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Generalization of Some Qualitative Behaviour of Solutions of Third Order Nonlinear Differential Equations

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Abstract

Criteria for uniform asymptotic stability, boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of the most general third order nonlinear differential equations with the restoring nonlinear terms depending explicitly on the independent real variable t are established. The construction a complete Lyapunov function, Lyapunov's second method, the technique introduced by Antoisewicz [9] and the limit point of Yoshizawa [29] are used to obtain the results. The most recent results of Ademola and Arawomo [1, 2, 3, 4] and results on third order nonlinear differential equations which have been discussed in [18] are particular cases of our results.

Keywords: Third order nonlinear differential equations; Uniform asymptotic stability; Boundedness; Asymptotic behaviour of solutions

1 Introduction

The theory of differential equations of higher order have been recognized to be invaluable tools in the modelling of many phenomena in various fields of science and engineering. In particular, stability, boundedness and asymptotic behaviour of solutions of nonlinear third order differential equations have in the past and also recently been researched. See for instance Reissig *et al,* [18], Rouche *et al,* [19], Yoshizawa [29, 30] and the papers of Ademola and Arawomo [1, 2, 3, 4], Ademola *et al,* [5]-[7], Afuwape and Adesina [8], Chukwu [10], Ezeilo [11]-[13], Ezeilo and Tejumola [14], Mehri and Shadman [15], Ogundare [16], Qian [17], Swick [20, 21], Tejumola [22, 23], Tunç [24, 25], Yamamoto [26]-[28]. These works were done with the aid of Lyapunov functions except in [7] where frequency domain approach was used.

However, the problem of uniform asymptotic stability, boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions where the restoring nonlinear terms depend explicitly on the independent real variable t remain unresolved. The aim of this article therefore is to establish criteria for uniform asymptotic stability, boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of the third order nonlinear differential equation

$$\dddot{x} + f(t, x, \dot{x}, \ddot{x})\ddot{x} + g(t, x, \dot{x}) + h(t, x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}) \qquad (1.1)$$

or its equivalent system

$$\dot{x} = y, \dot{y} = z, \dot{z} = p(t, x, y, z) - f(t, x, y, z)z - g(t, x, y) - h(t, x, y, z), \qquad (1.2)$$

where $f, h, p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. It is assumed that the functions f, g and p depend on the argument shown and the derivatives: $\partial f(t, x, y, z)/\partial t = f_t(t, x, y, z)$, $\partial f(t, x, y, z)/\partial x = f_x(t, x, y, z)$, $\partial f(t, x, y, z)/\partial y = f_y(t, x, y, z)$, $\partial f(t, x, y, z)/\partial z = f_z(t, x, y, z)$, $\partial g(t, x, y)/\partial t = g_t(t, x, y)$, $\partial g(t, x, y)/\partial x = g_x(t, x, y)$, $\partial h(t, x, y, z)/\partial t = h_t(t, x, y, z)$, $\partial h(t, x, y, z)/\partial x = h_x(t, x, y, z)$, $\partial h(t, x, y, z)/\partial y = h_y(t, x, y, z)$, $\partial h(t, x, y, z)/\partial z = h_z(t, x, y, z)$, $\partial^2 h(t, x, y, z)/\partial t \partial x = h_{tx}(t, x, y, z)$ exist and are continuous for all t, x, y and z. As usual, condition for uniqueness of solutions will be assumed and the dots as elsewhere stand for differentiation with respect to real variable t. Motivation for this study, comes from the works of Yamamoto [26]-[28] and the recent works of Ademola and Arawomo [1, 2, 3, 4], where results on uniform stability, uniform ultimate boundedness and asymptotic behaviour of solutions were proved.

2 Some Preliminaries

Definition 1 *(i) A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, strictly increasing with $\phi(0) = 0$ is said to be a function of class $\mathbb{K}$ for such function, we shall write $\phi \in \mathbb{K}$.*

(ii) If in addition to (i) $\phi(r) \to +\infty$ as $r \to \infty$, ϕ is said to be a function of class $\mathbb{K}^*$.

The following lemmas are very important in the proof of the main results. Consider the system of differential equations

$$\frac{dX}{dt} = F(t, X), \tag{2.1}$$

where $X = X(t)$ is an $n-$vector. Suppose that $F(t, X)$ is continuous in (t, X) on $\mathbb{R}^+ \times D$, D is a connected open set in $\mathbb{R}^n$. Let C be the class of solutions of (2.1) which remain in D.

Lemma 1 [30] Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+, \|X\| < H$ which satisfies the following conditions:

(i) $V(t, 0) \equiv 0$;

(ii) $a(\|X\|) \le V(t, X) \le b(\|X\|)$, $a, b \in \mathbb{K}$;

(iii) $\dot{V}_{(2.1)}(t, X) \le -c(\|X\|)$ for all $(t, X) \in \mathbb{R}^+ \times D$, where $c \in \mathbb{K}$.

Then the trivial solution $X(t) \equiv 0$ of (2.1) is uniformly asymptotically stable.

Now consider the system (2.1) and suppose that $F(t, X)$ is defined and continuous on $\mathbb{R}^+ \times \mathbb{R}^n$, we have the following lemma.

Lemma 2 [30] Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+, \|X\| \ge \rho$ where ρ may be large, which satisfies the following conditions:

(i) $a(\|X\|) \le V(t, X) \le b(\|X\|)$, $a \in \mathbb{K}^*$, $b \in \mathbb{K}$;

(ii) $\dot{V}_{(2.1)} \le -c(\|X\|)$ for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$, where $c \in \mathbb{K}$.

Then the solutions of (2.1) are uniformly ultimately bounded.

Next, consider a system of differential equations

$$\frac{dX}{dt} = F(t, X) + G(t, X). \tag{2.2}$$

Let Q be an open set in $\mathbb{R}^n$ and suppose that F, G are continuous on $\mathbb{R}^+ \times Q$. Moreover, suppose that X is continuous and bounded on $\mathbb{R}^+$, that is, for a compact set $Q^* \subset Q$, $X \in Q^*$ for all $t \in \mathbb{R}^+$, then we have

$$\int_0^\infty \|G(t, X(t))\| dt < \infty. \tag{2.3}$$

Lemma 3 Suppose that $F(t, X)$ of the system (2.2) is bounded for all t when X belongs to an arbitrary compact set in Q. Moreover, suppose that there exists a nonnegative Lyapunov function $V(t, X)$ such that

$$\dot{V}_{(2.2)}(t, X) \leq -W(X), \tag{2.4}$$

where $W(X)$ is positive definite with respect to a closed set Ω in the space Q. Then every bounded solution of (2.2) approaches Ω as $t \to \infty$.

3 Statement of Results

The main results in this paper are the following:

Theorem 1 *In addition to the basic assumptions on the functions f, g, h and p appearing in* (1.2)*, suppose that $a, a_1, b, b_0, b_1, c, c_0, \delta, \delta_0$ are positive constants and for all $t \geq 0$:*

(i) $a \leq f(t, x, y, z) \leq a_1$ *for all* x, y *and* z;

(ii) $b \leq g(t, x, y)/y \leq b_1,\ g_t(t, x, y)/y \leq b_0$ *for all* x *and* $y \neq 0$;

(iii) $h(t, 0, 0, 0) = 0,\ \delta_0 \leq h(t, x, y, z)/x \leq c,\ h_t(t, x, 0, 0)/x \leq \delta$ *for all* $x \neq 0, y, z$ *and* $c < ab$;

(iv) $h_t(t, 0, 0, 0) = 0,\ c_0 \leq h_{tx}(t, x, 0, 0)$ *for all* x;

(v) $yf_z(t, x, y, z) \geq 0,\ h_y(t, x, y, 0) \geq 0,\ h_z(t, x, 0, z) \geq 0$ *for all* x, y *and* z;

(vi) $f_t(t, x, y, 0) \leq 0,\ g_x(t, x, y) \leq 0,\ yf_x(t, x, y, 0) \leq 0$ *for all* x *and* y;

(vii) $\int_0^\infty |p(t, x, y, z)|dt < \infty$.

Then the solution $(x(t), y(t), z(t))$ of (1.2) *is uniformly ultimately bounded.*

Theorem 2 *If in addition to the hypotheses of Theorem 1, $g(t, 0, 0) = p(t, 0, 0, 0) = 0$, then the solution $(x(t), y(t), z(t))$ of* (1.2) *is uniformly bounded and satisfy*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0. \tag{3.1}$$

Theorem 3 *Under the hypotheses of Theorem 1, any solution $(x(t), y(t), z(t))$ of* (1.2) *with initial condition*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \tag{3.2}$$

for $t_0 = 0$ *satisfies*

$$|x(t)| \leq D, \quad |y(t)| \leq D, \quad |z(t)| \leq D, \tag{3.3}$$

for all $t \geq 0$*, where the constant* $D > 0$ *depends on* a, b, c, c_0, δ_0 *as well as on* $t_0 = 0, x_0, y_0, z_0$ *and on the forcing term* p *appearing in* (1.2).

If $p(t, x, y, z) \equiv 0$, Eq. (1.2) reduces to a particular case

$$\dot{x} = y, \dot{y} = z, \dot{z} = -f(t, x, y, z)z - g(t, x, y) - h(t, x, y, z), \tag{3.4}$$

with the following result.

Theorem 4 *If* $g(t, 0, 0) = 0$ *and hypotheses (i)-(vi) of Theorem 1 hold true, then the trivial solution of* (3.4) *is uniformly asymptotically stable.*

Corollary 1 *(i) If* $p(t, x, y, z) = p(t) \neq 0$, $p : \mathbb{R}^+ \to \mathbb{R}$*, the results in Theorem 1, Theorem 2 and Theorem 3 hold true for the particular case*

$$\dot{x} = y, \dot{y} = z, \dot{z} = p(t) - f(t, x, y, z)z - g(t, x, y) - h(t, x, y, z).$$

(ii) Under the hypotheses of Theorem 1 the solutions of (1.2) *are ultimately bounded and satisfies* (3.3) *for all* $t \geq 0$ *with* D *independent of the initial data.*

Remark 1 *(i) If* (1.1) *is a constant coefficient differential equation* $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$*, then conditions (i)-(vii) of Theorem 1 reduces to the Routh-Hurwitz conditions* $a > 0$, $ab > c$ *and* $c > 0$*. To see this, we set* $f(t, x, y, z) = a$, $g(t, x, y) = by$, $h(t, x, y, z) = cx$ *and* $p(t, x, y, z) = 0$.

(ii) The results of Qian [17], Yamamoto [26]-[28], Tunç [24] and the recent works of Ademola and Arawomo [1, 2, 3, 4] and Ademola et al, *[6, 7] are particular case of these results. Besides, the main tools used in [17, 24, 26, 27] and [28] are incomplete Lyapunov functions. Moreover, the complete Lyapunov functions used in [1]-[6] are special cases of the main tool used in this investigation.*

The main tool used in this paper is the continuously differentiable function $V = V(t, x, y, z)$ defined as

$$V = e^{-P_*(t)}U \tag{3.5a}$$

where

$$P_*(t) = \int_0^t |p(\mu, x, y, z)| d\mu \tag{3.5b}$$

and the function $U = U(t,x,y,z)$ is defined by

$$\begin{aligned} 2U &= 2(\alpha+a)\int_0^x h(t,\xi,0,0)d\xi + 4\int_0^y g(t,x,\tau)d\tau + 2\beta xz \\ &+ 2(\alpha+a)yz + 2(\alpha+a)\int_0^y \tau f(t,x,\tau,0)d\tau + 4yh(t,x,0,0) \\ &+ b\beta x^2 + \beta y^2 + 2z^2 + 2a\beta xy, \end{aligned} \tag{3.5c}$$

where, as usual, α and β are positive constants satisfying

$$b^{-1}c < \alpha < a \tag{3.5d}$$

and

$$0 < \beta < \min\{(ab-c)a^{-1}, (ab-c)\eta_1^{-1}, \frac{1}{2}(a-\alpha)\eta_2^{-1}\} \tag{3.5e}$$

where

$$\eta_1 := 1 + a + \delta_0^{-1}\left(\frac{g(t,x,y)}{y} - b\right)^2 \text{ and } \eta_2 := 1 + \delta_0^{-1}[f(t,x,y,z) - a]^2.$$

Remark 2 The inequalities in (3.5d) and (3.5e) hold true if b and c are replaced by positive constants b_0 and c_0 respectively.

Next, we shall show that the function V and its time derivative satisfy certain inequalities as discussed in the following lemma.

Lemma 4 Under the hypotheses of Theorem 1, there exist positive constants D_1 and D_2 such that

$$D_1(x^2(t) + y^2(t) + z^2(t)) \le V(t,x,y,z) \le D_2(x^2(t) + y^2(t) + z^2(t)) \tag{3.6a}$$

and

$$V(t, x(t), y(t), z(t)) \to +\infty \text{ as } x^2(t) + y^2(t) + z^2(t) \to \infty. \tag{3.6b}$$

Furthermore, there exists a constant $D_3 > 0$ such that along a solution $(x(t), y(t), z(t))$ of (1.2)

$$\dot{V}(t,x,y,z) \le -D_3(x^2(t) + y^2(t) + z^2(t)). \tag{3.6c}$$

Proof: Clearly $V(t,0,0,0)=0$, and since $h(t,0,0,0)=0$ for all $t\geq 0$, the function U defined in (3.5c) can be presented in the form

$$\begin{aligned} U &= b^{-1}\int_0^x\Big[(\alpha+a)b-2h_x(t,\xi,0,0)\Big]h(t,\xi,0,0)d\xi+\frac{1}{2}\beta(b-\beta)x^2 \\ &+2\int_0^y\left(\frac{g(t,x,\tau)}{\tau}-b\right)\tau d\tau+b^{-1}(h(t,x,0,0)+by)^2+\frac{1}{2}(\alpha y+z)^2 \\ &+\int_0^y[(\alpha+a)f(t,x,\tau,0)-(\alpha^2+a^2)]\tau d\tau+\frac{1}{2}\beta y^2+\frac{1}{2}(\beta x+ay+z)^2. \end{aligned} \tag{3.7}$$

From the hypotheses of Theorem 1: $h_x(t,x,y,z)\leq c$ for all x; $h(t,x,0,0)\geq\delta_0 x$ for all $x\neq 0$; $g(t,x,y)\geq by$ for all $x,y\neq 0$ and $f(t,x,y,0)\geq a$ for all $t\geq 0,x,y$ so that (3.7) becomes

$$\begin{aligned} U &\geq \frac{1}{2}(\alpha b-c+ab-c)b^{-1}\delta_0x^2+b^{-1}(\delta_0x+by)^2+\frac{\beta}{2}(b-\beta)x^2 \\ &+\frac{1}{2}[\alpha(a-\alpha)+\beta]y^2+\frac{1}{2}(\alpha y+z)^2+\frac{1}{2}(\beta x+ay+z)^2. \end{aligned} \tag{3.8}$$

By (3.5d) and (3.5e): $\alpha b>c$; $ab>c$; $a>\alpha$ and $b>\beta$ respectively, so that the right hand side of (3.8) is positive definite, hence there exists a positive constant $\delta_1=\delta_1(a,b,c,\alpha,\beta,\delta_0)$ such that

$$U\geq\delta_1(x^2+y^2+z^2) \tag{3.9a}$$

for all $t\geq 0,x,y$ and z. Furthermore, (3.5b) and condition (vii) of the Theorem 1, imply the existence of a positive constant P_0 such that

$$0\leq P_*(t)\leq P_0 \tag{3.9b}$$

for all $t\geq 0$.Using estimates (3.9) in (3.5a), we have

$$V\geq\delta_2(x^2+y^2+z^2) \tag{3.10a}$$

for all $t\geq 0,x,y$ and z, where $\delta_2=\delta_1e^{-P_0}>0$. From (3.10a), $V(t,x,y,z)=0$ if and only if $x^2+y^2+z^2=0$, $V(t,x,y,z)>0$ if and only if $x^2+y^2+z^2\neq 0$, it follows that

$$V(t,x,y,z)\to+\infty \text{ as } x^2+y^2+z^2\to\infty. \tag{3.10b}$$

Moreover, applying $f(t,x,y,0)\leq a_1$, $g(t,x,y)\leq b_1y$ $(y\neq 0)$, $h(t,x,0,0)\leq cx$ $(x\neq 0)$, the inequalities $2|x||y|\leq x^2+y^2$, $2|x||z|\leq x^2+z^2$ and $2|y||z|\leq y^2+z^2$, in (3.5c) there exist positive constants $\eta_3:=(\alpha+a+2)c+(a+b+1)\beta$, $\eta_4:=(\alpha+a)(a_1+1)+\beta(a+1)+2(b_1+c)$ and $\eta_5:=\alpha+\beta+2$ such that

$$U\leq\delta_3(x^2+y^2+z^2) \tag{3.11a}$$

for all $t \geq 0, x, y$ and z where $\delta_3 := \frac{1}{2}\max\{\eta_3, \eta_4, \eta_5\}$. In view of (3.9b) and (3.11a), (3.5a) yields

$$V \leq \delta_3(x^2 + y^2 + z^2) \tag{3.11b}$$

for all $t \geq 0, x, y$ and z.

Furthermore, let $(x(t), y(t), z(t))$ be any solution of (1.2). The derivative of the function V with respect to t along a solution of (1.2) is

$$\dot{V}_{(1.2)} = -e^{-P_*(t)}\Big[U|p(t,x,y,z)| - \dot{U}_{(1.2)}\Big] \tag{3.12}$$

where $P_*(t)$ and U are the functions defined by (3.5b) and (3.5c) respectively and the derivative of the function U with respect to t along a solution of (1.2) is defined as

$$\begin{aligned}\dot{U}_{(1.2)} =& U_1 + U_2 + a\beta y^2 + 2\beta yz + [\beta x + (\alpha + a)y + 2z]p(t,x,y,z)\\ &- U_3 - U_4 - \beta\left[\frac{g(t,x,y)}{y} - b\right]xy - \beta[f(t,x,y,z) - a]xz\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}&U_1 := (\alpha + a)\int_0^x h_t(t,\xi,0,0)d\xi + 2\int_0^y g_t(t,x,\tau)d\tau + 2yh_t(t,x,0,0)\\ &U_2 := (\alpha + a)\int_0^y \tau f_t(t,x,\tau,0)d\tau + 2y\int_0^y g_x(t,x,\tau)d\tau\\ &+ (\alpha + a)y\int_0^y \tau f_x(t,x,\tau,0)d\tau\\ &U_3 := [(\alpha + a)y + 2z][h(t,x,y,z) - h(t,x,0,0)]\\ &+ (\alpha + a)yz[f(t,x,y,z) - f(t,x,y,0)]\\ &\text{and}\\ &U_4 := \beta xh(t,x,y,z) + \left[(\alpha + a)\frac{g(t,x,y)}{y} - 2h_x(t,x,0,0)\right]y^2\\ &+ [2f(t,x,y,z) - (\alpha + a)]z^2\end{aligned}$$

Since $h_t(t,0,0,0) = 0$, the right hand sides of U_1 can be recast in the form

$$\int_0^x [(\alpha + a) - 2b_0^{-1}h_{tx}(t,\xi,0,0)]h_t(t,\xi,0,0)d\xi + 2\int_0^y \left(\frac{g_t(t,x,\tau)}{\tau} - b_0\right)\tau d\tau$$

$$+b_0^{-1}\Big(h_t(t,x,0,0) + b_0 y\Big)^2.$$

Now, since $h_t(t,x,0,0) \leq \delta x$ for all $x \neq 0$, $g_t(t,x,y) \leq b_0 y$ for all $t \geq 0, x, y \neq 0$, and $c_0 \leq h_{tx}(t,x,0,0)$ for all $t \geq 0$ and x, we have

$$U_1 \leq \frac{\delta}{2b_0}[(\alpha + a)b_0 - 2c_0]x^2 + b_0^{-1}(\delta x + b_0 y)^2.$$

In view of Remark 2, there exists a constant $\delta_4 = \delta_4(a, b_0, c_0, \alpha, \delta) > 0$ such that

$$U_1 \leq \delta_4(x^2 + y^2 + z^2)$$

for all $t \geq 0, x, y$ and z.
Furthermore, by hypothesis (vi) of Theorem 1, we have

$$U_2 \leq 0$$

for all $t \geq 0, x$ and y.
Also, by the mean value theorem, we have

$$U_3 = (\alpha + a)y^2 h_y(t,x,\theta_1 y, 0) + 2z^2 h_z(t,x,0,\theta_2 z) + (\alpha + a)yz^2 f_z(t,x,y,\theta_3 z) \geq 0$$

$0 \leq \theta_i \leq 1$ $(i = 1,2,3,)$, for all $t \geq 0, x, y \neq 0$ and $z \neq 0$, but $U_3 = 0$ when $y = 0 = z$. Finally, $h(t,x,y,z) \geq \delta_0 x$ $(x \neq 0)$, $g(t,x,y) \geq by$ and $h_x(t,x,0,0) \leq c$ for all x, and $f(t,x,y,z) \geq a$ for all $t \geq 0, x, y$ and z, we obtain

$$U_4 \geq \beta\delta_0 x^2 + [(\alpha + a)b - 2c]y^2 + (a - \alpha)z^2.$$

Using estimates U_i $(i = 1,2,3,4.)$ in (3.13), we obtain

$$\begin{aligned}
\dot{U}_{(1.2)} &\leq \delta_4(x^2 + y^2 + z^2) + \delta_5(|x| + |y| + |z|)|p(t,x,y,z)| - \frac{1}{2}\beta\delta_0 x^2 \\
&- [\alpha b - c + ab - c - \beta(1 + a)]y^2 - (a - \alpha - \beta)z^2 - \frac{1}{4}\beta\delta_0 x^2 \\
&- \beta\left(\frac{g(t,x,y)}{y} - b\right)xy - \frac{1}{4}\beta\delta_0 x^2 - \beta[f(t,x,y,z) - a]xz.
\end{aligned}$$

where $\delta_5 = \max\{\beta, \alpha + a, 2\}$. This inequality, after completing the squares, can be rearranged in the form

$$\begin{aligned}
&\dot{U}_{(1.2)} \leq \delta_4(x^2 + y^2 + z^2) + \delta_5((|x| + |y| + |z|)|p(t,x,y,z)| - \frac{1}{2}\beta\delta_0 x^2 \\
&- \left\{ab - c - \beta\left[1 + a + \delta_0^{-1}\left(\frac{g(t,x,y)}{y} - b\right)^2\right]\right\}y^2 - (\alpha b - c)y^2 \\
&- \left\{\frac{1}{2}(a - \alpha) - \beta\left[1 + \delta_0^{-1}\left(f(t,x,ty,z) - a\right)^2\right]\right\}z^2 - \frac{1}{2}(a - \alpha)z^2 \\
&- \frac{1}{4}\beta\delta_0\left[x + 2\delta_0^{-1}\left(\frac{g(t,x,y)}{y} - b\right)y\right]^2 - \frac{1}{4}\beta\delta_0\left[x + 2\delta_0^{-1}[f(t,x,y,z) - a]z\right]^2.
\end{aligned}$$

Using estimates (3.5d) and (3.5e) in this inequality, we obtain

$$\dot{U}_{(1.2)} \leq -\delta_7(x^2+y^2+z^2) + \delta_5(|x|+|y|+|z|)|p(t,x,y,z)|, \tag{3.14}$$

where $\delta_6 = \min\{\frac{1}{2}\beta\delta_0, \alpha b - c, \frac{1}{2}(a-\alpha)\} > 0$ chosen sufficiently large such that $\delta_7 = \delta_6 - \delta_4 > 0$. On gathering estimates (3.9a) and (3.14) in (3.12), we have

$$\begin{aligned}\dot{V}_{(1.2)} \leq -e^{-P_*(t)}\Big[\delta_1(x^2+y^2+z^2) - \delta_5(|x|+|y|+|z|)\Big]|p(t,x,y,z)| \\ - \delta_7 e^{-P_*(t)}(x^2+y^2+z^2).\end{aligned} \tag{3.15}$$

Choosing $(x^2+y^2+z^2)^{1/2} \geq 3^{1/2}\delta_1^{-1}\delta_5$, using hypothesis (vii) of Theorem (1), we have

$$\dot{V}_{(1.2)} \leq -\delta_8(x^2+y^2+z^2), \tag{3.16}$$

for all $t \geq 0, x, y$ and z, where $\delta_8 = \delta_7 e^{-P_*(\infty)} > 0$ is a constant. This completes the proof of the lemma.

4 The Proof of The Main Results

Proof of Theorem 1: Let $(x(t), y(t), z(t))$ be any solution of (1.2). In view of estimates (3.10), (3.11b) and (3.16), hypotheses of Lemma 2 are satisfied. Hence by Lemma 2, all solutions of (1.2) are uniformly ultimately bounded. This completes the proof of the theorem.

Proof of Theorem 2: Let $(x(t), y(t), z(t))$ be any solution of (1.2). We shall show that all hypotheses of Lemma 3 hold true. Now, from (3.16) $\dot{V}_{(1.2)}(t,x,y,z) \leq 0$ for all $t \geq 0, x, y$ and z, thus from this inequality and estimates (3.10) and (3.11b) the solutions of (1.2) are uniformly bounded (see [30] Theorem 10.2 pp. 38-39).

Next, let $W(X) = \delta_8(x^2+y^2+z^2)$, obviously $W(X) \geq 0$ for all $X = (x,y,z) \in \mathbb{R}^3$. Consider the set

$$\Omega := \{X \in \mathbb{R}^3 | W(X) = 0\}.$$

From the continuity of $W(X)$, the set Ω is closed and $W(X)$ is positive definite with respect to Ω and

$$\dot{V}_{(1.2)}(t,X) \leq -W(X)$$

for all $(t,X) \in \mathbb{R}^+ \times \mathbb{R}^3$.

Moreover, system (1.2) can be arranged in the form (2.2) where $X = (x,y,z)^T$, $F(t,X) = (y, z, -zf(t,x,y,z) - g(t,x,y) - h(t,x,y,z))^T$ and $G(t,X) =$

$(0, 0, p(t, x, y, z))^T$. In view of condition (vii) of Theorem 1, estimate (2.3) holds. From the continuity and boundedness of the functions f, g and h, the function $F(t, X)$ is bounded for all $t \in \mathbb{R}^+$ when X belongs to any arbitrary compact set in $\mathbb{R}^3$. By Lemma 3 the solutions of (1.2) approaches Ω as $t \to \infty$.
Next, we show that the set $\{(0, 0, 0)\}$ is contained in Ω as $t \to \infty$. Since $W(X) = 0$ on Ω, we have $x(t) = y(t) = z(t) = 0$, and system (1.2) becomes

$$(\dot{x}, \dot{y}, \dot{z})^T = (0, 0, -g(t, 0, 0) - h(t, 0, 0, 0))^T + (0, 0, p(t, 0, 0, 0))^T. \tag{4.1}$$

Since $g(t, 0, 0) = h(t, 0, 0, 0) = p(t, 0, 0, 0) = 0$, (4.1) has solution

$$(x(t), y(t), z(t))^T = (K_1, K_2, K_3)^T, \tag{4.2}$$

where K_i $(i = 1, 2, 3)$ is a constant. For (4.2) to remain in Ω as $t \to \infty$ we must have $K_1 = K_2 = K_3 = 0$. Hence (3.1) follows.

Proof of Theorem 3: Let $(x(t), y(t), z(t))$ be any solution of (1.2), since $|x| \leq 1 + x^2$, $|y| \leq 1 + y^2$ and $|z| \leq 1 + z^2$, estimate (3.15) becomes

$$\dot{V}_{(1.2)} \leq \delta_5 e^{-P_*(t)}[3 + (x^2 + y^2 + z^2)]|p(t, x, y, z)|.$$

Using estimates (3.9b) and (3.10a) this inequality yields

$$\dot{V}_{(1.2)} - \delta_2^{-1}\delta_5|p(t, x, y, z)|V \leq 3\delta_5|p(t, x, y, z)|.$$

Solving this first order differential inequality using integrating factor $\exp[-\delta_2^{-1}\delta_5 P_*(t)]$, we obtain

$$V(t, x, y, z) \leq \delta_9 \tag{4.3}$$

where $\delta_9 = [V(t_0, x_0, y_0, z_0) + 3\delta_5 P_0]\exp[\delta_2^{-1}\delta_5 P_0] > 0$ is a constant. Using estimate (4.3) in (3.10a) and the fact that $|x| < \|X\|^2$, $X \in \mathbb{R}^3$, we have

$$|x(t)| \leq \delta_{10}, \quad |y(t)| \leq \delta_{10}, \quad |z(t)| \leq \delta_{10},$$

for all $t \geq 0$, where $\delta_{10} = \delta_2^{-1}\delta_9$. This proves the theorem.

Proof of Theorem 4: The proof of this theorem is similar to the proof of Theorem 2.1 in [7], hence it is omitted.

Example 1 *Consider a particular case of* (1.1) *i.e the third order nonlinear differential equation*

$$\begin{aligned} &\dddot{x} + \left[4 + \left(1 + t^2 + |x\dot{x}| + \exp\left[(1 + |\dot{x}\ddot{x}|)^{-1}\right]\right)^{-1}\right]\ddot{x} \\ &+ 5\dot{x} + \left[1 + \exp\left[(1 + t)^{-1}\right] + |x\dot{x}|\right]^{-1}\dot{x} + 2x \\ &+ x\left[1 + \exp\left[(1 + t + |x\ddot{x}|\dot{x}^2)^{-1}\right]\right]^{-1} = \dot{x}^2\left[1 + t^2 + x^2 + \dot{x}^2 + \ddot{x}^2\right]^{-1} \end{aligned} \tag{4.4}$$

or its equivalent system

$$
\begin{aligned}
&\dot{x} = y, \dot{y} = z, \dot{z} = y^2/(1+t^2+x^2+y^2+z^2) \\
&- \left[4 + \left[1 + t^2 + |xy| + \exp\left[(1+|yz|)^{-1}\right]\right]^{-1}\right] z \\
&- \left[5 + \left[1 + \exp\left[(1+t)^{-1}\right] + |xy|\right]^{-1}\right] y \\
&- \left[2 + \left[1 + \exp\left[(1+t+|xz|y^2)^{-1}\right]\right]^{-1}\right] x
\end{aligned}
\tag{4.5}
$$

Now, from (1.2) *and* (4.5)*, we obtain the following relations:*

(a) the function

$$f(t,x,y,z) = 4 + \frac{1}{1+t^2+|xy|+\exp[\frac{1}{1+|yz|}]},$$

and we have the following inequalities:
(i) since $0 \le \frac{1}{1+t^2+|xy|+\exp[\frac{1}{1+|yz|}]} \le 1$*, for all* $t \ge 0, x, y$ *and* z*, this implies that*

$$4 \le f(t,x,y,z) \le 5$$

for all $t \ge 0, x, y$ *and* z*, where* $a = 4 > 0$ *and* $a_1 = 5 > 0$*;*
(ii) the partial derivative of the function f *with respect to* t *is*

$$f_t(t,x,y,z) = \frac{-2t}{[1+t^2+|xy|+\exp[\frac{1}{1+|yz|}]]^2} \le 0$$

for all $t \ge 0, x, y$ *and* z*;*
(iii) for $x > 0$*, we have*

$$yf_x(t,x,y,z) = \frac{-y^2}{[1+t^2+|xy|+\exp[\frac{1}{1+|yz|}]]^2}$$

for all $t \ge 0, x, y$ *and* z*;*
(iv) for $z > 0$*,*

$$yf_z(t,x,y,z) = \frac{y^2\exp[(1+|yz|)]}{(1+|yz|)^2[1+t^2+|xy|+\exp[\frac{1}{1+|yz|}]]^2} \ge 0$$

for all $t \ge 0, x, y$ *and* z*;*

(b) the function

$$g(t,x,y) = 5y + \frac{y}{1+\exp[(1+t)^{-1}]+|xy|},$$

(i) noting that $0 \leq \frac{1}{1+\exp[(1+t)^{-1}]+|xy|} \leq 1$ *for all* $t \geq 0, x$ *and* y*, it follows that*

$$5 \leq \frac{g(t,x,y)}{y} \leq 6$$

for all $t \geq 0, x$ *and* $y \neq 0$ *from where we have* $b = 5 > 0$ *and* $b_1 = 6 > 0$;
(ii) for $x > 0$*, we have*

$$g_x(t,x,y) = \frac{-y^2}{[1+\exp[(1+t)^{-1}]+|xy|]^2} \leq 0$$

for all $t \geq 0, x$ *and* y;
(iii) also,

$$\frac{g_t(t,x,y)}{y} = \frac{\exp[(1+t)^{-1}]}{(1+t)^2[1+\exp[(1+t)^{-1}]+|xy|]^2}$$

since,

$$0 \leq \frac{\exp[(1+t)^{-1}]}{(1+t)^2[1+\exp[(1+t)^{-1}]+|xy|]^2} \leq 1$$

for all $t \geq 0, x$ *and* y*, it follows that*

$$\frac{g_t(t,x,y)}{y} \leq 1$$

for all $t \geq 0, x$ *and* $y \neq 0$*, where* $b_0 = 1 > 0$;

(c) the function

$$h(t,x,y,z) = 2x + \frac{x}{1+\exp[(1+t+|xz|y^2)^{-1}]} \tag{4.6}$$

is defined with the following inequalities:
(i) since $0 \leq \frac{1}{1+e^u} \leq 1$ *for all* $t \geq 0, x, y$ *and* z *where* $u = \frac{1}{1+t+|xz|y^2}$*, this implies that*

$$2 \leq h(t,x,y,z)/x \leq 3 \tag{4.7}$$

$t \geq 0, x \neq 0, y$ *and* z*. Also, from* (4.7) *we have*

$$h_x(t,x,y,z) \leq 3$$

for all $t \geq 0, x, y$ *and* z, *where we have* $\delta_0 = 2 > 0$ *and* $c = 3 > 0$ *and* $ab > c$ *implies that* $20 > 3$;
(ii) it is also clear from (4.6) *that* $h(t, 0, 0, 0) = 0$ *and*

$$h_t(t, x, y, z) = \frac{xe^u}{[1 + t + |xz|y^2]^2[1 + e^u]^2}$$

so that

$$h_t(t, 0, 0, 0) = 0$$

if $x = y = z = 0$;
(iii) also,

$$h_{tx}(t, x, 0, 0) = \frac{e^v}{(1 + e^v)^2(1 + t)^2}$$

for all $t \geq 0, x$ *where* $v = \dfrac{1}{1+t}$. *Now since* $0 < \frac{e^v}{(1+e^v)^2(1+t)^2} < 1$ *for all* $t \geq 0$, *it follows that the constant* $c_0 > 0$ *exists such that*

$$h_{tx}(t, x, 0, 0, 0) \geq c_0$$

for all ≥ 0 *and* x.
(iv) furthermore,

$$h_y(t, x, y, z) = \frac{2x^2|yz|e^u}{[1 + t + |xz|y^2]^2[1 + e^u]^2} \geq 0$$

for all $t \geq 0, x, y$ *and* z,
(v) moreover,

$$h_z(t, x, y, z) = \frac{x^2y^2e^u}{[1 + t + |xz|y^2]^2[1 + e^u]^2} \geq 0$$

for all $t \geq 0, x, y$ *and* z.
(vi) and

$$h_x(t, x, 0, 0) = 2 + \frac{1}{(1 + e^v)(1 + t)^2}$$

since $\frac{1}{(1+e^v)(1+t)^2} \leq 1$ *for all* $t \geq 0$, *it follows that*

$$h_x(t, x, 0, 0) \leq 3$$

for all $t \geq 0$ *and* x *where* $c = 3 > 0$ *as defined (c)(i)*

(d) *Finally, for the function* $p(t, x, y, z)$, *we have*

$$p(t, x, y, z) = \frac{y^2}{1 + t^2 + x^2 + y^2 + z^2}.$$

Clearly, $p(t, 0, 0, 0) = 0$ *and it is not difficult to show that*

$$\int_0^\infty \left| \frac{y^2}{1 + t^2 + x^2 + y^2 + z^2} \right| dt < \infty.$$

Hence, all the assumptions of the theorems are satisfied and the conclusions follow.

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