Journal of Science Research. Volume 5[1] 1999 pp 32-37

ISSN 1117 9333

Units of burnside rings of cyclic groups

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Abstract

Computations showing that

$|\Omega(C_{2^n})^*| = 2^2$ and $|\Omega(G)^*| = 2^4$,

were obtained respectively if G is a cyclic group C_{2^n} of order 2^n and when $G = C_2 \oplus C_4$

Introduction

Let G be a finite group, $\Omega(G)$ the Burnside ring of G, that is, the Grothendieck ring obtained from the semi-ring of G-isomorphism classes of finite G-sets under addition and multiplication induced respectively by the disjoint union and the Cartesian product. The goal of this paper is to study the structure of the group $\Omega(G)^*$ of units of $\Omega(G)$.

In section 1 of this paper, we investigate the structure of units of Burnside rings for $G = C_{2^n}$, a cyclic group of order 2^n and show that $|\Omega(C_{2^n})^*| = 2^2$ while in section 2 we study $\Omega(G)^*$, $G = C_2 \oplus C_4$ where we obtain $|\Omega(G)^*| = 2^4$.

§ 1. The structure of the group of units of the Burnside ring for G a cyclic group of order 2^n : C_{2^n}.

1.1 Let

$$i := 2^n = [G : 1].$$

We can enumerate all divisors of i in an increasing sequence of numbers, say,

$$i_0 := 1, i_1 := 2, i_2 := 4, i_3 := 8, \ldots, i_n = 2^n.$$

For each divisor i_j of i, there is a unique subgroup $H_j \subseteq G$ such that $|H_j| = 2^j$, and hence $[G : H_j] = 2^{n-j}$.

1.2 Let α denote a generator of G and put $\alpha_j := \alpha^{2^{n-j}}$ so that $H_0 := <\alpha_0>, H_j := <\alpha_j>, j \neq 0, j = 1,2,\ldots,n$

with

$1 = <\alpha_0 \leq <\alpha_1>, \ldots \leq <\alpha_n> = G$

Now since

$N_G(<\alpha_j>) = G$ for all j, since G is commutative.

Then we have the following list of distinct conjugate classes denoted by $Cl(C_{2^n})$, list of distinct subgroups.

$Cl(C_{2^n}) = \{<\alpha_0>, <\alpha_1>, \ldots, <\alpha_n>\}$

1.3 Now let g be an arbitrary element of G, then $g = \alpha^k$ for all $k = 1, \ldots, 2^n$, it also follows from above relations that $<g> = H_j$, for some j that is, $<\alpha^k> = <\alpha^{2^{n-j}}>$. So we can rewrite each member in $Cl(C_{2^n})$ in terms of its set of generators as follows:

Let A_i be set of generators of H_i, $i = 0, 1, 2, \ldots, n$ then we have

$A_0 := \{\alpha^{2^n}\} = \{e\}$ e = identity of G.
$A_1 := \{\alpha^{2^{n-1}}\}$
$\vdots \quad \vdots \quad \vdots$
$A_{n-1} := \{\alpha^2, \alpha^6, \ldots, \alpha^{2^n-6}, \alpha^{2^n-2}\}$

$A_n := \{a, a^3, \ldots, a^{2^n-3}, a^{2^n-1}\}$

where

$\#A_0 = 1, \#A_1 = 1, \ldots, \#A_{n-1} = 2^{n-2}, \#A_n = 2^{n-1}$

Also, we obtain the following sequence of indexes in G:

$$|G| = (G:H_0) = 2^n$$
$$(G:H_1) = 2^{n-1}$$
$$(G:H_2) = 2^{n-2}$$
$$\vdots \quad \vdots$$
$$(G:H_{n-1}) = 2$$

1.4 Now we know, for $\gamma \supseteq \mathbb{Z}Cl(G)$, G a finite group, that

$$\gamma \in \Omega(G) \Leftrightarrow \sum_{g \in G} \gamma(<g>) \equiv 0\ (|G|)$$

$$\sum_{gS \in N_G(S)/S} \gamma(<g>S) \equiv 0(N_G(S):S))$$

so the above sum formula implies

$$\gamma(H_0)+\gamma(H_1)+2\gamma(H_2)+4\gamma(H_3)+\ldots+2^{n-2}\gamma(H_{n-1})+2^{n-1}\gamma(G) \equiv 0(2^n)$$

$$\gamma(H_1)+\gamma(H_2)+2\gamma(H_3)+\ldots+2^{n-3}\gamma(H_{n-1})+2^{n-2}\gamma(G) \equiv 0(2^{n-1})$$

$$\vdots \qquad \vdots \qquad \ldots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\gamma(H_{n-1}) + \gamma(G) \equiv 0(2)$$

Now since for all $H \leq G$.

$\gamma(H) \equiv \{\pm 1\}$ in case $\gamma \in \Omega(G)^*$

we obtain the following table with respect to the conjugate classes of G.

1.5 Table for $\Omega(C_{2^n})^*$

H_0	H_1	H_2	...	H_{n-1}	G
+1	+1	+1	...	+1	+1
-1	-1	-1	...	-1	-1
-1	-1	-1	...	-1	+1
+1	+1	+1	...	+1	-1

That means:

$|\Omega(C_{2^n})^*| = 2^2$

By the above table for $\Omega(C_{2^n})^*$ we observe the following claim:

1.5.1 Claim

Assume that $\gamma(H_i) \in \{\pm 1\}$ for $i = 0, \ldots, n-1$ then

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{j-1}\gamma(H_{i+j}) + \ldots$$

$$+2^{n-i-2}(H_{n-1}) + 2^{n-i-1}\gamma(H_n) \equiv 0\ (2^{n-i}) \text{ for all } i = 0, 1, \ldots, n-1$$

$$\Leftrightarrow \gamma(H_0) = \gamma(H_1) = \ldots = \gamma(H_{n-1}) = \pm\gamma(H_n)$$

Proof "$\Rightarrow$" It is easy to see, since

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{n-i-2}\gamma(H_{n-1}) = 2^{n-i-1}\gamma(H_n)$$

and by assumption we must have that

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{n-i-1}\gamma(H_n) \equiv 0\ (2^{n-i})$$

for all i

To see "$\Leftarrow$" We use method of induction on $n - i$:

For $n - i = 0 \Rightarrow i = n$ it is easy to see that

$$\gamma(H_0) = \gamma(H_n)$$

Similarly for $i = n - 1$

Now assume that the induction hypothesis is true for $i < n - 1$, that is, $n - i > 1$, so that we have

$$\gamma_0 := \gamma(H_{i+1}) = \gamma(H_{i+2}) = \ldots = \gamma(H_{n-1}) = \pm\gamma(H_n)$$

Then we obtain by hypothesis

$$\gamma(H_i) + (2^{n-i-1} - 1)\gamma_0 \pm 2^{n-i-1}\gamma(H_n) \equiv 0\ (2^{n-i})$$

this implies,

$$\gamma(H_i) + 2^{n-i-1}(\gamma_0 \pm \gamma(H_n)) - \gamma_0 \equiv 0\ (2^{n-i})$$

But since $(\gamma_0 \pm \gamma(H_n))$ is either 0 or ± 2 we get that

$$2^{n-i-1}(\gamma_0 \pm \gamma(H_n)) \equiv 0\ (2^{n-i})$$

and

$$\gamma(H_i) - \gamma_0 \equiv 0\ (2^{n-i})$$

also since $n - i > 1$, $\gamma(H_i) = \{\pm 1\}$, $\gamma_0 := \{\pm 1\}$ we cannot get that $+1 \equiv -1\ (4)$ for instance, so it follows that

$\gamma(H_i) = \gamma_0$

Therefore the proof of the claim is complete.

§ 2. The structure of the group of units of the Burnside ring for G := $C_2 \oplus C_4$

2.1 We derive its set of subgroups as follows:

Sub $(C_2 \oplus C_4)$:= $\{<1> := \{(1,1)\}; H_{21} := \{(1,1),(1,b^2)\} = <(1,b^2)>;$

$H_{22} := \{(1,1),(a,1)\} = <(a,1)>;$

$H_{23} := \{(1,1),(a,b^2)\} < (a,b^2)>;$

$H_{41} := \{(1,1),(1,b^2),(a,1),(a,b^2)\} = <(1,b^2),(a,1)>;$

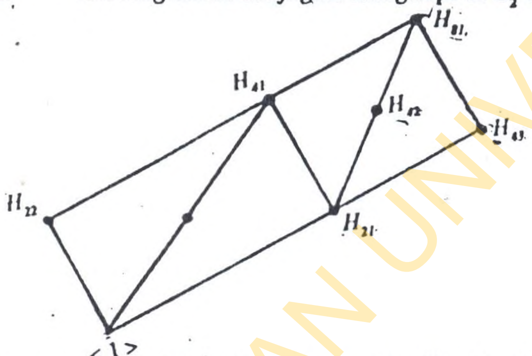
$H_{42} := \{(1,1),(1,b),(1,b^2),(1,b^3)\} = <(1,b)>;$

$H_{43} := \{(1,1),(a,b),(1,b^2),(a,b^3)\} = <(a,b)>;$

$H_{81} := C_2 \oplus C_4;$

$:= \{(1,1),(1,b^2),(1,b),(1b^3),(a,1),(a,b),(a,b^2),(a,b^3)\}\}$

2.2 The diagram of conjugate subrgoups of $C_2 \oplus C_4$ is:



The normalizer of each of the groups $<1>, H_{21}, H_{22}, H_{23}, H_{41}, H_{42}, H_{43}$ and H_{81} is H_{81}.

Then, by the congruence (Tom Dieck):

2.3 $\Sigma'|N(H)/N(H) \cap N(K)||(K/H)^*|\gamma(K)) \equiv 0 \bmod$ K

(N (H) : H) [4]

for all (H) ∈ Sub (G), where Σ is over all N (H)-conjugate classes (K) such that H is normal in K and K/ H is cylic, and $(K/H)^*$ is the set of generators of K / H. γ: Sub(G) → {±1}. That means, the following congruences:

(i) $H := H_{81}, (N(H_{81}) : H_{81}) = 1, \gamma(H_{81}) \equiv 0\ (1).$

(ii) $H := H_{41}, (N(H_{41}) : H_{41}) = 2, \gamma(H_{81}) + \gamma(H_{41}) \equiv 0\ (2)$

(iii) $H := H_{42}, (N(H_{42}) : H_{42}) = 2, \gamma(H_{81}) + \gamma(H_{42}) \equiv 0\ (2)$

(iv) $H := H_{43}, (N(H_{43}) : H_{43}) = 2, \gamma(H_{81}) + \gamma(H_{43}) \equiv 0\ (2)$

(v) $H := H_{21}, (N(H_{21}) : H_{21}) = 4$, the set of subgroups between H_{81} and H_{21} is $\{H_{81}, H_{41}, H_{42}, H_{43}, H_{21}\}$.

We must compute for

$|(H_{81}/H_{21})^*|, |(H_{41}/H_{21})^*|, |(H_{42}/H_{21})^*|$ and $|(H_{43}/H_{21})^*|$.

Clearly $|(H_{21}/H_{21})^*| = 1$, and since $H_{41}/H_{21} \cong \mathbb{Z}_2$, $H_{42}/H_{21} \cong \mathbb{Z}_2$, $H_{43}/H_{21} \cong \mathbb{Z}_2$, it follows that

$|(H_{41}/H_{21})^*| = 1$, $|(H_{42}/H_{21})^*| = 1$, and $|(H_{43}/H_{21})^*| = 1$.

Now, let us compute the factor group H_{81}/H_{21} as follows: Recall $(H_{21} := \{(1,1),(1,b^2)\}; H_{81} := C_2 \oplus C_4$.

Here the first factor C_2 of $C_2 \oplus C_4$ is left alone. The C_4 factor, on the other hand, is essentially collapsed by a subgroup of order 2. That means,

$H_{81}/H_{21} \cong C_2 \oplus C_2$

which is abelian but not cyclic, hence

$|(H_{81}/H_{21})^*| = 0.$

So then we obtain

$\gamma(H_{41}) + \gamma(H_{42}) + \gamma(H_{43}) + \gamma(H_{21}) \equiv 0\ (4)$

(vi) $H := H_{22}, (N(H_{22}) : H_{22}) = 4$, K := H_{81}, H_{41}, H_{22}

Clearly, $|(H_{81}/H_{22})^*| = 2$, since $H_{81}/H_{22} \cong \mathbb{Z}_4$ which has two generators

$|(H_{41}/H_{22})^*| = 1$, since $H_{41}/H_{22} \cong \mathbb{Z}_2$

$|(H_{22}/H_{22})^*| = 1$, so we obtain

$2\gamma(H_{81}) + \gamma(H_{41}) + \gamma(H_{22}) \equiv 0\ (4)$

(vii) $H := H_{23}, (N(H_{23}) : H_{23}) = 4, K := H_{81}, H_{41}, H_{23}$
Since $(1,b)H_{23}$ is of order 4 in the factor group H_{81}/H_{23}, that means,

$H_{81}/H_{23} \cong \mathbb{Z}_4$ and so $|(H_{81}/H_{23})^*| = 2.$

Also, $|(H_{41}/H_{23})^*| = 1$, $|(H_{23}/H_{23})^*| = 1$,
hence, we have

$2\gamma(H_{81}) + \gamma(H_{41}) + \gamma(H_{23}) \equiv 0\ (4)$

(viii) $H := < 1 >, (N(< 1 >) : < 1 >) = 8,$

K $:= H_{81}, H_{41}, H_{42}, H_{43}, H_{21}, H_{22}, H_{23}, < 1 >.$

$|(H_{81}/ < 1 >)^*| = 0$, since H_{81} is not cyclic

$|(H_{41}/ < 1 >)^*| = 0$, since H_{41} is not cyclic

$|(H_{42}/ < 1 >)^*| = 2$, since $H_{42} \cong \mathbb{Z}_4$ which has two generators.

$|(H_{43}/ < 1 >)^*| = 2$, since $H_{43} \cong \mathbb{Z}_4$

$|(H_{21}/ < 1 >)^*| = 1$, since $H_{21} \cong \mathbb{Z}_2$

$|(H_{22}/ < 1 >)^*| = 1$, since $H_{22} \cong \mathbb{Z}_2$

$|(H_{23}/ < 1 >)^*| = 1$, since $H_{23} \cong \mathbb{Z}_2$

$|(< 1 > / < 1 >)^*| = 1$, so we have

$$0\gamma(H_{81}) + 0\gamma(H_{41}) + 2\gamma(H_{42}) + 2\gamma(H_{43}) + \gamma(H_{21}) + \gamma(H_{22}) + \gamma(H_{23}) + \gamma(< 1 >) \equiv 0\ (8)$$

2.4 The ring $\Omega(C_2 \oplus C_4)$ contains precisely the following units

H_{81}	H_{41}	H_{42}	H_{43}	H_{21}	H_{22}	H_{23}	$< 1 >$
+1	+1	+1	+1	+1	+1	+1	+1
-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	+1	+1	-1	-1	-1	-1
+1	+1	-1	-1	+1	+1	+1	+1
+1	-1	+1	+1	-1	-1	-1	-1
-1	+1	-1	-1	+1	+1	+1	+1
+1	+1	-1	+1	-1	+1	+1	-1
-1	-1	+1	-1	+1	-1	-1	+1
-1	+1	-1	+1	-1	+1	+1	-1
+1	-1	+1	-1	+1	-1	-1	+1
+1	+1	+1	-1	-1	+1	+1	-1
-1	-1	-1	+1	+1	-1	-1	+1
-1	+1	+1	-1	-1	+1	+1	-1
+1	-1	-1	+1	+1	-1	-1	+1
-1	+1	+1	+1	+1	+1	+1	+1
+1	-1	-1	-1	-1	-1	-1	-1

That is,

$|\Omega(C_2 \oplus C_4)^*| = 2^4$

An alternative method

The above table for $\Omega(C_2 \oplus C_4)^*$ is equivalent to the following lemma.

2.5 Let U be an arbitrary element of Sub $(C_2 \oplus C_4)$ and $\gamma(U) \in \{\pm 1\}$. Then the congruences

(i) $\gamma(H_{81}) \equiv 0\ (1),$

(ii) $(\gamma H_{81}) + \gamma(H_{41}) \equiv 0\ (2),$

(iii) $\gamma(H_{81}) + \gamma(H_{42}) \equiv 0\ (2),$

(iv) $\gamma(H_{81}) + \gamma(H_{43}) \equiv 0\ (2),$

(v) $\gamma(H_{41}) + \gamma(H_{42}) + \gamma(H_{43}) + \gamma(H_{21}) \equiv 0\ (4),$

(vi) $2\gamma(H_{81}) + \gamma(H_{41}) + \gamma(H_{22}) \equiv 0\ (4),$

(vii) $2\gamma(H_{81}) + \gamma(H_{41}) + \gamma(H_{23}) \equiv 0\ (4),$

(viii) $2\gamma(H_{42}) + 2\gamma(H_{43}) + \gamma(H_{21}) + \gamma(H_{22}) + \gamma(H_{23}) + \gamma(< 1 >) \equiv 0\ (8)$ imply the following results:

Firstly, in view of

$2\gamma(U) = \pm 2 \equiv 2\ (4)$

and, hence,

$\gamma(U) \equiv 2 - \gamma(U)\ (4),$

for every U $\in$ Sub $(C_2 \oplus C_4)$, the equations (vi) and (vii) imply

$\gamma(H_{22}) \equiv 2 - \gamma(H_{41}) \equiv \gamma(H_{41})\ (4)$

and

$\gamma(H_{23}) \equiv 2 - \gamma(H_{41}) \equiv \gamma(H_{41})\ (4)$

and therefore

(ix) $\gamma(H_{22}) = \gamma(H_{23}) = \gamma(H_{41}).$

From (v), we have

(x) $2\gamma(H_{41}) + 2\gamma(H_{42}) + 2\gamma(H_{43}) + 2\gamma(H_{21}) \equiv 0\ (8)$, and substituting (ix) in (x) and (viii), respectively, we get

(xi) $2\gamma(H_{22}) + 2\gamma(H_{42}) + 2\gamma(H_{43}) + 2\gamma(H_{21}) \equiv 0\ (8)$

(xii) $2\gamma(H_{22}) + 2\gamma(H_{42}) + 2\gamma(H_{43}) + 2\gamma(H_{21}) + \gamma(<1>) \equiv 0\ (8)$

and subtracting (xii) from (xi), we obtain

$\gamma(H_{21}) - \gamma(<1>) \equiv 0$ (8),

this implies

$\gamma(H_{21}) \equiv \gamma(<1>)$ (8),

also, this implies

$\gamma(<1>) = \gamma(H_{21})$.

Next, we have the following claim:

2.6 $\gamma(U) \in \{\pm 1\}$ for all $U \leq G$

and

$\gamma(U_1) + \gamma(U_2) + \gamma(U_3) + \gamma(U_4) \equiv 0$ (4)

for some $U_1, U_2, U_3, U_4 \leq G$ implies

$\gamma(U_1) \cdot \gamma(U_2) \cdot \gamma(U_3) \cdot \gamma(U_4) = 1$

that is

$\gamma(U_4) = \gamma(U_1) \cdot \gamma(U_2) \cdot \gamma(U_3)$

To verify this claim, all we have to do is to show that no two of

$\{\gamma(U_1), \gamma(U_2), \gamma(U_3), \gamma(U_4)\}$

are congruent modulo 4. To see this is easy: Suppose that for any pair (i, j), $i \neq j = 1,2,3,4$

$\gamma(U_i) \equiv \gamma(U_j)$ (4)

That means,

$\gamma(U_i) \equiv \gamma(U_j) = \pm 1$

and so our assumption is affected because $2\gamma(U) \equiv -1$ (4) is not possible. Hence if

$\gamma(U_i) \neq \gamma(U_j)$,

then $\gamma(U_i) \not\equiv \gamma(U_j)$ (4),

and no two elements of

$\{\gamma(U_1), \gamma(U_2), \gamma(U_3), \gamma(U_4)\}$

are congruent modulo 4. Therefore we can have

$\gamma(U_4) = \gamma(U_1) \cdot \gamma(U_2) \cdot \gamma(U_3)$

Finally, we can choose

$\gamma(H_{81}) = a$, $\gamma(H_{41}) = b$, $\gamma(H_{42}) = c$ and $\gamma(H_{21}) = d$,

where a, b, c, d, $\in \{\pm 1\}$ and obtain the following equivalent table

2.7	H_{81}	H_{41}	H_{42}	H_{43}	H_{21}	H_{22}	H_{23}	$<1>$
	a	b	c	c	d	b	b	d

so then we have

$|\Omega(C_2 \oplus C_4)^*| = 2^4$.

Acknowledgement

I am sincerely grateful to my Ph.D thesis supervisors Professor (Dr.) Andreas Dress and Professor Aderemi O. Kuku for their guidance, patience and generosity.

References

Alawode, M. A (1999). The group of units of Burnside rings of various finite groups. Ph.D thesis, University of Ibadan, Ibadan, Nigeria.

Araki, R (1982). Equivariant stable homotopy theory and idempotents of Burnside rings, *Publ. R.I.M.S, Kyoto Univ.*, *18*, 1193-1212

Bender, H (1970)., On groups with abelian sylow 2-subgroups *Math. Z.*, *117*, 164-176.

Curtis, C. W. and Reiner, I (1981). *Method of Representation Theory*, Wiley-interscience Publ., New York, Vol. 1 & 2.

Dieck, T (1979). Transformation Groups and Representation Theory. *Lecture Notes in Math.*, 766, Springer.

Dress, A. A (1969). Characterization of solvable groups, *Math. Z.*, *110(1969)*, 213-217.

Dress, A (1971). Operations in representation rings, In *Pro Symposia in Pure Math.*, pp. 39-45.

Dress, A.(1973) Contributions to the theory of induced representations, in., "Algebraic K-theory II", Proc. Battle Institute Conf., *1972 Lecture Notes in Math.*, *342*, Springer, pp. 183-240

Dress, A. (1971) Notes on the theory of representations of finite groups. *Bielefeld Notes.*

Dress, A. and Kuchler, M. Zur (1970) Darstellungstheorie endlicher Gruppen I. *Bielefeld Notes*,

Feit, W. and Thompson, J. (1963) Solvability of groups of odd order. *Pacific J. Math.*, *13*, 775-1029.

Gluck, D (1981). Idempotent formula for the Burnside algebra with applications to the P-subgroup Simplicial Complex. *Illinois J. Math.*, *25*, 63 - 67.

Gorenstein, D (1968). *Finite Groups*, Harper & Row, New York.

Green, J. A (1971). Axiomatic representation theory for finite groups. *J. Pure Appl. Algebra 1*, 41-77.

Greub, W. H (1967). *Multilinear Algebra*, Springer-Verlag Berlin

Heidelberg, New York.

Gustafson, W. H (1977). Burnside rings which are Gorenstein Comm. Algebra 5, 1-16.

Keown, R (1975). *An introduction to Group Representation Theory.*

Kuku, A. O (1985). Axiomatic theory of induced representation of finite groups. In: A. O. Kuku (Ed.) Group Representation and its applications; *les cours du C.I.M.P.A.*

Li, I (1978), Burnside algebra of a finite inverse semigroup, Zap. Nauc. Steklov Inst. 46(1974), 41-52; *J. Soviet Math. 9,* 322-331.

Matsuda, T (1982). On the unit groups of Burside rings, Japanese *J. Math. (New Series) 8 (1982),* 71-93.

Matsuda, T. (1986) A note on the unit groups of the Burnside rings as Burnside ring Modules. *J. Fac. Sci., Shinshu Univ., Vol. 21, No. 1,*

Matsuda, T. and Miyata, T (1983). On the unit groups of the Burnside rings of finite groups, *J. Math. Soc. Japan, 35,, 345-354.*

Sasaki, H.(1982) Green correspondence and transfer theorems of Wielandt type for G-functors. *J. Algebra 79,* 98-120.

Serre, J. P. *Linear Representation of finite Groups.* GTM, 42 Springer-Verlag New York, Heidelberg, Berlin.

Walter, J. H. (1969) Finite groups with abelian sylow 2-subgroups, *Ann. of Math. 89,* 405-514.

Yoshida, T. (1978) Character-theortic transfer, *J. Algebra, 52,* 1-38.

Yoshida, T. On G-functors I: Transfer theorems for cohomological G-functors, *Hokkaido Math. J., 9(1980),* 222-257.

Yoshida, T. (1983) Idempotents of Burnside rings and Dress induction theorem, *J. Algebra, 80* . 90-105

Yoshida, T. (1985) Idempotents and transfer theorems of Burnside rings, character rings and span rings In. *Algebraic and Topological Theories*, Kinokuniya, Tokyo pp. 589-615

Yoshida, T. (1990) On the unit groups for Burnside rings. *J. Math. Soc. Japan, Vol. 42, No. 1.*