundamental chaotic representation property of the Azema martingales have been discovered (Mayer, 1993; Hudson, 2001). However, it is well known that the subject of quantum probability is far from being reduced to a simple non commutative extension of classical probability theory. It is not exclusive to finding non-commutative analogues of the classical theorems. Its connection with quantum field theory is very deep as earlier stated.

7.5. Fundamental Concepts and Structures

Among my contributions, I employed the formulations of the H-P quantum stochastic calculus as described briefly in what follows. Let D be an inner product space and H, the completion of D. We denote by $L^+(D,\ H)$, the set

$$\{X : D \to H\ /X \text{ is a linear map with } D \subseteq DomX^*\},$$

where X^* is the operator adjoint of X. We remark that $L^+(D,\ H)$ is a linear space under the usual notions of addition and scalar multiplications of operators.

If H is a Hilbert space, then the Boson Fock space $\Gamma(H)$ determined by H is the Hilbert space direct sum given by :

$$\Gamma(H) = \bigoplus_{n=0}^{\infty} H^{(n)}$$

where $H^{(0)} = \mathbb{C}$. For $n \geq 1$, $H^{(n)}$ is the subspace of the n-fold Hilbert space tensor product of H with itself comprising all symmetric tensors:

$$H^{(n)} = (H \otimes \cdots \otimes H)_{\text{sym}}$$

For each $f \in H$, an element of the form:

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f$$

is called an exponential or coherent vectors in $\Gamma(H)$. Here $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$. The element $e(0)$ in $\Gamma(H)$ is called the vacuum vector.

It is well established that the exponential vectors enjoy a number of properties. Its linear span is dense in the Fock space, the set of exponential vectors is linearly independent among other properties. These properties aid the use of exponential vectors for the development of the H-P QSC calculus.

For an arbitrary inner product space $I\!D$ and its completion $\mathcal{R}$, and γ some fixed Hilbert space we write

$$\mathcal{A} = L^{+}(I\!D \underline{\otimes} I\!E, \mathcal{R} \otimes \Gamma(L^2_{\gamma}(I\!R_{+}))) \tag{7.51}$$

where $L^2_{\gamma}(I\!R_{+})$ is the Hilbert space of square integrable, γ-valued maps on $I\!R_{+}$ and $\underline{\otimes}$ denotes the algebraic tensor product. Many other relevant spaces are similarly defined and employed. Motivated by the first and second fundamental formula of H-P (Hudson and Parthasarathy, 1984), Ekhaguere (1992) equipped the linear space $\mathcal{A}$ with a locally convex topology generated by a family of seminorms

$$\{x \to \|x\|_{\eta\xi} = |<\eta, x\xi>|, \; x \in \mathcal{A}, \eta, \xi \in I\!D \underline{\otimes} I\!E.\}$$

The space $\mathcal{A}$ is similarly equipped with another locally convex topology generated by the family of semi norms:

$$\{x \to \|x\|_\xi = \|x\xi\|, \xi \in I\!D\underline{\otimes}I\!E\}$$

following the second fundamental formula of H-P. The completions $\tilde{\mathcal{A}}$ of these spaces are employed in my research activities in this field. Thus, we define a quantum stochastic process as an $\tilde{\mathcal{A}}$ valued map on some intervals contained in the positive segment of the real number line.

For the coefficients of the QSDE (resp. QSDI) E, F, G, H : $[0,T] \times \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ (resp.E, F, G, H : $[0,T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$) belonging to appropriate spaces and with the solutions $X(t)$ enjoying suitable properties, the following is a brief account of my contributions.

7.6. MY RESEARCH CONTRIBUTIONS

(1). Numerical Schemes for Solving Quantum Stochastic Differential Equations.

Mr. Vice Chancellor Sir, I proposed, developed and studied multi-step schemes for solving numerically Lipschitzian quantum stochastic differential equation (LQSDE) (Ayoola, 2000a) of the form:

$$\begin{aligned} dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^+(t) \\ +G(t, X(t))dA_g(t) + H(t, X(t))dt \end{aligned} \tag{7.61}$$

in an interval $[t_0, T]$ with initial condition $X(t_0) = X_0$. The driving processes Λ_π, A_f^+, A_g are the stochastic integrators in the Boson Fock space quantum stochastic calculus. Convergence of the discrete schemes to the exact solutions and error estimates were obtained for explicit scheme of class A in the locally convex space of solutions. Results in (Ayoola, 2000a) contain the Euler-Maruyama schemes for Ito stochastic differential equations as special cases and numerical examples were given. Explicit and exact solutions of

LQSDE (7.61) are rarely available, making the search for approximate solutions a necessary and worthwhile endeavour. Prior to the publication of Ayoola (2000a), very little, if any at all, was known about the features of numerical solutions of LQSDE (7.61). As the LQSDE is a non-commutative generalisation of the classical Ito stochastic differential equation (Ito SDE), driven by Brownian motion, the implementation of the multi-step schemes and other discrete schemes developed in my subsequent works completely eliminated the need for the computation of random increments by random number generators as obtained in the implementation of stochastic Taylor schemes for simulation of sample paths and functional of solutions of classical Ito SDE. This paper, (Ayoola, 2000a) has opened further research directions concerning the refinement of the schemes in several ways, as well as the study of numerical stability associated with the multi-step schemes. The convergence and stability of a general multi-step scheme for (7.61) was considered in Ayoola (2000b). For the *Mathematical Reviews* database, published by the American Mathematical Society (AMS), the reviews of the articles (Ayoola, 2000a,b) were respectively written in the years 2001 and 2004. The first was written by one of the founding fathers of quantum stochastic calculus, **Emeritus Professor K. R. Parthasarathy** of the Indian Statistical Institute, and the other written by **Professor Henri Schurz** of Southern Illinois University, Carbondale, USA. The assigned review numbers are respectively: MR 2001e:81065 and MR: 2004: 81072.

(2). **One Step Schemes for Solving Quantum Stochastic Differential Equations (QSDE)**

My article in Ayoola (2001a) was concerned with the development, analysis and applications of several one-step schemes for computing weak solutions of LQSDE (7.61). The work was accomplished in the framework of Hudson and Parthasarathy formulation of quantum stochastic calculus and subject to the matrix elements of solution being sufficiently differentiable. The results here concern non-commutative generalisation of the usual Euler scheme, Runge-

Kutta schemes and an integral scheme for computing solutions of the LQSDE. The paper contains results for the Ito SDE as a special case with Ito processes as multiplication valued operators in a simple Fock space. The schemes exhibit important implementation benefits as in Ayoola (2000a,b). The article in Ayoola (2001a) is 40 page long and contains the main existence results of Ayoola (1998) as appendix, as well as some numerical experiments to illustrate the main features of the different schemes and their error estimates. The one step schemes here also generalise discrete schemes reported in Ayoola (1999a) and Ayoola (1999b). Extension of the results here to the case of continuous time Euler approximation scheme and a computational scheme under Caratheodory conditions was undertaken in Ayoola (2002b). The findings in Ayoola (2001a) has created further research questions involving extensions to LQSDE (7.61) of various improvements already established for classical discrete schemes in the finite dimensional setting. The mathematical review of the article, Ayoola (2001a), was written by **Professor Rolando Rebolledo Berroe** of Zentralblatt Mathematics Database, Germany with review number: Zbl 0998.60056 and the Abstract is listed in the *AMS Mathematical Review* with number MR 2002f:65017.

(3). **Kurzweil Equations Associated with QSDE**

In Ayoola (2001b), I introduced and studied Kurzweil equations associated with LQSDE (7.61) and I established the non-commutative quantum extensions of classical Kurzweil integrals and some technical results. In addition, I proved the interesting equivalence between LQSDE (7.61) in integral form and the Kurzweil equation of the form:

$$\frac{d}{d\tau} < \eta, X(\tau)\xi >= D\Phi(X(\tau), t)(\eta, \xi)$$

on $[t_0, T]$ and for $t \in [t_0, T]$, for a suitable map Φ and η, ξ belonging to an appropriate class. Investigation of approximate solutions of LQSDE (7.61) by utilising established results on Kurzweil integrals and equations was afforded by the equivalence results. It was shown in the paper that the associated Kurzweil equation may be used to obtain reasonably high accurate solutions of the LQSDE. This paper extends established relationship between Lebesgue and Kurzweil integrals to quantum stochastic integrals. This particular study generalised some numerical results in Ayoola (2000a, b) since the results in Ayoola (2001b) hold under pure Caratheodory conditions where the matrix elements of solutions need not be differentiable more than once. The results also generalised several analogous results for classical initial value problems to the non commutative quantum setting involving unbounded linear operators on a Hilbert space. Further research problems have thus been opened by the paper. One of this is the issue of variational stability of LQSDE (7.61).

The review of the article (Ayoola (2001b)) was written in the year 2002 by **Professor Debashish Goswami** of Indian Statistical Institute for the *AMS Mathematical Reviews* with review number MR 2002g: 81078.

(4). Construction of Approximate Attainability Set for Quantum Stochastic Differential Inclusions

I presented similarly, a numerical method for constructing with a specified accuracy, attainability set $R^{(T)}(x_0)(\eta,\xi)$ (Ayoola (2001c) defined by

$$R^{(T)}(x_0)(\eta,\xi) = \{< \eta, \Phi(T)\xi >: \Phi(\cdot) \in S^{(T)}(x_0)\} \qquad (7.62)$$

for the Lipschitzian quantum stochastic differential inclusion (LQSDI) in integral form:

$$X(t) \in x_0 + \int_{t_0}^{t} \Big(E(s,X(s))d\lambda_\pi(s) + F(s,X(s))dA_f(s)$$

$$+ G(s,X(s))dA_g^+(s) + H(s,X(s)\Big)\, ds,\ t \in [t_0,T]. \qquad (7.63)$$

where $S^{(T)}(x_0)$ is the set of solutions to LQSDI (7.63).

An algorithm is described for numerically approximating the attainability set within any prescribed accuracy. Results in this

paper generalised an analogous classical result of Komarov and Pevchikh to non-commutative quantum stochastic differential inclusion (7.63). Attainability sets are important for several characterisation of the set of trajectories of LQSDI (7.63). In (Ayoola (2008b), I established the existence of solutions of QSDI (7.63) satisfying a general Lipschitz condition. The Lipschitz condition of Ayoola (2001c) is a special case and extension of the numerical algorithm of this paper to general case is still open.
The AMS review of (Ayoola (2001c) was written by **Professor Volker Wihstutz** of North Carolina State University, Charlotte for the *AMS Mathematical Reviews* with review number MR 2002f:65018.

(5). Lagrangian Quadrature for Computing Solutions of QSDE

Investigations in Ayoola (2002a) concerned the analysis of the Lagrangian quadrature schemes for computing weak solutions of LQSDE (7.61) with matrix elements that are sufficiently smooth. Results concerning the convergence of Lagrangian schemes to exact solutions were obtained. Precise estimates for an error term were given in the case when the nodes of approximations are chosen to be roots of the Chebyshev polynomials. Some important features of the quadrature schemes are the conversion of LQSDE (7.61) to solvable algebraic equations in term of the nodal values and that the nodes need not be equally spaced. This paper established the possibility of applying numerous results in linear and computer algebra for investigating numerical solutions to LQSDE (7.61). Numerical experiments were performed by solving associated linear systems taking into consideration, computational complexity of the algorithm and round off errors.
The AMS review of this paper was written by **Professor Vassili N. Kolokoltsov** of Warwick University, UK, for the *AMS Mathematical Reviews* with review number MR 2003e:60121.

(6). **Topological Properties of Solution Sets of QSDI**

In Ayoola (2008c), I established a continuous mapping of the space of the matrix elements of an arbitrary nonempty set of quasi solutions of Lipschitzian QSDI (7.63) into the space of the matrix elements of its solutions. As a corollary, I furnished a generalisation of my previous selection result in Ayoola (2004b). In particular, when the coefficients of the inclusion are integrally bounded, it was shown that the space of the matrix elements of solutions is an absolute retract, contractible, locally and integrally connected in an arbitrary dimension. As usual, we employ the Hudson and Parthasarathy formulation of quantum stochastic calculus.
The AMS review of this paper was written by **Professor Vassili N. Kolokoltsov** of Warwick University, UK for the *AMS Mathematics Reviews* with assigned number MR 2008k: 81174.

(7). **Existence of Strong Solutions of LQSDE**

In Ayoola (2002c), the existence, uniqueness and stability of strong solutions of LQSDE (7.61) were established. The locally convex topology on the space of quantum stochastic processes in this case is generated by a family of semi-norms induced by the norm of the Fock space. The second fundamental formula of Hudson and Parthasarathy concerning the estimate of the square of the norm of the values of stochastic processes on exponential vectors facilitates the existence results by method of successive approximations. Results here generalise analogous results concerning classical SDE driven by Brownian motion. Convergence in the sense of this paper generalise the root mean square convergence of successive approximation in the case of classical Ito process considered as quantum stochastic process in a simple Fock space. The study of Ayoola and Gbolagade (2005) happened to be a continuation of Ayoola (2002c) concerning the existence and stability of solutions of QSDE satisfying a general Lipschitz condition in the strong topology. Ayoola and Gbolagade (2005) established a class of Lipschitzian QSDE where the coefficients are merely continuous on the locally convex space of the quantum observables. The AMS Mathematical Reviews of paper Ayoola (2002c) and Ayoola and Gbolagade (2005) were written respectively by **Professor Vassili N. Kolokoltsov** of Warwick University, UK and **Professor Debashism Goswami** of Indian Statistical Institute for the *AMS Mathematical Reviews* with review numbers respectively given by MR 2003b: 60081 and MR 2005m: 81179.

(8). Exponential Formula for the Reachable Set of QSDI

Ayoola (2003a) was my second major work on quantum stochastic differential inclusions (QSDI) (7.63). The paper is a continuation of my previous work Ayoola (2001c) concerning the QSDI, where the coefficients are assumed to have suitable regularity properties. The basic set-up of the paper is that of multi-valued functions with appropriately defined multi-valued stochastic integrals. By endowing the family of closed subsets of the locally convex space of quantum observables with a Hausdorff topology, the paper established the following exponential formula:

$$R^{(T)}(x_0) = \lim_{N\to\infty} \left(I + \frac{T}{N}P\right)^N (x_0) \qquad (7.64)$$

where $R^{(T)}(x_0)$ is the reachable set of QSDI (3), I is the identity multifunction.

Repeated composition of multi-functions is understood in some sense and the limit in Equation (7.64) is interpreted as the Kuratowski limit of sets. Equation (7.64) has a direct consequence for the convergence, to the exact value, of discrete approximations to the reachable set. The basic motivation for considering QSDI (7.63) concerns the need to develop a reasonable numerical scheme for solving QSDE (7.61) with discontinuous coefficients since many of such interesting QSDE can be reformulated as QSDI with regular coefficients.

The AMS review of Ayoola (2003a) was written by **Professor Debashish Goswami** of Indian Statistical for the *AMS Mathematical Reviews* with review number MR 2004e:81073.

(9). **Discrete Approximation of Solutions of QSDI**

Ayoola (2003b) was a continuation of my study of discrete approximation of QSDI (7.63). This paper is concerned with the error estimates involved in the solution of a discrete version of QSDI (7.63). The main results relied on some properties of the averaged modulus of continuity for multi-valued sesquilinear forms associated with QSDI (7.63). The paper established a sharp estimate for the Hausdorff distance between the set of solutions of QSDI (7.63) and the set of solutions of its discrete approximations. This paper however, extended the result of Dontchev and Farkhi (1989) concerning classical differential inclusions to the present non-commutative quantum setting involving inclusions in certain locally convex spaces.
The AMS review of this paper was written by **Professor Habib Querdiane** of the University of Tunis El Manar, Tunisia for the *AMS Mathematics Reviews* with review number MR 2005a: 60109.

(10). **Existence of Continuous Selections of Solution and Reachable Sets**

Ayoola (2004b) established the existence of continuous selections of solution set of Lipschitzian QSDI (7.63). The paper precisely proved that if

$$S^{(T)}(a)(\eta,\xi) =: \{t \to <\eta, X(t)\xi>, \; X \in S^{(T)}(a)\}$$

is a set of absolutely continuous complex valued functions associated with the set of solutions $S^{(T)}(a)$ of QSDI (7.63), then the multifunction $<\eta, a\xi> \to S^{(T)}(a)(\eta,\xi)$ admits a continuous selection for all $a \in \mathcal{A}$ given that the set of matrix elements $A(\eta,\xi)$ of A is compact in the field of complex numbers.
As corollaries to the main result, I proved that the solutions set map, as well as the reachable sets of QSDI (7.63) admitted some continuous representations. A search in the AMS Mathematical Reviews and the ISI Web of Science databases showed that Ayoola (2004a) was the first known selection result concerning QSDI

(7.63) in the framework of the Hudson -Parthasarathy formulation of quantum stochastic calculus. Consequently, the paper opened further research questions in respect of the refinement, generalisation and applications of the selection results in parallel with the classical cases of differential inclusions in finite dimensional Euclidian spaces. Ayoola (2008a) was a follow up publication, where I showed that a continuous selection from the set of solutions exists directly defined on the space of stochastic processes with values in the space of adapted weakly absolutely continuous solutions.
As a corollary, the reachable set multifunction admits a continuous selection. Ayoola and Adeyeye (2007) also extended the selection results in Ayoola (2004a) as an interpolation to cover a finite number of trajectories. The AMS Mathematical reviews of Ayoola (2004a), Ayoola and Adeyeye (2007) and Ayoola (2008a) were written by **Professor C. R. Belton** of Mathematics Department, Lancaster University, UK and **Professor Vitonofrio Crismale** of Muhammad Ibn University Saudi Arabia, for the *AMS Mathematical Reviews* with review numbers MR 2005i: 81076, MR 2008m:65012 and MR 2011d:81177 respectively.

(11). **Mayer Problem for Quantum Stochastic Control**
In the framework of Ekhaguere's formulation of the multi-valued analog of H-P quantum stochastic calculus, Ogundiran and Ayoola (2010), published jointly with my former doctoral student Dr. Michael Ogundiran, concerns some results on quantum stochastic control. In particular, we studied the regularity properties of the value function inherited from the multi-valued quantum stochastic processes involved. We showed that under the assumption of directional differentiability of the value function, the associated Mayer problem had at least one optimal solution. Our theory covered earlier work on quantum stochastic control by Belavkin and by Andreas Boukas.
The AMS Review of this paper was written by **Professor Andreas Boukas** of Southern Illinois University, Carbondale, USA, with Review number 2011d:81180.

As evidence of the high impact factors of the journals where these publications appeared, abstracts of the papers can be accessed in the database of the well-known Thomson ISI Web of Science. The web consists of abstracts of articles published by the world's most prestigious and influential journals in the basic and applied sciences. It is gratifying to know that through the help of the Postgraduate School, for which we are very grateful to the current Dean, Professor Olorunisola and other officers of the Postgraduate School, our university currently holds a 2-year subscription to the MATHSCINET database, the online version of the AMS Mathematical Reviews. This aids the research works of both postgraduate students and staff as we can easily get informed of the reviews of thousands of new mathematics results that are regularly posted to the database. We can also search the database for old but useful results that are up to 60 years old and more. The site is also useful for the assessment of research accomplishments of applicants for sabbatical and regular positions in mathematics and for fellowship applications all over the world.

From the database, we are able to get information about the works of our senior colleagues which serve as important motivations for postgraduate students and upcoming mathematicians. An author's publication search in the MATHSCINET reveals Professor Ekhaguere's and my articles as leading major publications in quantum stochastic differential equations and inclusions, confirming our research leadership in the field. As can be seen by any visitor to the website anywhere in the world, our publications have received very positive and commendable reviews and comments from experts in all continents of the world. However, we do not yet have access to the Thomson ISI Web of Knowledge/Science, which serves similar purposes as MATHSCINET for all disciplines. I am therefore, using this opportunity to renew my old recommendation that our university Library subscribe to the web as a matter of urgency as we have in most standard universities all over the world.

8. SOME PROBLEMS CONFRONTING MATHEMATICAL SCIENCES IN NIGERIA

Mr Vice-Chancellor, from the foregoing, it is clear that basic scientific research as exemplified by Mathematics is intimately connected with development. As a nation, we should strive to develop our resources in all positive direction i.e. apart from producing Scientists and Technologists who are pursuing developmental research; we should endeavour to develop a critical mass of Mathematical Scientists who can contribute to frontier knowledge. As the general scientific concern, the far more serious issue is projecting a much greater effort in getting funds for educating and motivating young people to embrace mathematics. We must be asking questions and proposing solutions to our problems. According to the famous mathematician Abel, *a problem that seems insurmountable is just seemingly so because we have not asked the right questions. You should always ask the right questions and then you can solve the problem.*

For the sake of emphasis, I will proceed to highlight some specific problems that are making giant strides in Mathematical Sciences elusive in this country:

(i). Inadequate Access to Journals, Books, Teaching and Research Facilities

It cannot be over-emphasised that mathematical, scientific and technological research can only be productive in an atmosphere of adequate journal, books and facilities. Most scientific researches in Nigeria are carried out in the Universities where the library, teaching and research facilities are less than adequate. Online digital library system, which is now in vogue all over the world, is either grossly underdeveloped or virtually non-existent in most universities in Nigeria. Well known professional databases (e.g. Thomson ISI web of knowledge / science) that aid collaborative works are rare in our libraries. Because of low value of the Nigerian currency, imported books at all levels are very expensive. In Nigeria, primary and secondary school books are written and published locally, thus reducing cost; however, most tertiary mathematics texts are imported, and so are unaffordable by students, teachers and sometimes libraries. Inadequate funding of education and research institutions result in lack of good libraries, infrastructures, computers, teaching aids and other facilities.

(ii). Nigerian Mathematical Scientists Work in Relative Isolation

Specialties in various areas of Mathematical Sciences are spread thin all over the country. Relatively, few universities have viable Research groups. Collaborative research with other scientists, engineers, clinicians is rare and un-encouraged in most of our universities. While access to the Internet has improved of recent, it is still a far cry from the situation in Europe and America where every lecturer has access to high quality information technology and highly efficient internet network devoid of electric power failure. Since no library on its own can stock all needed books and journals at all times, networking of libraries is a common things in advanced countries providing interlibrary digital loan facilities. Lack of library networking in Nigeria further expands the problem of isolation of researchers.

(iii). Poor Preparation and Shortage of Mathematics Teachers at All Levels

Many primary and secondary schools have no choice but to employ teachers who have no specialist training in mathematics because of shortage of mathematics staff. Many University Mathematics Departments in Nigeria are under-staffed. As a result, mathematics courses suffer quite a lot in term of quality of instruction, inevitability of large classes, inadequate tutorials, etc. Those of them offering graduate courses have to stretch their meagre human resources and facilities beyond reasonable limits. Research in mathematics in our department covers about 30 sections out of the 97 sections in the AMS 2010 subject classification and this is about the highest coverage in any Nigerian University.

(iv). **Brain Drain of Nigerian Mathematicians**

Many well trained Nigerian Mathematicians prefer working abroad where their contributions and expertise are well rewarded and appreciated. Other reasons are lack of enough facilities for productivity, poor remuneration and lack of access to meaningful research grants. In the US, Europe, Japan, China, other emerging Asian countries and South Africa, there are many research foundations that support mathematical research. Institutions such as National Science Foundation (NSF) in the US, South African Research Foundation support frontier mathematical research in all areas without restriction to applications. Some institutions such as the US Army, Navy, NSA give generous grants for mathematical research. The reverse is the case in Nigeria with the exception of government agencies like Petroleum Trust Development Fund (PTDF) and Education Tax Funds (ETF) that are motivating applied research. In addition, many multi-national companies in Nigeria fail to meaningfully contribute to our science and mathematics capacity building efforts through their unwillingness to consider Nigerian Universities for the awards of Research and Development contracts in these subjects and related disciplines.

(v).**Environmental Problems**

Environmental handicaps in the form of malfunctioning public utilities (such as electricity, pipe-borne water) porous security of life and properties make it an uphill task to practice as a Mathematician. Moreover, quite a number of Mathematicians trained in sophisticated difficult areas of mathematics have problems continuing with their research in those areas in Nigeria. So, they are forced to abandon those areas of research or quit the country. A related problem is lack of interest in the study of basic sciences and mathematics by capable and talented students. Many prefer applied disciplines like engineering, medicine and law because of the Nigerian reward systems and limited employment opportunities in the Nigerian economy. Many students studying mathematics and other basic sciences are in most cases rejects from other areas and therefore not well motivated. High level of corruption in the polity where hard work is not adequately rewarded compared to mediocrity further compounds the problem.

9. RECOMMENDATIONS

Professor Aderemi Kuku in his 2004 N.A.S Quarterly lecture (Kuku, 2004) and other mathematics inaugural lecturers have made a number of appropriate and well-thought out recommendations for the improvements of Science, Technology and Mathematics learning, research and applications in Nigeria. These recommendations are still very relevant today. Precisely, the various recommendations have touched on:

(a) The need to intensify the popularisation of science, mathematics and inculcation of scientific culture in the society.
There is a need for an attitudinal change to Mathematics in particular and all the basic sciences in general by the government and all of our policy makers. The need for popularisation of Mathematics is particularly compelling. From the historical development of Mathematics, it is clear that most people who have studied Mathematics at secondary-school level know little more than was known in the Seventeenth century. Most tertiary education in Mathematics hardly goes beyond what was discovered in the nineteenth century. Much spectacular and fundamental Mathematics discovered in the last 100 years has only been circulating among relatively few initiates, with a consequent global illiteracy in contemporary Mathematics. The antipathy towards Mathematics by administrators in government, many parents, funding agencies and the general public, sustained by lack of appreciation of what Mathematicians do, dictates a pressing need for a more aggressive and well-organised popularisation programme for Mathematics at local and global levels in Nigeria. Due to the present ugly trend, it is generally feared that the present inadequate number of active professional mathematicians may be very difficult to replace if the situation is not arrested quickly.
Others include:

(b) The need for massive improvements in teaching and research facilities.
(c) The need for the intensification of scientific, industrial and technology self reliance in the country.
(d) The need for increased funding and fund raising for mathematics, science and technology.
(e) The need for a massive mobilisation effort for tertiary scientific and mathematics textbooks development.
(f) The need for an overhaul and renovation of mathematics curriculum at all levels.
(g) The need for closer links between universities, research institutes, government and industry.
(h) The need for Nigeria to spearhead the overall industrial and technological development of Africa.
(i) Aggressive program of financial incentives for talented mathematicians, students of mathematics and other basic
sciences comparable at least to that of the medical profession in the country.
(j) The need for the identification of contemporary areas of mathematics to be developed.
(k) The need to generate good students for research and career in the mathematical sciences.
(l) The need to stimulating interests of our people in the applications of mathematics.

It is clear that Nigeria is yet to have a critical mass of mathematical scientists for development purposes. As earlier stated, the problem continues to get worse everyday as many people are opting for professional courses in our universities.

It is important for us to do good quality and contemporary mathematics that will realign Nigerian Mathematicians with the rest of the world. There was a time when our Faculty was one of the best ten in the Commonwealth as our effort in the past led to the recognition of what was then known as Ibadan School of Differential equations, Ibadan School of Functional Analysis, Ibadan Algebra and lately Ibadan School of Non-Commutative Quantum Stochastic Analysis and Applications. As highlighted before, many areas

of human activities now require rather sophisticated Mathematics for their in-depth study and even several simply posed problems have only been solvable through highly sophisticated, abstract and powerful techniques. Nigerian scholars, government officials, politicians and members of the public should see frontier mathematical and scientific research as development oriented, relevant and consistent with our culture and aspirations.
It is important that Nigerian experts in mathematics should continue to enlighten the public about how knowledge of frontier mathematics are being translated into further development in the already developed countries. Nigeria cannot afford to distance itself from the frontiers if only to register our capacity to do other deep things apart from music, sports, entertainment, literature etc in the eyes of the world. We have to leapfrog into the various frontiers. We thank the present University of Ibadan Administration for encouraging us by recently approving a Mathematical Research Centre named after the first Nigerian Professor of Mathematics, the late Professor Adegoke Olubummo. It is envisaged that given the needed supports, top level mathematical research that will realign us with the rest of the world, will be done at the Centre.

Further recommendations worthy of consideration are:

(a) Just like the Education Tax Fund (E. T. F) and other laws, there is need for legal instruments compelling multinational companies to use certain percentage of their annual profits to support research and development in Mathematics and other basic sciences in Nigeria. This is the practice in some front line developing nations like Japan, China and India.

(b) The Federal Government as a matter of urgency is advised to assemble a pool of top level active professionals in Mathematics and the Basic Sciences to form a think-tank on the directions of Nigerian Science and Mathematics. This is needed to leapfrog Nigeria to the 21st century research and applications of the subjects in line with vision 20:2020. The group will serve as a consultancy concerning the contemporary use of mathematical and other scientific techniques as employed in other lands and offer useful and workable advice for our domestic applications in Government, Business and Industry.

(c) In particular, efforts must be made to make use of recent mathematical and scientific results and methods to address our pressing security and defence problems as in the US NSA and the US defence industry. It should be known that no nation will spare the best of her manpower in security issues for our country due to the classified and expensive nature of security duties and training of personnel. This is particularly important because of the reality of terrorism in our nation in recent times.

(d) Any restriction that ties down research grants in mathematics to only application areas must be removed. There is no area of mathematics no matter how abstract that cannot be applied to real life problems one day.

(e) Nigerian defence, that is, the Army, the Navy, the Air Force and the SSS, must be made to emulate their US counterparts by sponsoring top level mathematical and scientific research that aids the defence mission and development of the country.

(f) The establishment of a national research foundation in Nigeria is long overdue. A division of mathematical sciences must be created to cater for the diverse kinds of problems facing mathematics teaching and learning in Nigeria analogous to the Division of the Mathematical Sciences (DMS) of the US National Science Foundation. According to the report by William Kirwan and his team in the Notices of the American Mathematical Society, Volume 57, March 2010, in the 1980s and 1990s, there was concern within the American mathematical sciences community that post secondary education in the mathematical sciences was in trouble. A series of challenges was identified in important national reports, including inadequate funding, insufficient numbers of students interested in mathematics, and shortcoming in the shape and direction of post-secondary mathematics education. The reports raised many issues among which are: the number of students receiving degrees; the lack of racial and gender diversity among the mathematics graduate student body; the declining fraction of US citizens receiving advanced degrees in mathematics and lack of sufficient postdoctoral fellowships for new doctorates.

Four issues were identified with respect to the structure of training in the mathematical sciences: the need for increased breadth; providing a better balance of education and research; decreasing the time period to acquire the degree and creating a more positive learning experience. These led the DMS of the NSF to establish the Grants for Vertical Integration of Research and Education in the Mathematical Sciences (VIGRE) program. The goals of the VIGRE program have changed from year to year but consistently included: integration of research and education; enhanced interaction across undergraduates, graduates, postdoctoral fellows and faculty; broadened educational experiences of students to include workforce and early research opportunities; and more students motivated to study mathematics and statistics. Giant strides in the development of all facets of the mathematical sciences in the US are eloquent testimonies of the success of this intervention by the DMS program of NFS. Nigeria should emulate these progressive programs and policy as witnessed in the US. All the problems that have been identified concerning the mathematical sciences in the US are much more relevant here in Nigeria and even more. The activities of similar research foundation in South Africa are some of the secrets behind the current top rankings of South African universities in the African and whole world categories.

(g) It is recommended that our university include mathematics service courses as part of the curriculum for our Pan African University hub for Earth and Life Sciences because as we have seen, Mathematics is a strong tool for the earth and life sciences if applied appropriately.

10. ACKNOWLEDGEMENTS

Mr Vice Chancellor, this inaugural lecture will not be complete without acknowledging those who have contributed strongly to my career development and advancement. First, I am eternally grateful to God for His unfailing guidance, protection and care. I thank God for ordering my footsteps in the wilderness of life. To Him be all glory.

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I appreciate all my academic mentors. I specially thank Professor H. O. Tejumola, for refusing to sign my change of course form away from mathematics in 1981 as the then Head of Department, after my preliminary year in Mathematics. God used you sir to retain me in my career of destiny. I appreciate Professor S. A. Ilori whom

God used to attract and retain me as a career mathematician here in Ibadan, I cannot thank you enough. The blessing and favour of God shall never cease in your life and family in Jesus name. I remember Professor John Adeyeye, my M.Sc project supervisor who left for the USA in 1989, late Professor Gabriel Oluremi Olaofe, for his brief but eventful supervision and encouragement he gave me before his death. May his soul continue to rest in peace.

I sincerely thank Professor G. O. S. Ekhaguere, for accepting to supervise my doctoral work in January 1993. The foundation of my career accomplishments was firmly laid on the thorough supervision and motivation that I received from him. I pray that the Lord continue to bless you and make you a blessing to the global mathematical community. I thank all my other teachers in Mathematics. I remember the late Professors Adegoke Olubummo and C.O. A. Sowunmi. I cannot forget Professors Akangbe Kenku, Aderemi Kuku, Victor Babalola, Olusola Akinyele, Olabisi Ugbebor and S.A. Adeleke. I remember Professor Bayo Okunade for useful advice and long-time profitable friendship. You all contributed to what I am today. God bless you all. I cannot forget Dr Adenike Ogunshe for proof reading the manuscript of this lecture and for offering other useful advice. Dr M. Egwe and P. Adeyemo, I thank you for your numerous assistance.

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I thank and praise God for giving me God fearing and obedient children; I appreciate you all: Oludare, Oladayo, Olayinka and OreOfe. God bless you. You shall all be blessings to the Kingdom of God and to your generation and future generations in Jesus name. To my loving and amiable wife Olufunmilayo, a veritable *help mate.* I thank God for giving me the *bone of my bone and the flesh of my flesh.* I thank you my wife for keeping the home front and for continually lifting me and the children to the Lord in prayers. Your labour of love shall never be in vain. Together we shall be victorious in the battle of life and make Heaven in Jesus name.

Mr Vice - Chancellor sir, distinguished ladies and gentlemen, thank you for your patience and for listening to my inaugural lecture. God bless.

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Bio Data of Professor Ezekiel Olusola Ayoola

Professor Ezekiel Olusola Ayoola was born on September 26, 1958 to the family of Late Papa Samson Ayoola and Late Madam Felicia Ayoola of Ile Jagun, Apaara, Oyo town in Oyo East Local Government Area of Oyo State.

He attended St. Michael's Anglican Primary School, Oke Esinele, Oyo for his primary education starting from January 1963 to December 1968. He subsequently proceeded to Anglican/Methodist Secondary Modern School, Ajagba, Oyo from January 1969 to December 1971. According to the wish of his late father, Professor Ayoola attended the then Government Trade Centre, Oyo (now Government Technical College) where he studied Painting and Decoration. He received the City and Guilds Craft Training Certificate, Federal Government Craft Certificate and Labour Trade Test Grade II in Painting and Decoration. Not satisfied with this level of education, while still at the Technical College, he studied for General Certificate of Education, Ordinary Level with the aid of the defunct Exam Success Correspondence College, Yaba, Lagos. He worked briefly as a Survey Assistant, Ministry of Lands and Housing, Oyo State between 1977 and 1980. During this period, he had the opportunity of being trained as a Land Surveyor at Survey School, Oyo. He gained admission into the University of Ibadan to study Mathematics in October 1980.

Professor Ayoola graduated in Mathematics with Second Class Honor, Upper Division in July 1984. He had the best result in the mathematics graduating class of 1984. He proceeded to the then Gongola State (now Adamawa and Taraba States) for the one year compulsory National Youth Service Corps (NYSC) programme from August 1984 to July 1985. After his NYSC program, he was appointed a Lecturer at the then St. Andrew's College of Education, Oyo in January 1986. He worked at the College for three years after which he assumed the position of an Assistant Lecturer in Mathematics, University of Ibadan in February 1989, having successfully completed the M.Sc. degree program at the University during the academic year 1986 - 1987 as a part-time postgraduate student.

Professor E. O. Ayoola was awarded the degree of Doctor of Philosophy in Mathematics by this university effective January 1999. He conducted his doctoral research under the supervision of Professor G.O.S. Ekhaguere, our world-acclaimed Professor of Mathematical Physics and Non-commutative Stochastic Analyst of international repute. Professor Ekhaguere started the supervision in January 1993, after Ayoola had been a Lecturer in the Department for about four years. The progress of his Ph.D. work suffered initial setback because of the brain-drain that made his first supervisor to relocate abroad towards the end of 1989.

After the completion of the doctoral work, Professor Ayoola was promoted to the position of a Lecturer Grade I in October 1999, Senior Lecturer in the year 2002, Reader in the year 2005 and finally Professor effective October 1, 2008. Thus, he was able to make professorship within 10 years of the award of his doctorate degree.

Professor Ayoola proceeded to the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy during the year 2000 - 2001 for his post-doctoral research work. The ICTP is a major UNESCO - sponsored research centre with the mandate of assisting Mathematics and Physics researchers based in developing and least developed countries of the world. Professor Ayoola was a Swedish Institute Guest Scholar at the School of Mathematics, Chalmers University of Technology, Goteborg, Sweden from October 2004 to July 2005. He was granted a research fellowship as a Regular Associate of the ICTP starting from January 1, 2003 to December 31, 2010. Professor Ayoola was a Visiting Professor, Mathematics Department, University of Pretoria, South Africa (November - December 2006) and Winston Salem State University, North Carolina, USA (August - December 2005). He has served as an External Examiner in many Universities in Nigeria, Ghana and South Africa.

Professor Ayoola's major area of mathematical research is Stochastic Analysis and Applications. In particular, he has made substantial contributions in the field of Quantum Stochastic Differential Equations and Inclusions. He has published research articles in the most competitive and best mathematical journals in the world. The reviews of his research articles in the *Mathematical Reviews* database published by the **American Mathematical Society (AMS)** show major contributions in his field of research. This earned him recognition and appointment as a Reviewer in January 2009 by the AMS. He is a member of the Nigeria Mathematical Society, Nigerian Association of Mathematical Physics and an Affiliate Member of the American Mathematical Society.

Professor Ayoola has served this university as a member of many important committees. Among many others, he has been one of the Congregation Representatives in Senate for many years before he became a professor and member of many subcommittees of Council. He is one of the Staff Advisers, MFM Campus Fellowship, he has served as Warden, Tedder Hall of residence, he is the current Head of Mathematics Department and one of the internal members of our Governing Council recently elected to represent Senate of the University.

Professor Ayoola is happily married with children.

END