Interval Valued Extension of Wendroff Type Integral Inequalities.

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Abstract

In this paper we extend the theory of integral inequalities to interval valued mappings in two independent variables. By this, set inclusions are used instead of partial order relation and this enables us to obtain simultaneous upper and lower bounds instead of a pair of inequalities.
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1. Introduction

The use of integral inequalities in the establishment of boundedness, estimate of bounds and determination of the behaviour of solutions of differential equations which are hitherto not solvable in closed forms cannot be overemphasised. Moore [6] considered the extension of the fundamental Gronwall's integral inequality and some of its variants to interval value mappings. In this paper we consider the interval extension of some Wendroff type integral inequalities which are the two variable generalisation of Gronwall inequality. A host of authors including Bondge and Pachpatte [1], Bondge, Pachpatte and Walter [2], Fink [4], Sergiy Borysenko and Gerudo Iovane [8], Wing Sun Cheung [9], and Wing Sun Cheung, Qing-Hua Ma and Shiojenn Tsenghave [10] have considered various extensions of the Wendroff integral inequality, however, the extension to interval valued functions are not readily available in existing literature.

Let IR denote the set of real valued intervals $X = [\underline{x}, \overline{x}] = \{x : \underline{x} \leq x \leq \overline{x},\ x \in R\}$ and let interval valued functions be defined by $F(X) = [\underline{F(X)}, \overline{F(X)}],\ X \in IR$, from which we have extensions of real value functions f given by $f(X) = \{f(x) : x \in X\}$ as special cases. The basic theory of interval analysis have been extensively developed in recent years, details can be found in Moore[5], Rall [7], and Caprani et al [3]. However, we highlight some properties that are peculiar to operations in interval analysis which are needed in the establishment of our results here. For $X,\ Y,\ Z \in IR$ we have $Z(X+Y) \subseteq ZX + ZY$. If $X \subseteq Y$, and f is an interval function defined on IR, then $f(X) \subseteq f(Y)$. Also if $X(t)$ and $Y(t)$ are interval functions defined on $0 \leq t \leq x$ such that $X(t) \subseteq Y(t)$, then their interval integrals exist and satisfy, $\int_0^x X(t)dt \subseteq \int_0^x Y(t)dt$ [3]. Next in the main result we recast a type of Wendroff inequality in two independent variables given in [2] and extend the result to interval valued mappings.

2. Main Result.

The main result reported here is based on the work of Bondge et al [2] which is a sharper general version of of the Wendroff's inequality and it is given in the theorem below.

Theorem 2.1

Let c, v, g, g_x, g_y and $g_{xy} \in C(R)$, where R is the first quadrant $x \geq 0, y \geq 0$, or a rectangle $0 \leq x \leq a, 0 \leq y \leq b, c \geq 0, g_x \geq 0, g_y \geq 0$ and $g_{xy} \geq 0$.

$$v(x,y) \leq g(x,y) + \int_0^x \int_0^y c(s,t)v(s,t)dsdt \tag{2.1}$$

Then

$$v(x,y) \leq g(x,y)e^{C(x,y)}, \tag{2.2}$$

where

$$C(x,y) = \int_0^x \int_0^y c(s,t)dsdt$$

We recast the theorem in the new form:

Theorem 2.2

Let c, v and $g \in C(R)$, where R is the first quadrant $x \geq 0, y \geq 0$, or a rectangle $0 \leq x \leq a,\ 0 \leq y \leq b,\ c(x,y) \geq 0.$ and

$$v(x,y) \leq g(x,y) + \int_0^x \int_0^y c(s,t)v(s,t)dsdt \ \forall\ (x,y) \in R \tag{2.1}$$

Then

$$V(x,y) \leq g(x,y)\left(1 + \int_0^x \int_0^y c(s,t)dsdt + \int_0^x \int_0^y c(s,t) \int_0^s \int_0^t c(s_1,t_1)ds_1dt_1dsdt + ...\right) \tag{2.3}$$

Proof:

Setting the right hand side of (2.1) to be $w(x,y)$, we then have $v(x,y) \leq w(x,y)$

and using this repeatedly n times in (2.1) gives

$$\begin{aligned}
v(x,y) &\le g(x,y) + \int_0^x \int_0^y c(s,t)w(s,t)dsdt \\
&\le g(x,y) + \int_0^x \int_0^y c(s,t)\left(g(s,t) + \int_0^s \int_0^t c(s_1,t_1)w(s_1,t_1)ds_1dt_1\right)dsdt \\
&\le g(x,y)\left(1 + \int_0^x \int_0^y c(s,t)dsdt + \int_0^x \int_0^y c(s,t)\int_0^s \int_0^t c(s_1,t_1)ds_1dt_1dsdt\right. \\
&+ \left. \ldots + \int_0^x \int_0^y c(s,t)\ldots \int_0^{s_n} \int_0^{t_n} c(s_{n+1},t_{n+1})ds_{n+1}dt_{n+1}\ldots dsdt \right) \\
&+ \ldots + \int_0^x \int_0^y c(s,t)\ldots \int_0^{s_n} \int_0^{t_n} c(s_{n+1},t_{n+1})v(s_{n+1},t_{n+1})ds_{n+1}dt_{n+1}\ldots dsdt
\end{aligned}$$

Let $M \ge ||v(x,y)||$ and $K \ge ||c(x,y)||\ \forall (x,y) \in R$, then we have

$$\begin{aligned}
&||\int_0^x \int_0^y c(s,t)\ldots \int_0^{s_n} \int_0^{t_n} c(s_{n+1},t_{n+1})v(s_{n+1},t_{n+1})ds_{n+1}dt_{n+1}\ldots dsdt|| \\
&\le MK^n \int_0^x \int_0^y \ldots \int_0^{s_n} \int_0^{t_n} ds_{n+1}dt_{n+1}\ldots dsdt \\
&\le M\frac{(abK)^n}{(n!)^2} \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

and hence the result follows.
An extension of this result to interval valued mappings is given in the following theorem.

Theorem 2.3

Let $P(x,y)$, $A(x,y)$, $B(x,y)$ and $U(x,y)$ be bounded interval valued functions defined on $I \times I$, where $I = [0,\tau]$ such that $||P(x,y)|| \le \alpha$, $||A(x,y)|| \le a$, $||B(x,y)|| \le b$ and $||U(x,y)|| \le \lambda\ \forall\,(x,y) \in I \times I$. If

$$P(x,y) \subseteq U(x,y) + A(x,y)\int_0^x \int_0^y B(s,t)P(s,t)dsdt \tag{2.4}$$

holds for $0 \le x \le \tau$ and $0 \le y \le \tau$
Then

$$P(x,y) \subseteq \sum_{k=0}^{\infty} P_k(x,y) \tag{2.5}$$

where

$$P_0(x,y) = U(x,y) \tag{2.6}$$

and

$$P_k(x,y) = A(x,y)\int_0^x \int_0^y B(s,t)P_{k-1}(s,t)dsdt,\ k = 1,2,\ldots \tag{2.7}$$

Proof

We shall show by induction that for $n = 1, 2, ...$

$$P(x,y) \subseteq \sum_{k=0}^{n} P_k(x,y)$$
$$+\int_0^x\int_0^y R(s,t)\int_0^s\int_0^t R(s_1,t_1)...\int_0^{s_{n-1}}\int_0^{t_{n-1}} R(s_n,t_n)P(s_n,t_{n)}ds_n dt_n...dsdt \quad (2.8)$$

where

$$R(s,t) = A(x,y)B(s,t)$$

Let J_0 = rhs of (2.4) and J_n = rhs of (2.8). By the hypothesis (2.4) of the theorem, (2.8) is true for $n = 0$. For $n = 1$, we have,

$$\begin{aligned}
J_1(x,y) &= P_0(x,y)+P_1(x,y)+\int_0^x\int_0^y R(s,t)\int_0^s\int_0^t R(s_1,t_1)P(s_1,t_1)ds_1dt_1dsdt \\
&= U(x,y)+A(x,y)\int_0^x\int_0^y B(s,t)U(s,t)dsdt \\
&+ \int_0^x\int_0^y A(x,y)B(s,t)\int_0^s\int_0^t A(s,t)B(s_1,t_1)P(s_1,t_1)ds_1dt_1dsdt \\
&\supseteq U(x,y)+A(x,y)\int_0^x\int_0^y B(s,t)J_0(s,t)dsdt \\
&\supseteq J_0(x,y) \supseteq P(x,y).
\end{aligned}$$

Thus the hypothesis is true for n=1. We now assume it is true for n = m, so for n = m+1 we have

$$J_{m+1} = \sum_{k=0}^{m+1} P_k(x,y)$$
$$+\int_0^x\int_0^y R(s,t)\int_0^s\int_0^t R(s_1,t_1)...\int_0^{s_m}\int_0^{t_m} R(s_{m+1},t_{m+1})$$
$$\times P(s_{m+1},t_{m+1)}ds_{m+1}dt_{m+1}...dsdt$$
$$\supseteq \sum_{k=0}^{m} P_k(x,y)+\int_0^x\int_0^y R(s,t)\int_0^s\int_0^t R(s_1,t_1)...\int_0^{s_{m-1}}\int_0^{t_{m-1}} R(s_m,t_m)$$
$$\times J_0(s_m,t_{m)}ds_m dt_m...dsdt$$
$$\supseteq J_0(x,y) \supseteq P(x,y)$$

Hence, $P(x,y) \subseteq J_n \; \forall \; n \in \mathbb{N}$ therefore (2.8) holds true.

Now taking the norm of the iterated integral in (2.8) we have

$$\left.\begin{aligned}&\left\|\int_0^x\int_0^y R(s,t)\int_0^s\int_0^t R(s_1,t_1)\dots\int_0^{s_{n-1}}\int_0^{t_{n-1}} R(s_n,t_n)P(s_n,t_n)ds_n dt_n\dots dsdt\right\|\\&\leq (ab)^{n+1}\alpha\int_0^x\int_0^y\dots\int_0^{s_{n-1}}\int_0^{t_{n-1}} ds_n dt_n\dots dsdt\\&\leq \frac{\alpha(ab\tau^2)^{n+1}}{[(n+1)!]^2}.\end{aligned}\right\} \tag{2.9}$$

This integral is uniformly bounded in the norm by the last expression in (2.9) and thus, as $n\to\infty$, it converges to zero.

The series in (2.5) converges, since from (2.6) $||P_0(x,y)||\leq\lambda$, and from (2.7) we have

$$\begin{aligned}||P_k(x,y)|| &= ||A(x,y)\int_0^x\int_0^y B(s,t)P_{k-1}(s,t)dsdt||\\&\leq \frac{\lambda(ab\tau^2)^k}{(k!)^2}\\&\to 0 \text{ as } k\to\infty.\end{aligned}$$

Hence, the result follows.

Remark 2.1

If we set $P(x,y)=[0,v(x,y)]$, $U(x,y)=[0,g(x,y)]$, $A(x,y)=[0,1]$ and $B(x,y)=[0,c(x,y)]$ in Theorem 2.3, we recover Theorem 2.2 as a special case with (2.5) given by

$$v(x,y)\leq g(x,y)+\int_0^x\int_0^y c(s,t)g(s,t)dsdt+\int_0^x\int_0^y c(s,t)\int_0^s\int_0^t c(s_1,t_1)g(s_1,t_1)ds_1dt_1dsdt+\dots$$

$$\leq g(x,y)\left(1+\int_0^x\int_0^y c(s,t)dsdt+\int_0^x\int_0^y c(s,t)\int_0^s\int_0^t c(s_1,t_1)ds_1dt_1dsdt+\dots\right)$$

which gives the inequality (2.3).

Remark 2.2

If we set $P(x,y)=[0,\phi(x,y)]$, $U(x,y)=[0,u(x,y)]$, $A(x,y)=[0,a(x,y)]$ and $B(x,y)=[0,b(x,y)]$ in Theorem 2.3, the expression (2.4) in Theorem 2.3 becomes the integral inequality (1) of Theorem 1 in

Bondge B. K. and Pachpatte B. G. [1]. Therefore, the result in the theorem is a special case of our theorem here with (2.5) given by:

$$\phi(x,y) \leq u(x,y)\Bigg(1 + \int_0^x \int_0^y a(x,y)b(s,t)dsdt$$
$$+ \int_0^x \int_0^y a(x,y)b(s,t) \int_0^s \int_0^t a(s,t)b(s_1,t_1)ds_1dt_1dsdt + \dots\Bigg)$$

as the result corresponding to the integral inequality (2) in Theorem 1 of [1].

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