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Counting Subgroup Formula for the Groups Formed by Cartesian Product of the Generalized Quaternion Group With Cyclic Group of Order Two

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Abstract. The main goal of this note is to determine an explicit formula of finite group formed by taking the Cartesian product of the generalized quaternion group of two power order with a order two cyclic group.

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1 Preliminaries

Counting subgroups of finite groups is one of the most important problems of combinatorial finite group theory.
In the last century, the problem was completely solved in the abelian case, by establishing an explicit expression of the number of subgroups of a finite abelian group (see [1]).

Unfortunately, in the nonabelian case a such expression can be given only for certain finite nonabelian groups (see [5]). Thus this paper derived an explicit formula for number of subgroups of nonmetacyclic groups of type: $Q_{2^{n-1}} \times C_2$ where $Q_{2^{n-1}}$ is the generalized quaternion group of order 2^{n-1}, $n \geq 4$, and C_2 is a cyclic group of order 2.

In the following if G is a group, then the set $L(G)$ consisting of all subgroups of G forms a complete lattice with respect to set inclusion, called the subgroup lattice of G.

More precisely, we prove the following result.

Theorem 1.2

For $n \geq 4$, the number of subgroups of the nonmetacyclic group $Q_{2^{n-1}} \times C_2$ is given by the following equality:

$$|L(Q_{2^{n-1}} \times C_2)| = \begin{cases} 19 & ;\text{if } n = 4 \\ 3\left(n + 1 + \sum_{k=2}^{n-3} 2^{n-k-1}\right) + 2^{n-2} & ;\text{if } n \geq 5 \end{cases}$$

Where $|L(Q_{2^{n-1}} \times C_2)|$ is the subgroup lattice of $Q_{2^{n-1}} \times C_2$
But in order to prove Theorem 1.2, we shall use the following auxiliary result, established in [2].

Lemma 1.1

For $n \geq 3$, the number of subgroups of the nonmetacyclic group $D_{2^{n-1}} \times C_2$ is given by the following equality:

$$|L(D_{2^{n-1}} \times C_2)| = \begin{cases} 16 & ;\text{if } n = 3 \\ 3\left(n + 1 + \sum_{k=2}^{n-2} 2^{n-k}\right) + 2^{n-1} & ;\text{if } n \geq 4 \end{cases}$$

Where $|L(D_{2^{n-1}} \times C_2)|$ is the subgroup lattice of $D_{2^{n-1}} \times C_2$, $D_{2^{n-1}}$ is a dihedral group of order 2^{n-1}, $n \geq 3$, and C_2 is a cyclic group of order 2.

Most of our notation is standard and will usually not be repeated here. For basic definitions and results on groups we refer the reader to [3], [4] and [6].

2 Proof of Theorem 1.2 :

Proof. Let $Q_{2^{n-1}} \times C_2 := \langle x, y : x^{2^{n-2}} = 1, y^2 = x^{2^{n-3}}, yxy^{-1} = x^{-1} \rangle \times \langle a \rangle$
The case $n = 4$ is clear. So we assume $n \geq 5$. An important property of this group is that its characteristic subgroup defined by

$$\mho_{n-3}(Q_{2^{n-1}} \times C_2) := \langle z^{2^{n-3}} : z \in Q_{2^{n-1}} \times C_2 \rangle$$

is of order 2. Also, for $n \geq 4$, we have:

$$\frac{Q_{2^{n-1}} \times C_2}{\mho_{n-3}(Q_{2^{n-1}} \times C_2)} \cong D_{2^{n-2}} \times C_2 \quad n \geq 5 \tag{1}$$

This follows from the epimorphism $\gamma : Q_{2^{n-1}} \times C_2 \longrightarrow D_{2^{n-2}} \times C_2$ defined by $\gamma(z) := z\,\langle(x^{2^{n-3}}, 1)\rangle$ where $z \in Q_{2^{n-1}} \times C_2$ and $z\,\langle(x^{2^{n-3}}, 1)\rangle \in Q_{2^{n-2}} \times C_2$. Clearly, the kernel of γ is $\mho_{n-3}(Q_{2^{n-1}} \times C_2) := \langle(x^{2^{n-3}}, 1)\rangle$ and from the first isomorphism theorem for groups.

Now, we observe that the trivial subgroup as well as all minimal subgroups of $Q_{2^{n-1}} \times C_2$ excepting $\mho_{n-3}(Q_{2^{n-1}} \times C_2)$ contains 3 such minimal subgroups, for all $n \geq 4$, and the join of any two distinct such subgroups includes $\mho_{n-3}(Q_{2^{n-1}} \times C_2)$.

Therefore one obtains that the number of subgroups of $Q_{2^{n-1}} \times C_2$ verifies the relation:

$$|L(Q_{2^{n-1}} \times C_2)| = |L(\frac{Q_{2^{n-1}} \times C_2}{\mho_{n-3}(Q_{2^{n-1}} \times C_2)})| + 3$$

or

$$|L(Q_{2^{n-1}} \times C_2)| = |L(D_{2^{n-2}} \times C_2)| + 3 \tag{2}$$

for all $n \geq 5$.
Thus by Lemma 1.1 and writing (2) for $n = 5, 6, \cdots$ and summing up these equalities,
we obtain an explicit expression as follows:

$$|L(Q_{2^{n-1}} \times C_2)| = \begin{cases} 19 & ;\text{if } n = 4 \\ 3\left(n + 1 + \sum_{k=2}^{n-3} 2^{n-k-1}\right) + 2^{n-2} & ;\text{if } n \geq 5 \end{cases}$$

Thus the proof of the theorem is complete. □
We end this note by indicating one open problem with respect to the above result.

Problem:

Determine the number of subgroups for other remarkable classes of nonabelian non-metacyclic finite p-groups.

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