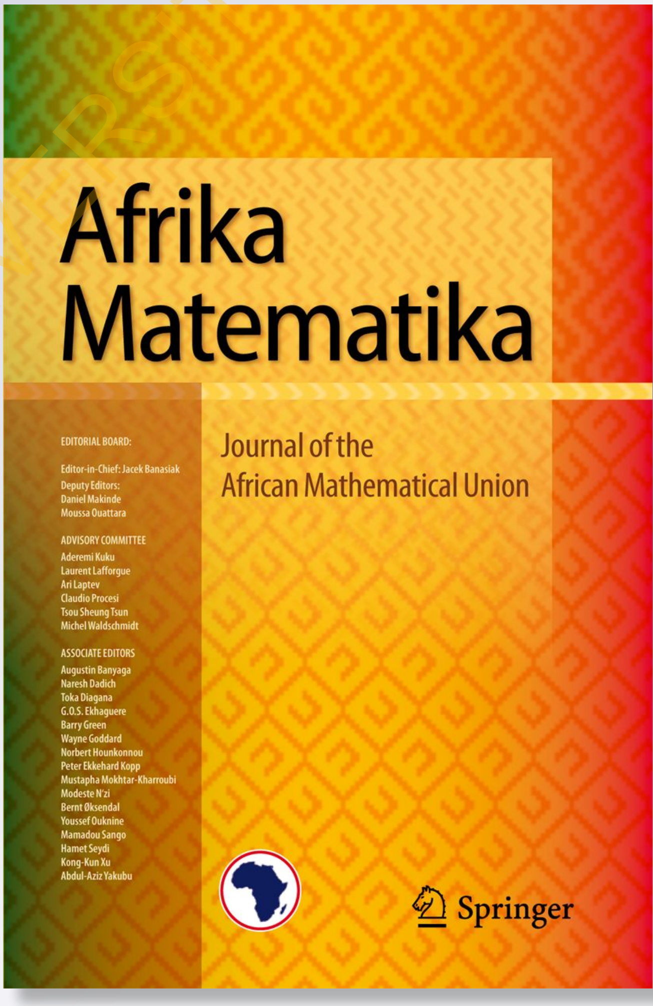
Boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order

A. T. Ademola & P. O. Arawomo

Afrika Matematika

ISSN 1012-9405
Volume 23
Number 2

Afr. Mat. (2012) 23:261-271
DOI 10.1007/s13370-011-0034-x



Springer

Afr. Mat. (2012) 23:261–271
DOI 10.1007/s13370-011-0034-x

Boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order

A. T. Ademola · P. O. Arawomo

Received: 6 October 2010 / Accepted: 10 June 2011 / Published online: 30 June 2011

Abstract In this paper, we use Lyapunov second method. A complete Lyapunov function was constructed and used to obtain criteria for boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order. Our results revise, improve and extend existing results on boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of third order nonlinear differential equations in the literature.

Keywords Third order · Nonlinear differential equation · Uniform ultimate boundedness · Asymptotic behaviour · Complete Lyapunov function

Mathematics Subject Classification (2000) 34C11 · 34A34 · 34D40

1 Introduction

Nonlinear ordinary differential equations of the second, third and higher order have been vastly studied with high degree of generality. In relevant literature, interesting results have been obtained concerning, stability, instability, boundedness, periodicity and asymptotic behaviour of solutions of these nonlinear differential equations. Outstanding authors that have contributed immensely to the qualitative behaviour of solutions include Reissig et al. [17] and Yoshizawa [26,27] which contain general results on the subject matter, Ademola et al. [1–4], Afuwape [5] Chukwu [7], Ezeilo [8–11], Hara [12], Ogundare et al. [13,14], Omeike [15,16], Swick [18] and Tunç et al. [19–25] which contain results on the qualitative behaviour of solutions of third order nonlinear differential equations.

A. T. Ademola (✉) · P. O. Arawomo
Department of Mathematics, University of Ibadan, Ibadan, Nigeria
e-mail: ademola672000@yahoo.com

P. O. Arawomo
e-mail: womopeter@gmail.com

Some of these works were done with the aid of Lyapunov functions which are either incomplete or contain signum functions. These we find unconvincing. However, the problem of boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of nonlinear, ordinary differential equations, in general, and those of third order, in particular, is not still solved for general nonlinearities. Using a complete Lyapunov function, the purpose of this paper therefore is to obtain criteria for boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of the third order nonlinear differential equation

$$\dddot{x} + f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x}) + h(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

or its equivalent system of first order differential equations

$$\dot{x} = y, \dot{y} = z, \dot{z} = p(t, x, y, z) - f(x, y, z)z - g(x, y) - h(x, y, z), \tag{1.2}$$

where $f, h \in C(\mathbb{R}^3, \mathbb{R})$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and the dots denote differentiation with respect to the independent variable t. The derivatives $f_x(x, y, z)$, $f_z(x, y, z)$, $h_x(x, y, z)$, $h_y(x, y, z)$, $h_z(x, y, z)$ and $g_x(x, y)$ exist and are continuous. Moreover, the existence and uniqueness of solutions of (1.1) will be assumed. Motivation for this study comes from the works of Tunç [25], Omeike [16] and Ademola and Arawomo [2] (the generalization of the works in [16] and [25]) where criteria for uniform stability and uniform ultimate boundedness of solutions of (1.1) were proved.

2 Preliminaries

The following lemmas are every important in the proofs of our results. Consider the system of differential equation

$$\dot{X} = F(t, X) \tag{2.1}$$

where X is an $n-$vector. Suppose that $F(t, X)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $\mathbb{R}^n$ an $n-$dimensional Euclidean space.

Lemma 1 [27] *Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+$, $\|X\| \geq \rho$ where $\rho > 0$ may be large which satisfies the following conditions:*

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, *where $a(r)$, $b(r)$ are continuous increasing and $a(r) \to +\infty$ as $r \to \infty$;*
(ii) $\dot{V}_{(2.1)}(t, X) \leq 0$, *for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$.*

Then the solutions of (2.1) are uniformly bounded.

Lemma 2 [27] *If assumption (i) of Lemma 1 holds and $\dot{V}_{(2.1)}(t, X) \leq -c(\|X\|)$, $c(r)$ is continuous and increasing for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then the solutions of (2.1) are uniformly ultimately bounded.*

Lemma 3 [26] *Let Q be an open set in $\mathbb{R}^n$. Consider the system*

$$\dot{X} = H(X) + G(t, X) \tag{2.2}$$

where H is continuous on Q, G is continuous on $\mathbb{R}^+ \times Q$ and for any continuous and bounded function $X(t)$ on $t \in [0, \infty)$

$$\int_0^\infty \|G(t, X)\| dt < \infty.$$

Assume that all solutions of (2.2) *are bounded, and that there exists a nonnegative continuous function* $V(t, X)$ *which satisfies locally, a Lipschitz condition with respect to* X *in* Q *such that*

$$\dot{V}_{(2.2)}(t, X) \leq -W(X),$$

where $W(X)$ *is positive definite with respect to a closed set* Ω *in* Q. *Then all the solutions of* (2.2) *approach, the largest semi invariant set contained in* Ω *of the equation*

$$\dot{X} = H(X)$$

on Ω.

3 Statement of results

We have the following results.

Theorem 4 *Further to the basic assumptions on the functions* f, g, h *and* p *appearing in* (1.2)*, suppose that* a, b, c, a_1, b_1 *and* δ_0 *are positive constants such that:*

(i) $a \leq f(x, y, z) \leq a_1$ *for all* x, y, z;
(ii) $b \leq \dfrac{g(x, y)}{y} \leq b_1$ *for all* x *and* $y \neq 0$;
(iii) $h(0, 0, 0) = 0$, $\delta_0 \leq \dfrac{h(x, 0, 0)}{x}$ *for all* $x \neq 0$;
(iv) $yf_x(x, y, 0) \leq 0$, $g_x(x, y) \leq 0$, $h_x(x, 0, 0) \leq c$ *for all* x, y *and* $c < ab$;
(v) $yf_z(x, y, z) \geq 0$, $h_y(x, y, 0) \geq 0$, $h_z(x, 0, z) \geq 0$ *for all* x, y, z;
(vi) $\int_0^{\infty} |p(t, x, y, z)| dt < \infty$,

then the solutions $x(t)$ *of* (1.2)*, its first and second derivatives are uniformly ultimately bounded.*

Theorem 5 *If all the hypotheses of Theorem* 4 *are satisfied and in addition* $g(0, 0) = 0$, *then every solution* $(x(t), y(t), z(t))$ *of* (1.2) *is uniformly bounded and satisfies*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0. \tag{3.1}$$

Theorem 6 *Suppose that* a, b, c *and* δ_0 *are positive constants such that:*

(i) *hypotheses (iii)–(vi) of Theorem* 4 *hold;*
(ii) $f(x, y, z) \geq a$ *for all* x, y, z;
(iii) $\dfrac{g(x, y)}{y} \geq b$ *for all* $x, y \neq 0$,

then any solution $(x(t), y(t), z(t))$ *of the system* (1.2) *with the initial conditions*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0 \tag{3.2}$$

satisfies

$$|x(t)| \leq D_0, \quad |y(t)| \leq D_0, \quad |z(t)| \leq D_0 \tag{3.3}$$

for all $t \geq 0$ *where the constant* $D_0 > 0$ *depends on* a, b, c, δ_0 *as well as on* t_0, x_0, y_0, z_0 *and the function* p *appearing in* (1.2).

Now if $p(t, x, y, z)$ in (1.2) is replaced by $p(t)$, $p \in C(\mathbb{R}^+, \mathbb{R})$, we have

$$\dot{x} = y, \dot{y} = z, \dot{z} = p(t) - f(x, y, z)z - g(x, y) - h(x, y, z). \tag{3.4}$$

Corollary 7 *If assumptions (i)–(v) of Theorem 4 hold and* $\int_0^\infty |p(t)|dt < \infty$, *then the solution* $(x(t), y(t), z(t))$ *of* (3.4) *is uniformly ultimately bounded.*

Corollary 8 *If the hypotheses of Corollary 7 hold, then every solution* $(x(t), y(t), z(t))$ *of* (3.4) *is uniformly bounded and satisfies* (3.1).

Corollary 9 *Assuming assumptions (iii)–(v) of Theorem 4, (ii), (iii) of Theorem 5 are satisfied and in addition* $\int_0^\infty |p(t)|dt < \infty$, *then any solution* $(x(t), y(t), z(t))$ *of the system* (3.4) *with the initial conditions* (3.2) *satisfies* (3.3) *for all* $t \geq 0$.

Remark 10 (i) The situation when $f(x, y, z) \equiv f(z), g(x, y) \equiv g(y)$ and $h(x, y, z) \equiv h(x)$, system (1.2) reduces to that discussed by Ademola et al. [3,4] and Tunç [25]. The assumption on the function $p(t, x, y, z)$ in [3] Theorem 3.5 for the solutions $x(t)$ its first and second derivatives to be uniformly ultimately bounded and the hypothesis that $\int_0^x h(\xi)d\xi \to +\infty$ as $|x| \to \infty$ assumed in [4] Theorem 2.6 for solutions to converge to zero as $t \to \infty$ are not required here. Furthermore, the basic assumptions that the functions f, g and h are not differentiable for the solutions to converge to zero as $t \to \infty$ and ultimately bounded for all $t \geq 0$ in [25] are also not applicable here.

(ii) Whenever $f(x, y, z) \equiv a$ $a > 0$ is a constant, $g(x, y) \equiv by$ or $yg(x)$ and $h(x, y, z) \equiv c$ $c > 0$ is a constant or $h(x)$, system (1.2) reduces to that studied by Ezeilo [8–11] and Swick [18]. Thus some of our hypotheses and conclusion coincide with those discussed in [8–11] and [18].

(iii) Also, when $f(x, y, z) \equiv \psi(x, y)$ and $h(x, y, z) \equiv 0$ or cx $c > 0$ is a constant or $h(x)$, system (1.2) specializes to that discussed by Omeike [16] and Tunç [22] and the references cited theirin.

(iv) Finally, our result in Theorem 4 revises and improves the most recent results given by Ademola and Arawomo [2], generalizes and extends the results of Omeike [15], Tunç [21] and all results on boundedness and asymptotic behaviour of solutions of nonlinear third order differential equations discussed by Reissig et al. [17].

The main tool employed in the proofs of our results is the continuously differentiable function $V \equiv V(t, x(t), y(t), z(t))$ defined as

$$V = e^{-P_*(t)}U, \tag{3.5a}$$

where

$$P_*(t) = \int_0^t |p(\mu, x, y, z)|\, d\mu \tag{3.5b}$$

and the function $U \equiv U(x(t), y(t), z(t))$ is given by

$$\begin{aligned} 2U &= 2(\alpha + a)\int_0^x h(\xi, 0, 0)\, d\xi + 4\int_0^y g(x, \tau)d\tau + 4yh(x, 0, 0) \\ &\quad + 2(\alpha + a)yz + 2z^2 + 2(\alpha + a)\int_0^y \tau f(x, \tau, 0)\, d\tau + \beta y^2 + b\beta x^2 \\ &\quad + 2a\beta xy + 2\beta xz \end{aligned} \tag{3.5c}$$

where α and β are positive fixed constants satisfying

$$b^{-1}c < \alpha < a \tag{3.5d}$$

and

$$0 < \beta < \min\left\{(ab-c)a^{-1}, (ab-c)\delta_0\lambda_0^{-1}, \frac{1}{2}(a-\alpha)\delta_0\lambda_1^{-1}\right\} \tag{3.5e}$$

where $\lambda_0 := (1+a)\delta_0 + (\frac{g(x,y)}{y} - b)^2$, $(1+a)\delta_0 \neq (\frac{g(x,y)}{y} - b)^2$, $\frac{g(x,y)}{y} \neq b$, $y \neq 0$, $\lambda_1 := \delta_0 + (f(x,y,z)-a)^2$, $\delta_0 \neq (f(x,y,z)-a)^2$ and $f(x,y,z) \neq a$. We have the following results.

Remark 11 If $t = 0$ in (3.5b), (3.5a) reduces to that employed in [2].

Lemma 12 *If all the hypotheses of Theorem 4 are satisfied and $P_0 > 0$ is a constant, then for the function V defined in (3.5a) there exist positive constants $D_1 = D_1(a, b, c, \alpha, \beta, \delta_0, P_0)$ and $D_2 = D_2(a, b, c, a_1, b_1, \alpha, \beta)$ such that*

$$D_1(x^2(t) + y^2(t) + z^2(t)) \leq V(t, x, y, z) \leq D_2(x^2(t) + y^2(t) + z^2(t)) \tag{3.6a}$$

and

$$V(t, x(t), y(t), z(t)) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{3.6b}$$

Furthermore, there exist positive constants $D_3 = D_3(a, b, c, \alpha, \beta, \delta_0)$, $D_4 = D_4(a, b, c, \alpha, \beta, \delta_0)$ and $D_5 = D_5(a, \alpha, \beta)$ such that a solution of (1.2)

$$\dot{V} \leq -e^{-P_*(t)}\left\{D_3(x^2+y^2+z^2) + \left[D_4(x^2+y^2+z^2) - D_5(|x|+|y|+|z|)\right]\right.$$
$$\left.\times|p(t,x,y,z)|\right\} \tag{3.6c}$$

Proof Since $h(0,0,0) = 0$, (3.5c) can be recast in the form

$$2U = 2b^{-1}\int_0^x [(\alpha+a)b - 2h_\xi(\xi,0,0)]h(\xi,0,0)\,d\xi$$
$$+4\int_0^y \left(\frac{g(x,\tau)}{\tau} - b\right)\tau\,d\tau + 2b^{-1}\Big(h(x,0,0) + by\Big)^2$$
$$+\Big(\beta x + ay + z\Big)^2 + \beta(b-\beta)x^2 + (\alpha y + z)^2$$
$$+2\int_0^y [(\alpha+a)f(x,\tau,0) - (\alpha^2+a^2)]\tau\,d\tau + \beta y^2.$$

Using the hypotheses of Theorem 4, we have the following estimate

$$U \geq \frac{1}{2}\Big[[(\alpha+a)b - 2c]b^{-1}\delta_0 + \beta(b-\beta)\Big]x^2 + \frac{1}{2}(\alpha y + z)^2$$
$$+\frac{1}{2}[\alpha(a-\alpha)+\beta]y^2 + b^{-1}(\delta_0 x + by)^2 + \frac{1}{2}\Big(\beta x + ay + z\Big)^2. \tag{3.7}$$

In view of estimates (3.5d) and (3.5e), the quadratic in the right hand side of (3.7) is positive definite, hence there exists a constant $\lambda_2 = \lambda_2(a, b, c, \alpha, \beta, \delta_0) > 0$ such that

$$U \geq \lambda_2(x^2 + y^2 + z^2) \tag{3.8}$$

for all x, y and z. Equation (3.5b) and hypothesis (vi) of Theorem 4, imply the existence of a constant $P_0 > 0$ such that

$$0 \leq P_*(t) \leq P_0 \tag{3.9}$$

for all $t \geq 0$. From (3.9) and estimate (3.8), Eq. (3.5a) yields

$$V \geq \delta_1(x^2 + y^2 + z^2) \tag{3.10a}$$

for all $t \geq 0$, x, y and z, where $\delta_1 := \lambda_2 e^{-P_0}$. Furthermore, from estimate (3.10a), we have

$$V(t, x, y, z) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{3.10b}$$

Next, since $h(0, 0, 0) = 0$, $h_x(x, 0, 0) \leq c$ for all x implies that $h(x, 0, 0) \leq cx$ for all $x \neq 0$. This inequality, the hypotheses of Theorem 4 and the fact that $2qr \leq q^2 + r^2$, Eq. (3.5c) becomes

$$U \leq \delta_2(x^2 + y^2 + z^2) \tag{3.11a}$$

for all x, y and z, where

$$\delta_2 := \frac{1}{2}\max\{\alpha + a + 2c + \beta(1 + a + b), (\alpha + a)(1 + a_1) \\ + \beta(1 + a) + 2(b_1 + c), 2 + \alpha + \beta + a\}.$$

Using estimates (3.11a) and (3.9), Eq. (3.5a) yields

$$V \leq \delta_2(x^2 + y^2 + z^2) \tag{3.11b}$$

for all x, y and z. Moreover, the derivative of V with respect to t along a solution $(x(t), y(t), z(t))$ of (1.2) is

$$\dot{V}_{(1.2)} = -e^{-P_*(t)}\Big[U|p(t, x, y, z)| - \dot{U}_{(1.2)}\Big] \tag{3.12a}$$

where U is defined in (3.5c) and after simplifying

$$\dot{U}_{(1.2)} := -\beta\frac{h(x, y, z)}{x}x^2 - \left[(\alpha + a)\frac{g(x, y)}{y} - 2h_x(x, 0, 0)\right]y^2 \\ -[2f(x, y, z) - (\alpha + a)]z^2 + W_1 - W_2 - \beta\left(\frac{g(x, y)}{y} - b\right)xy \\ -\beta[f(x, y, z) - a]xz + [\beta x + (\alpha + a)y + 2z]p(t, x, y, z) \tag{3.12b}$$

where

$$W_1 := a\beta y^2 + 2\beta yz + 2y\int_0^y g_x(x, \tau)d\tau + (\alpha + a)y\int_0^y \tau f_x(x, \tau, 0)\,d\tau;$$

$$W_2 := (\alpha + a)yz[f(x, y, z) - f(x, y, 0)] + [(\alpha + a)y + 2z][h(x, y, z) - h(x, 0, 0)].$$

Now, since $g_x(x, y) \leq 0$, $yf_x(x, y, 0) \leq 0$ for all x, y and the fact that $2yz \leq x^2 + y^2$, we have

$$W_1 \leq \beta[(1 + a)y^2 + z^2] \tag{3.13a}$$

for all y and z. Also, applying the mean value theorem and hypothesis (v) of Theorem 4 we obtain

$$W_2 := (\alpha + a)yz^2 f_z(x, y, \theta_1 z) + (\alpha + a)y^2 h_y(x, \theta_2 y, 0) + 2z^2 h_z(x, 0, \theta_3 z) \geq 0 \quad (3.13b)$$

for all x, y, z where $0 \leq \theta_i \leq 1 (i = 1, 2, 3)$ and $W_2 = 0$ if $y = 0 = z$. Using estimates (3.13) and the hypotheses of Theorem 4, we obtain

$$\begin{aligned}\dot{U}_{(1.2)} \leq{}& \lambda_3(|x| + |y| + |z|)|p(t, x, y, z)| - \frac{1}{2}\beta\delta_0 x^2 - (\alpha b - c)y^2 - \frac{1}{2}(a - \alpha)z^2 \\ & - \left[x + 2\delta_0^{-1}\left(\frac{g(x, y)}{y} - b\right)y\right]^2 - \left[x + 2\delta_0^{-1}\left(f(x, y, z) - a\right)\right]^2 \\ & - \left\{ab - c - \beta\left[1 + a + \delta_0^{-1}\left(\frac{g(x, y)}{y} - b\right)^2\right]\right\}y^2 \\ & - \left\{\frac{1}{2}(a - \alpha) - \beta\left[1 + \delta_0^{-1}\left(f(x, y, z) - a\right)^2\right]\right\}z^2,\end{aligned}$$

where $\lambda_3 := \max\{\beta, \alpha + a, 2\}$. Applying estimates (3.5d) and (3.5e), this inequality yields

$$\dot{U}_{(1.2)} \leq -\lambda_4(x^2 + y^2 + z^2) + \lambda_3(|x| + |y| + |z|)|p(t, x, y, z)| \quad (3.14)$$

for all x, y, z where $\lambda_4 := \min\{\frac{1}{2}\beta\delta_0, \alpha b - c, \frac{1}{2}(a - \alpha)\}$. Using estimates (3.8) and (3.14) in (3.12a), we obtain

$$\begin{aligned}\dot{V}_{(1.2)} \leq -e^{-P_*(t)}\Big\{&\lambda_4(x^2 + y^2 + z^2) + \lambda_2(x^2 + y^2 + z^2)|p(t, x, y, z)| \\ & -\lambda_3(|x| + |y| + |z|)|p(t, x, y, z)|\Big\} \quad (3.15)\end{aligned}$$

for all $t \geq 0$, x, y and z. This completes the proof of the lemma. □

Proof of Theorem 4 Let $(x(t), y(t), z(t))$ be any solution of (1.2). Now, since $(|x| + |y| + |z|)^2 \leq 3(x^2 + y^2 + z^2)$, choosing $(x^2 + y^2 + z^2)^{1/2} \geq 3^{1/2}\lambda_2^{-1}\lambda_3$ estimate (3.15) becomes

$$\dot{V}_{(1.2)} \leq -\lambda_4 e^{-P_*(t)}(x^2 + y^2 + z^2)$$

for all $t \geq 0$, x, y and z. In view of (3.5b) and hypothesis (vi) of the theorem, we have

$$\dot{V}_{(1.2)} \leq -\delta_3(x^2 + y^2 + z^2) \quad (3.16)$$

for all $t \geq 0$, x, y and z where $\delta_3 := \lambda_4 e^{-P_*(\infty)} > 0$. From estimates (3.10), (3.11b) and (3.16), the hypotheses of Lemma 2 hold; hence, by Lemma 2 the solutions of (1.2) are uniformly ultimately bounded. □

Proof of Theorem 5 The proof of this theorem depends on the function defined in (3.5). By Theorem 4 the solutions of (1.2) are bounded. Now, let $W(X) \equiv \delta_3(x^2 + y^2 + z^2)$, obviously $W(X) \geq 0$ for all $X \in \mathbb{R}^3$. Consider the set

$$\Omega = \{X = (x, y, x) \in \mathbb{R}^3 | W(X) = 0\}.$$

Since $W(X)$ is continuous, the set Ω is closed and $W(X)$ is positive definite with respect to Ω and

$$\dot{V}_{(1.2)} \leq -W(X) \quad (3.17)$$

for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^3$. Moreover the system (1.2) can be recast in the form (2.2) where $X = (x, y, z)^{\mathrm{T}}$, $F(X) = (y, z, -f(x, y, z)z - g(x, y) - h(x, y, z))^{\mathrm{T}}$ and $G(t, X) = (0, 0, p(t, x, y, z))^{\mathrm{T}}$. From the continuity and boundedness of the functions f, g and h the function $F(X)$ is bounded. Thus, by (3.17) all solutions of (2.2) approach the largest semi invariant set contained in Ω of the equation

$$\dot{X} = F(X) \tag{3.18}$$

on Ω. Since $W(X) = 0$ on Ω, we have $x = y = z = 0$, and by hypotheses of Theorem 5 $g(0, 0) = h(0, 0, 0) = 0$, it follows from (3.18) that

$$X = (K_1, K_2, K_3)^{\mathrm{T}}$$

where K_1, K_2 and K_3 are constants. For X to remain in Ω, $K_1 = K_2 = K_3 = 0$. Hence, by Lemma 3 equations (3.1) follows, this completes the proof of the theorem. □

Proof of Theorem 6 Let $(x(t), y(t), z(t))$ be any solution of (1.2). We shall use the method introduced in [6]. Now, since $\lambda_2(x^2 + y^2 + z^2)|p(t, x, y, z)| \geq 0$, $\lambda_4(x^2 + y^2 + z^2) \geq 0$, $|x| \leq 1 + x^2$, $|y| \leq 1 + y^2$ and $|z| \leq 1 + z^2$, estimate (3.15) becomes

$$\dot{V}_{(1.2)} \leq \lambda_3 e^{-P_*(t)}(3 + x^2 + y^2 + z^2)|p(t, x, y, z)|,$$

for all $t \geq 0$, x, y and z. From estimates (3.9) and (3.10a), this inequality becomes

$$\dot{V}_{(1.2)} - \lambda_5|p(t, x, y, z)|V \leq 3\lambda_3|p(t, x, y, z)|$$

where $\lambda_5 := e^{-P_0}\delta_1^{-1}\lambda_3$. Solving this first order differential inequality using the integrating factor $\exp[-\lambda_5 P_*(t)]$ and estimate (3.9), we obtain

$$V \leq \lambda_6 \tag{3.19}$$

where $\lambda_6 := [V(t_0, x_0, y_0, z_0) + 3\lambda_3 P_0]e^{\lambda_5 P_0}$. From (3.10a) and (3.19) estimate (3.3) follows with $D_0 \equiv \delta_1^{-1}\lambda_6$ for all $t \geq 0$. This completes the proof of the theorem. □

Example 1 As a special case of equation (1.1), consider the following third order nonlinear ordinary differential equation

$$\dddot{x} + 4\ddot{x} + \frac{\ddot{x}}{1 + |x\dot{x}| + \exp\left(1/(1 + |\dot{x}\ddot{x}|)\right)} + 3\dot{x} + \frac{\dot{x}}{1 + |x\dot{x}|} + 5x + \frac{x}{1 + \exp\left(1/(1 + |\dot{x}| + |\ddot{x}|)\right)} = \frac{1}{1 + t^2 + x^2 + \dot{x}^2 + \ddot{x}^2} \tag{3.20}$$

(3.20) is equivalent to the systems of first order ordinary differential equations

$$\begin{aligned} \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = & \tfrac{1}{1+t^2+x^2+y^2+z^2} \\ & - \left[4 + 1/\left(1 + |xy| + \exp(1/(1 + |yz|))\right)\right]z \\ & - \left[3 + 1/(1 + |xy|)\right]y - \left[5 + 1/(1 + \exp(1/(1 + |y| + |z|)))\right]x. \end{aligned} \tag{3.21}$$

Comparing (1.2) and (3.21), we have:

(a) the function $f(x, y, z)$, is

$$4 + \frac{1}{1 + |xy| + \exp(1/(1 + |yz|))}. \tag{3.22}$$

(i) Since

$$0 \leq \frac{1}{1 + |xy| + \exp(1/(1 + |yz|))} \leq 1 \quad \forall x, y, z$$

it follows that

$$4 \leq f(x, y, z) \leq 5$$

for all x, y and z where $a = 4 > 0$ and $a_1 = 5 > 0$.

(ii) From (3.22) we obtain

$$yf_x(x, y, z) = \frac{-y^2}{[1 + |xy| + \exp(1/(1 + |yz|))]^2} \leq 0$$

for all x, y and z.

(iii) Similarly

$$yf_z(x, y, z) = \frac{y^2 \exp(1/(1 + |yz|))}{\left[(1 + |yz|)[1 + |xy| + \exp(1/(1 + |yz|))]\right]^2} \geq 0$$

for all x, y and z.

(b) The function $g(x, y)$, is defined as

$$3y + \frac{y}{1 + |xy|} \tag{3.23}$$

(i) Also, since

$$0 \leq \frac{1}{1 + |xy|} \leq 1$$

for all x and y, it follows that

$$3 \leq \frac{g(x, y)}{y} \leq 4$$

for all x and $y \neq 0$, where $b = 3 > 0$ and $b_1 = 4 > 0$ and

(ii) By (3.23)

$$g_x(x, y) = \frac{-y^2}{1 + |xy|} \leq 0$$

for all x and y.

(c) The function $h(x, y, z)$ is also defined as

$$5x + \frac{x}{1 + \exp(1/(1 + |y| + |z|))} \tag{3.24}$$

(i) Since

$$\frac{1}{1+\exp(1/(1+|y|+|z|))} \geq 0$$

for all y and z, it follows that

$$\frac{h(x,y,z)}{x} \geq 5$$

for all $x \neq 0$, y and z, where $\delta_0 = 5 > 0$.

(ii) Moreover, from (3.24), we obtain

$$h_x(x,y,z) \leq 6$$

for all x, y, z where $c = 6 > 0$ and $ab > c$ implies that $2 > 1$.

(iii) Furthermore, in view of (3.24), we have

$$h_y(x,y,z) = \frac{|x|\exp(1/(1+|y|+|z|))}{\left[(1+|y|+|z|)[\exp(1/(1+|y|+|z|))]\right]^2} \geq 0$$

for all x, y and z.

(iv) Simiarly

$$h_z(x,y,z) = \frac{|x|\exp(1/(1+|y|+|z|))}{\left[(1+|y|+|z|)[\exp(1/(1+|y|+|z|))]\right]^2} \geq 0$$

for all x, y and z.

(d) Finally, for the function $p(t,x,y,z)$, we have

$$p(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2}.$$

It is not difficult to show that

$$\int_0^\infty \left|\frac{1}{1+t^2+x^2+y^2+z^2}\right| dt < \infty.$$

Hence, all the assumptions of the theorems are satisfied and the conclusions follow.

Acknowledgments The authors would like to thank the referees for their several helpful suggestions.

References

1. Ademola, A.T., Arawomo, P.O.: Boundedness and stability of solutions of some nonlinear differential equations of the third order. Pacif. J. Sci. Technol. **10**(2), 187–193 (2009)
2. Ademola, A.T, Arawomo, P.O.: Stability and ultimate boundedness of solutions to certain third order differential equations. Appl. Math. E-Notes **10**, 61–69 (2010)
3. Ademola, A.T., Ogundiran, M.O., Arawomo, P.O., Adesina, O.A.: Boundedness results for a certain third order nonlinear differential equations. Appl. Math. Comput. **216**, 3044–3049 (2010)
4. Ademola, A.T., Ogundiran, M.O., Arawomo, P.O., Adesina, O.A.: Stability results for the solutions of a certain third order nonlinear differential equation. Math. Sci. Res. J. **12**(6), 124–134 (2008)

5. Afuwape, A.U: Uniform ultimate boundedness results for some third order nonlinear differential equations. Int. Centre Theor. Phys. **IC/90/405/**, 1–14 (1990)
6. Antoisewicz, H.A.: On nonlinear differential equations of the second order with integrable forcing term. J. Lond. Math. Soc. **30**, 64–67 (1955)
7. Chukwu, E.N.: On boundedness of solutions of third order differential equations. Ann. Mat. Pura. Appl. **104**(4), 123–149 (1975)
8. Ezeilo, J.O.C.: A boundedness theorem for a certain third order differential equation. Proc. Lond. Math. Soc. **13**(3), 99–124 (1963)
9. Ezeilo, J.O.C.: An elementary proof of a boundedness theorem for a certain third order differential equation. J. Lond. Math. Soc. **38**, 11–16 (1963)
10. Ezeilo, J.O.C., Tejumola, H.O.: Boundedness theorems for certain third order differential equations, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **55**(10), 194–201 (1973)
11. Ezeilo, J.O.C.: Further results for the solutions of a third order differential equation. Proc. Camb. Phil. Soc. **59**, 111–116 (1963)
12. Hara, T.: On the uniform ultimate boundedness of solutions of certain third order differential equations. J. Math. Anal. Appl. **80**(2), 533–544 (1981)
13. Ogundare, B.S.: Boundedness and stability properties of solutions to certain third order nonlinear nonautonomous differential equations. Int. J. Pure Appl. Math. **38**(3), 309–323 (2007)
14. Ogundare, B.S., Okecha, G.E.: On the boundedness and the stability of solution to third order non-linear differential equations. Ann. Differ. Equ. **24**(1), 1–8 (2008)
15. Omeike, M.O.: New result in the ultimate boundedness of solutions of a third order nonlinear ordinary differential equation. J. Inequal. Pure Appl. Math. **9**(1):8 (Art. 15, 2008)
16. Omeike, M.O.: New results on the asymptotic behaviour of a third order nonlinear differential equations. Differ. Equ. Appl. **2008**, 1–13 (2008)
17. Reissig, R., Sansone, G., Conti, R.: Nonlinear Differential Equations of Higher Order. Noordhoff International Publishing, Leyeden (1974)
18. Swick, K.E.: Boundedness and stability for a nonlinear third order differential equation. Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **56**(20), 859–865 (1974)
19. Tunç, C., Ates, M.: Boundedness and stability of solutions to a kind of nonlinear third order differential equations. J. Appl. Funct. Anal. **5**(3), 242–250 (2010)
20. Tunç, C., Ergören, H.: On boundedness of a certain non-linear differential equation of third order. J. Comput. Anal. Appl. **12**(3), 687–694 (2010)
21. Tunç, C.: Boundedness of solutions of a third order nonlinear differential equation. J. Inequal. Pure Appl. Math. **6**(1), 1–6 (2005)
22. Tunç, C.: On the asymptotic behaviour of solutions of certain third order nonlinear differential equations. J. Appl. Math. Stoch. Anal. **1**, 29–35 (2005)
23. Tunç, C.: Some new results on the boundedness of solutions of a certain nonlinear differential equation of third order. Int. J. Nonlinear Sci. **7**(2), 246–256 (2009)
24. Tunç, C.: The boundedness of solutions to nonlinear third order differential equations. Nonlinear Dyn. Syst. Theory **10**(1), 97–102 (2010)
25. Tunç, E.: On the convergence of solutions of certain third order differential equations. Discret. Dyn. Nat. Soc. **2009**, 1–12 (2009)
26. Yoshizawa, T.: Asymptotic behaviour of solutions of a system of differential equations. Contributions Differ. Equ. **1**, 371–387 (1963)
27. Yoshizawa, T.: Stability theory by Liapunov's second method. The Mathematical Society of Japan, Japan (1966)