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DETERMINING THE NUMBER OF DISTINCT FUZZY SUBGROUPS FOR THE ABELIAN STRUCTURE: $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$,n >2

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Abstract

The problem of classification of fuzzy subgroups can be extended from finite p-groups to finite nilpotent groups. Accordingly, any finite nilpotent group can be uniquely written as a direct product of p-groups. In this paper, we give explicit formulae for the number of distinct fuzzy subgroups of the Cartesian product of two abelian groups of orders 2^{n-1} and 4 respectively for every integer n >2.

Keywords: *Finite p-Groups, Nilpotent Group, Fuzzy subgroups, Dihedral Group, Inclusion-Exclusion Principle, Maximal subgroups. SUBJECT CLASSIFICATION MSC. 2010.*

1. Introduction

In the fuzzy group theory, the classification of the fuzzy subgroups, most especially the finite p-groups cannot be underestimated. This aspect of pure Mathematics has undergone some dynamic developments over the years. For instance, many researchers have treated cases of finite abelian groups (see [1], [2]). The starting point for this concept all started as presented in [3] and [4] . Since then, the study has been extended to some other important classes of finite abelian and nonabelian groups such as the dihedral, quaternion, semidihedral, and Hamiltonian groups.

Although, the natural equivalence relation was introduced in [5], where a method to determine the number and nature of fuzzy subgroups of a finite group G was developed with respect to the natural equivalence. In [6] and [3], a different approach was applied for the classification. Here, an essential role in solving counting problems is played by adopting the Inclusion-Exclusion Principle. The process leads to some recurrence relations from which the solutions are then finally computed with ease.

2. Preliminaries

Suppose that $(G,\cdot,e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are sat.

(i) $\lambda(ab) \geq \min \{\lambda(a),\lambda(b)\}, \forall\ a,b \in G$; (ii) $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$.

Also, by this notation and definition, $\lambda(e) = \sup\lambda(G)$. [Marius [7]].

Theorem : The set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

We define the level subset:

$\lambda G_\beta = \{a \in G/\lambda(a) \geq \beta\}$ for each $\beta \in [0,1]$

The fuzzy subgroups of a finite p-group G are thus, characterized, based on these subsets. In the sequel, λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This theorem gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of G.

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite q-group G (see [4] and [5]). One of them is defined by: $\lambda \sim \gamma$ iff $(\lambda(a) > \lambda(b) \Longleftrightarrow v(a) > v(b), \forall\ a,b \in G)$. Also, two fuzzy subgroups λ,γ of G and said to be distinct if $\lambda \times v$.

As a result of this development, let G be a finite p-group and suppose that$\lambda : G \longrightarrow [0,1]$ is a fuzzy subgroup of G. Put $\lambda(G) = \{\beta_1,\beta_2,...,\beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in G is determined by λ.

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$\lambda G_{\beta 1} \subset \lambda G_{\beta 2} \subset \cdots \subset \lambda G_{\beta k} = G$ (*a*)

Also, we have that:

$\lambda(a) = \beta_t \Longleftrightarrow t = \max\{r/a \in \lambda G_{\beta r}\} \Longleftrightarrow a \in \lambda G_{\beta t} \backslash \lambda G_{\beta t-1}$, for any $a \in G$ and $t = 1,...,k$, where by convention, set $\lambda G_{\beta 0} = \varphi$.

3. METHODOLOGY

In the sequel, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, ..., M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) (\#)$$

In [1], (#) was used to obtain the explicit formulas for some positive integers n. **Theorem (*) [Marius]:** The number of distinct fuzzy subgroups of a finite p-group of order p^nwhich have a cyclic maximal subgroup is:

(i) $h(\mathbb{Z}pn) = 2^n$

(ii) $h(\mathbb{Z}p \times \mathbb{Z}pn_{-1}) = h(M_p n) = 2^{n-1}[2 + (n-1)p]$

4. THE NUMBER OF FUZZY SUBGROUPS FOR: $\mathbb{Z}_4 \times \mathbb{Z}_4$

Lemma(γ): Let G be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$Then,$h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$

Proof : By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = h(\mathbb{Z}_2 \times \mathbb{Z}_4)$(using(#)). And by theorem (*), $\mathbb{Z}_2 \times \mathbb{Z}_{2^2} = 24. \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$

Corollary : Following the last lemma, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5})$,$h((\mathbb{Z}_4 \times \mathbb{Z}_{2^6})$,$h((\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h((\mathbb{Z}_4 \times \mathbb{Z}_{2^8}) = 1536$, 4096, 10496 and 26112 respectively.

Proposition : Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}, n \ge 2$. Then, $h(G) = 2^n[n^2 + 5n - 2]$

Proof : G has three maximal subgroups of which two are isomorphic to$\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$.

Hence, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^1 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 2^2 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})$
$+ 2^3 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-3}}) + 2^4 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-4}}) + \cdots + 2^{n-2} h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2})$

$$= 2^{n+1}[2(n+1) + \sum_{j=1}^{n-2}[(n+1) - j]$$

$$= 2^{n+1}[2(n+1) + \tfrac{1}{2}(n-2)(n+3)] = 2^n[n^2 + 5n - 2], n \ge 2$$

We have that : $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = 2^{n-1}[(n-1)^2 + 5(n-1) - 2]$
$= 2^{n-1}[n^2 + 3n - 6], n > 2$ ■

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