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Existence Results for Some Interval Differential Equations

By

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Abstract

This paper is concerned with the problem of existence of solutions to interval differential equations which result from attempts to provide a concide interval solution to some differential equations with inexact data or parameters contained in an interval. An extension of the interval iterative scheme of Moore is used to establish the main results which include those of Sveloslav Merkov's.

1. Introduction

Consider the initial value problem [see 6]

$$x'(t) = ct^2 + x^b(t), x(0) = a \tag{1.1}$$

for real numbers a, b, c such that

$$0 \leq a \leq 0.1, 0.2 \leq b \leq 0.38, 3.3 \leq c \leq 3.6 \tag{1.2}$$

It could be seen that the i.v.p. satisfies the basic conditions for existence and uniqueness of solution, for each value taken by the numbers a, b and c, except for the case $a = 0$. However because of condition (1.2) there would exist infinitely many such unique solutions. This paper considers the existence of solution to some interval differential which result when problems of the form (1.1) & (1.2) are recast in the form of interval differential equations. Markov [4,5] considered a somewhat similar problem, however, his definitions of the basic interval arithmetic operations differ from those in [2,3,6,7,8] which are used here. His definitions do not always guarantee that if f, g are real valued functions and F, G interval valued functions such that $f \in F$ and $g \in G$, then $f.g \in F * G$, where . and $*$ are respectively the real and interval arithmetic operations. Whereas the definitions used here always guarantee this. Therefore the results considered here are more general than those of his.

Definition 1.3

Let $I\!R$ be the set of real numbers, $I I\!R$ the set of all compact intervals on $I\!R$ and

$$\mathcal{D} = \{(t, X) : t \in J = [t_o, t_o + a], X, A \in I I\!R, d(X, A) \leq b, t_o, a, b \in I\!R\}.$$

By an interval differential equation we mean an equation of the form

$$X'(t) = F(t, X(t)), X(t_o) = A, t \in J \tag{1.4}$$

where X is an interval function defined and continuous on $J, X'(t) = \{x'(t) : x(t) \in X(t), x(t_o) \in A\}$ and F an interval function defined on $\mathcal{D}$.

Definition 1.5

By a solution of the interval differential equation (1.4), we mean an interval function X defined and continuous on an interval $J_o \subseteq J$ such that the interval function X' is defined and satisfies.

$$X'(t) = F(t, X(t)), X(t_o) = A.$$

Readers that are not familiar with the fundamentals of interval arithmetic and interval analysis are referred to [1, 2, 3, 6, 7, 8] for details. In the next section we give some results in interval analysis which will be applied in the establishment of the main result.

2. Some Results in Interval Analysis

Lemma 2.1

Let X and Y be intervals in $I\mathbb{R}$, then

$$X \subseteq X + (Y - Y) \tag{2.2}$$

Equality holds if and only if Y is a degenerate interval in $I\mathbb{R}$.

Proof:. Let $X = [\underline{x}, \bar{x}]$ and $Y = [\underline{y}, \bar{y}]$, then

$$\begin{aligned} Y - Y &= [\underline{y} - \bar{y}, \bar{y} - \underline{y}] \\ &= w(Y)[-1, 1] \\ \therefore \quad X + (Y - Y) &= [\underline{x} - w(Y), \bar{x} + w(Y)] \end{aligned} \tag{2.3}$$

since $\underline{x} - w(Y) \leq \underline{x}$ and $\bar{x} \leq \bar{x} + w(Y)$, the result, (2.2) follows. Suppose Y is degenerate in $I\mathbb{R}$, then $w(Y) = 0$, so (2.3) gives

$$X + (Y - Y) = [\underline{x}, \bar{x}] = X.$$

□

Theorem 2.4 [1]. *Let P be an interval operator defined on $I\mathbb{R}$. Suppose that X_o is an interval in the domain of P such that*

$$|m(X_o) - m(P(X_o))| \leq \frac{1}{2}\{w(X_o) + w(P(X_o))\} \tag{2.5}$$

Then the interval iterative scheme defined by

$$X_{n+1} = P(V_n), \quad n \in N, \tag{2.6}$$

where

$$V_n = \bigcap_{k=0}^{n} X_k, \quad k \in N. \tag{2.7}$$

generates a nested sequence of interval functions $X_n, n = 0, 1, 2...$
If for each n, we have

$$|m(V_n) - m(P(V_n))| \leq \frac{1}{2}\{w(V_n) + w(P(V_n))\} \tag{2.8}$$

the sequence $\{X_n\}$ converges to the limit

$$X = \lim_{n\to\infty} X_n = \bigcap_{n=0}^{\infty} X_o \tag{2.9}$$

3. Existence Results

To enable us establish the existence results we need the following.

Theorem 3.1

Let the interval function F in equation (1.4) be continuous on D. Then any solution X of the interval integral equation

$$X(t) + \omega(A)[-1,1] = A + \int_{t_0}^{t} F(s, X(s))ds, t \in J \tag{3.2}$$

such that

$$X(t_0) = A \tag{3.3}$$

is also a solution of the interval differential equation (1.4) and conversely if X is an interval function satisfying (1.4), then X satisfies (3.2).

Proof. Let X be a solution of the integral equation (3.2). Since F is continuous the equation can be differentiated [see 2] to give

$$X'(t) = F(t, X(t)), \quad t \in J.$$

and by (3.2) $X(t_0) = A$.
This implies that X is a solution of (1.4). Conversely let X be an interval function which solves (1.4), then the equation can be integrated [see 2] to obtain

$$\int_{t_0}^{t} X(s)ds = \int_{t_0}^{t} F(s, X(s))ds, \quad t \in J.$$

i.e.

$$X(t) - A = \int_{t_o}^{t} F(s, X(s))ds$$

and this by Lemma 2.1 gives

$$X(t) + w(A)[-1,1] = A + \int_{t_o}^{t} F(s, X(s))ds, t \in J$$

as required. $\square$

Remark 3.4: The result of Theorem 3.1 generalists that of Theorem 1 [4] and Proposition 12 of [5], in the sense that X need not be continuous and if A is degenerate in $I\mathbb{R}$, the integral equation reduces to

$$X(t) = A + \int_{t_o}^{t} F(s, X(s))ds, \quad t \in J$$

as given in the said results.

Theorem 3.5. *Let the interval function F in equation (1.4) be defined and continuous on D. Then there exists an interval function X defined on an interval $J_o \subseteq J$ satisfying the interval differential equation (1.4).*

Proof:. By Theorem 3.1, the solution X of the interval integral equation (3.2) also satisfies the interval differential equation (1.4). Therefore it suffices to find a solution X of equation (3.2). $\square$

Define the operator P on D by

$$P(X(t)) = A + \int_{t_o}^{t} F(s, X(s))ds, \quad t \in J_o \subseteq J \tag{3.6}$$

and let the sequence $X_n, n = 0, 1, 2, ...$ of interval functions be iteratively defined by

$$X_{n+1}(t) = P(X_n(t)), \quad n = 0, 1, 2, ... \tag{3.7}$$

where X_o is chosen such that

$$\underline{X}_o(t) \leq \bar{A} + \int_{t_o}^{-t} F(s, X_o(s))ds \tag{3.8}$$

and

$$\bar{X}_o(t) \geq \underline{A} + \int_{-t_o}^{t} F(s, X_o(s))ds \tag{3.9}$$

for functions $\underline{X}_o, \bar{A}, \int_{-t_o}^{t} F(s, X_o(s))ds$ and $\bar{X}_o, \bar{A}$ and $\bar{\int}_{t_o}^{t} F(x, X_o(s))ds$ which are respectively the lower-end functions and upper-end functions of the interval functions X_o, A and $\int_{t_o}^{t} F(s, X_o(s))ds$. By Theorem 2.4, this sequence is nested if condition (2.5) is satisfied and the sequence converges if the condition (2.8) is also satisfied. Conditions (3.8) and (3.9) ensure that the sequence (3.7) converges as it is clearly seen from (3.8) that

$$\begin{aligned} m(X_o) - \tfrac{1}{2}w(X_o) &\leq M(A) + \tfrac{1}{2}w(A) + m\left(\int_{t_o}^{t} F(s, X_o(s))ds\right) \\ &\quad + \tfrac{1}{2}w\left(\int_{t_o}^{t} F(s, X_o(s))ds\right) \\ &= m(P(X_o)) + \tfrac{1}{2}w(P(X_o)) \end{aligned}$$

i.e.

$$m(X_o) - m(P(X_o)) \leq \frac{1}{2}\{w(X_o) + w(P(X_o))\} \tag{3.10}$$

and also from (3.9) that

$$\begin{aligned} m(X_o) + \tfrac{1}{2}w(X_o) &\geq m(A) - \tfrac{1}{2}w(A) + m\left(\int_{t_o}^{t} F(x, X_o(s))ds\right) \\ &\quad -\tfrac{1}{2}w\left(\int_{t_o}^{t} F(x, X_o(s))ds\right) \\ &= m(P(X_o)) - \tfrac{1}{2}w(P(X_o)) \end{aligned}$$

i.e.

$$-\frac{1}{2}\{w(X_o) + w(P(X_o))\} \leq m(X_o) - m(P(X_o)) \tag{3.11}$$

Combining (3.10) and (3.11) the condition (2.5) is obtained. By the definition (3.7), each term of the sequence $\{X_n\}$ satisfies conditions (3.8) and (3.9) with X_o replaced by X_n and hence condition (2.8) is satisfied accordingly.

The limit (2.9) gives the required solution since

$$\begin{aligned} X(t) &= \lim_{n\to\infty} P\left(\bigcap_{k=0}^{n} X_k\right) \\ &= A + \int_{t_o}^{t} F\left(s, \lim_{n\to\infty}\left(\bigcap_{k=0}^{n} X_k(s)\right)\right) ds \\ &= A + \int_{t_o}^{t} F(s, X(s))ds, \quad \in J_0 \subseteq J \end{aligned}$$

since F is continuous.

Definition 3.12

The interval function $F(t, X)$ is said to be Lipschitz in its domain D if for interval functions $X, Y \in D$

$$d(F(t, X), F(t, Y)) \leq kd(X, Y)$$

for a positive constant k.

Theorem 3.13. *If the function F in (1.4) is Lipschitz on D, then the solution X on the interval J_o is unique.*

Proof:. Suppose the solution X is not unique. Let Y be another solution, then from (3.6) we have

$$X(t) = P(X(t)) = A + \int_{t_o}^{t} F(s, X(s))ds, \quad t \in J_0$$

and

$$Y(t) = P(Y(t)) = A + \int_{t_0}^{t} F(s, X(s))ds, \quad t \in J_0$$

then

$$\begin{aligned} d(X(t), Y(t)) &= d\left(A + \int_{t_o}^{t} F(s, X(s))ds, \quad A + \int_{t_o}^{t} F(s, Y)s))ds\right) \\ &\leq d(A, A) + \int_{t_0}^{t} d(F(s, X(s)), F(s, Y(s)))ds \\ &\leq d(A, A) + k\int_{t_o}^{t} d(X(s), Y(s))ds \end{aligned}$$

which when integrated gives
$d(X(t), Y(t)) \leq d(A, A)\exp(k(t - t_0)) = 0$
$\Rightarrow d(X(t), Y(t)) = 0$
hence $X(t) = Y(t)$.

Definition 3.14

The interval function $F(t, X)$ is said to be w-Lipschitz on D if there exists a positive constant λ such that

$$w(F(t, X(t)) \leq kw(X(t)).$$

Theorem 3.15. *Suppose that the interval function F in (1.4) is w-Lipschitz on D then the solution X is degenerate only if the initial interval A in (1.4) is degenerate.*

Proof:. Let X be a solution of the equation (1.4), then by (3.6)

$$\begin{aligned} X(t) &= A + \int_{t_0}^{t} F(s, X(s))ds, \quad t \in J_0 \\ w(X(t)) &= w(A) + w\left(\int_{t_o}^{t} F(s, X(s))ds\right) \\ &\leq w(A) + \int_{t_0}^{t} w(F(s, X(s))ds \\ &\leq w(A) + \lambda \int_{t_0}^{t} w(X(s))ds \end{aligned}$$

□

upon integration this yields

$$\begin{aligned} w(X(t)) &\leq w(A)\exp(\lambda(t - t_o)) \\ &= 0 \text{ only if } w(A) = 0. \end{aligned}$$

then $w(X(t)) = 0$ and this would imply that X is degenerate.

Remark 3.16

The result in (3.5) and (3.13) generalise those in [4 and 5] which include some conditions on F which are not necessary.

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