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POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF NONLINEAR SECOND ORDER EIGENVALUE PROBLEMS

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Abstract: In this work, we consider a system of coupled nonlinear second order eigenvalue problems. Under suitable conditions, existence of positive solutions are established, for determined eigenvalues, by the use of abstract fixed-point.

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1. Introduction

The existence and multiplicity of positive solutions for nonlinear second order BVP of ordinary differential equations have attracted many authors' attention and concern.

Johnny Henderson and H. Wang [6] considered a nonlinear second order eigenvalue problem

$$\left.\begin{array}{rcl} u''(t)+\lambda a(t)f(u(t)) &=& 0, \quad 0<t<1, \\ u(0)=u(1) &=& 0. \end{array}\right\} \tag{1.}$$

They determined the value of λ (eigenvalue) for which there exist positive solutions to the BVP(1).

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Ling Hu and Lianglong Wang [7] studied the existence of multiple positive solutions for systems of nonlinear second order BVP.

$$\left.\begin{array}{rcl} -u''(x) &=& f(x,v), \\ -u''(x) &=& g(x,u), \\ \alpha u(0)-\beta u'(0) &=& 0, \ \ \gamma u(1)+\delta u'(1)=0, \\ \alpha v(0)-\beta v'(0) &=& 0, \ \ \gamma v(1)+\delta v'(1)=0. \end{array}\right\} \tag{2.}$$

By the application of Krasnosel'skii [8] fixed-point theorem, the existence of positive solutions of BVP (2) is established. Motivated by the works of [6] and [7], this paper is concerned with the existence of positive solutions for the coupled system of nonlinear second order eigenvalue problem

$$\left.\begin{array}{l} u''(t)+\lambda a(t)f(v(t))=0, \\ v''(t)+\mu b(t)g(u(t))=0, \\ \alpha u(0)-\beta u'(0)=0, \ \ \gamma u(1)+\delta u'(1)=0, \\ \alpha v(0)-\beta v'(0)=0, \ \ \gamma v(1)+\delta v'(1)=0, \end{array}\right\} \tag{3}$$

where $f,g \in C([0,1],I\!R_+)$, $\ a,b \in C([0,1],I\!R_+)$, $\ \alpha,\beta,\gamma,\delta \geq 0$ and
$\rho=\alpha\gamma+\beta\gamma+\alpha\delta>0$.
A fixed-point theorem due to Krasnosel'skil [8] is applied to obtain positive solution x s of the BVP(3), for each λ,μ belonging to an open interval.

2. Preliminary Notes

Obviously, $(u,v)\in C^2[0,1]\times C^2[0,1]$ is the solution of the $BVP(3)$ if and only if $(u,v)\in C[0,1]\times C[0,1]$ is the solution of the system of integral equations

$$\left.\begin{array}{l} u(t)=\lambda \displaystyle\int_0^1 G(t,s)a(s)f(v(s))ds \\ v(t)=\mu \displaystyle\int_0^1 G(t,s)b(s)g(u(s))ds \end{array}\right\} \tag{4}$$

where $G(t,s)$ is the Green's function defined as follows:

$$G(t,s)=\begin{cases} \dfrac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s), & 0\leq s\leq t\leq 1, \\ \dfrac{1}{\rho}(\beta+\alpha t)(\gamma+\delta-\gamma s), & 0\leq t\leq s\leq 1, \end{cases}$$

The integral equation (4) can be transferred to the nonlinear integral equation

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds,\ t \in (0,1) \quad (5)$$

Lemma 2.1. *(see [1], [3], [4], [7]): - The Green's function $G(t,s)$ satisfies*

(i) $G(t,s) \leq G(s,s)$, *for* $0 \leq t,\ \ s \leq 1$,

(ii) $G(t,s) \geq M \cdot G(s,s)$, *for* $\frac{1}{4} \leq t \leq \frac{3}{4}$, $0 \leq s \leq 1$,

where

$$M = \min\left\{\frac{\gamma + 4\delta}{4(\gamma+\delta)}\ ,\ \frac{\alpha + 4\beta}{4(\alpha+\beta)}\right\} < 1.$$

The proof of this lemma is standard and omitted.

Definition 2.2. The values of λ, μ for which there exist positive solutions to the BVP(3) are called eigenvalues and the corresponding solutions $u(t) > 0,\ \ v(t) > 0$ are called eigenfunctions.

Let $B = C[0,1]$ be a Banach space with norm $\|u\| = \max\limits_{0\leq t\leq 1} |u(t)|$. Define a cone K in B by

$$K = \left\{u \in B : u(t) \geq 0 \text{ and } \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} u(t) \geq M\|u\|.\right\}.$$

Define an integral operator $A : K \longrightarrow B$ by

$$Au(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds,\ u \in K \quad (6)$$

Lemma 2.3. *(see [7]) If the operator A is defined as in (6), then $A : K \longrightarrow K$ is completely continuous.*

Proof. : For each $u \in K$, $Au \geq 0$ since the functions G, a, b, f and g are non-negative. Hence $Au(t) \geq 0$. From lemma (1) and for $u \in K$,

$$\begin{aligned} Au(t) &= \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds \\ &\leq \lambda \int_0^1 G(s,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds \end{aligned}$$

By the non-negativity of the functions G, a, b, f and g, we have

$$\|Au\| \leq \lambda \int_0^1 G(s,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds \tag{7}$$

Also, for $u \in K$ and for $\frac{1}{4} \leq t \leq \frac{3}{4}$, we have

$$\begin{aligned}
\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} Au &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds \\
&\geq \lambda M \int_0^1 G(s,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds \\
&\geq M\|Au\|.
\end{aligned}$$

Hence, $Au \in K$ and consequently $A(K) \subset K$.
Since the functions G, a, b, f and g are continuous, it follows that $A : K \longrightarrow K$ is completely continuous. This completes the proof.

□

From the above arguments, we know that the existence of positive solutions of the BVP(3) is equivalent to the existence of positive fixed points of the operator A in the cone K.

3. Main Results

We begin this section by stating the Krasnosel'skii fixed-point theorem which is also given in ([2], [5], [8]) for it important in establishing our main result.

Theorem 3.1. *Let B be a Banach Space and $K \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B such that $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If $A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, or

(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$,

then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Next, the following conditions are assumed true:

C_1. $f : [0,\infty) \longrightarrow [0,\infty)$ and $g : [0,\infty) \longrightarrow [0,\infty)$ are continuous.

C_2. $a : [0,1] \longrightarrow [0,\infty)$ and $b : [0,1] \longrightarrow [0,\infty)$ are continuous and $a(t) \neq 0$, $b(t) \neq 0$ on any subinterval of $[0,1]$.

C_3. $\lim_{u\to 0^+} \frac{f(u)}{u} = f_0$ and $\lim_{u\to 0^+} \frac{g(u)}{u} = g_0$.

C_4. $\lim_{u\to \infty} \frac{f(u)}{u} = f_\infty$ and $\lim_{u\to \infty} \frac{g(u)}{u} = g_\infty$.

Theorem 3.2. *Assume that conditions C_1, C_2, C_3 and C_4 are satisfied and let*

$$\left(M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)ds\right) f_\infty > \left(\int_0^1 G(s,s)a(s)ds\right) f_0$$

and

$$\left(M \int_{1/4}^{3/4} G(r,r)b(r)dr\right) g_\infty > \left(\int_0^1 G(r,r)b(r)dr\right) g_0.$$

Then for each λ, μ satisfying

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)ds\right) f_\infty} < \lambda < \frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right) f_0} \tag{8}$$

and

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(r,r)b(r)dr\right) g_\infty} < \mu < \frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right) g_0}, \tag{9}$$

there exists at least one positive solution (u, v) of the BVP (3) in K.

Proof. : Let λ, μ be given as in (8) and (9). Choose $\varepsilon > 0$ such that

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)ds\right) (f_\infty - \varepsilon)} \le \lambda \le \frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right) (f_0 + \varepsilon)} \tag{10}$$

and

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(r,r)b(r)dr\right) (g_\infty - \varepsilon)} \le \mu \le \frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right) (g_0 + \varepsilon)} \tag{11}$$

Now consider f_0 and g_0: There exists a constant $H_1 > 0$ such that
$f(u) \leq (f_0 + \varepsilon)u, \;\; g(u) \leq (g_0 + \varepsilon)u, \;$ for $0 < u \leq H_1$.
For $u \in K$ with $\|u\| = H_1$, we have

$$Au(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds.$$

$$\|Au\| \leq \lambda \int_0^1 G(s,s)a(s)(f_0 + \varepsilon)\mu \int_0^1 G(s,r)b(r)g(u(r))drds.$$

$$\begin{aligned}
&\leq \;\; \lambda \int_0^1 G(s,s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)g(u(r))dr. \\
&\leq \;\; \lambda \int_0^1 G(s,s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)(g_0 + \varepsilon)udr. \\
&\leq \;\; \lambda \int_0^1 G(s,s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)(g_0 + \varepsilon) \cdot H_1 dr \\
&\leq \;\; \lambda \int_0^1 G(s,s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)(g_0 + \varepsilon)\|u\|dr.
\end{aligned}$$

Using (10) and (11), we have

$$\|Au\| \leq \|u\|.$$

If we set $\Omega_1 = \{u \in B : \|u\| < H_1\}$, then

$$\|Au\| \leq \|u\|, \text{ for } u \in (K \cap \partial\Omega_1).$$

Next, consider f_∞ and g_∞: There exists a constant $H_{2*} > 0$ such that
$f(u) \geq (f_\infty - \varepsilon)u$ and $g(u) \geq (g_\infty - \varepsilon)u$, for all $u \geq H_{2*}$.
Let $H_2 = \max\{2H_1, H_{2*}/M\}$.
Then for $u \in K$ with $\|u\| = H_2$, we have

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq M\|u\| \geq H_{2*} \text{ and}$$

$$\begin{aligned}
Au(\tfrac{1}{2}) \;\; &= \;\; \lambda \int_0^1 G(\frac{1}{2},s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right) ds. \\
&\geq \;\; \lambda \int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)(f_\infty - \varepsilon)\mu \int_{1/4}^{3/4} G(s,r)b(r)g(u(r))drds. \\
&\geq \;\; \lambda \int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)(f_\infty - \varepsilon)ds \cdot \mu m \int_{1/4}^{3/4} G(r,r)b(r)(g_\infty - \varepsilon)udr.
\end{aligned}$$

$$\geq \lambda \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)(f_\infty - \varepsilon)ds \cdot \mu M^2 \int_{1/4}^{3/4} G(r,r)b(r)(g_\infty - \varepsilon)\|u\|dr.$$
$$\geq \lambda M \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)(f_\infty - \varepsilon)ds \cdot \mu M \int_{1/4}^{3/4} G(r,r)b(r)(g_\infty - \varepsilon)\|u\|dr.$$

Using (10) and (11), we have

$$\left|Au\left(\frac{1}{2}\right)\right| \geq \|u\|.$$

Thus, $\|Au\| \geq \left|Au\left(\frac{1}{2}\right)\right| \geq \|u\| \Longrightarrow \|Au\| \geq \|u\|$.

If we set $\Omega_2 = \{u \in B : \|u\| < H_2\}$, then $\|Au\| \geq \|u\|$, for $u \in (K \cap \partial\Omega_2)$. By the first part of Theorem 1, it follows that the operator A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. □

Theorem 3.3. *Assume that conditions C_1, C_2, C_3 and C_4 are satisfied and let*

$$\left(M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)ds\right) f_0 > \left(\int_0^1 G(s,s)a(s)ds\right) f_\infty$$

and

$$\left(M \int_{1/4}^{3/4} G(r,r)b(r)dr\right) g_0 > \left(\int_0^1 G(r,r)b(r)dr\right) g_\infty.$$

Then for each λ, μ satisfying

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)ds\right) f_0} < \lambda < \frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right) f_\infty} \tag{12}$$

and

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(r,r)b(r)dr\right) g_0} < \mu < \frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right) g_\infty}, \tag{13}$$

there exists at least one positive solution (u,v) of the bvp (3) in K.

Proof. Let λ, μ be given as in (12) and (13). Choose $\varepsilon > 0$ such that

$$\frac{1}{\left(M \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)ds\right)(f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right)(f_\infty + \varepsilon)} \tag{14}$$

and

$$\frac{1}{\left(M\displaystyle\int_{1/4}^{3/4} G(r,r)b(r)dr\right)(g_0-\varepsilon)} \leq \mu \leq \frac{1}{\left(\displaystyle\int_0^1 G(r,r)b(r)dr\right)(g_\infty+\varepsilon)}. \tag{15}$$

Consider f_0 and g_0: There exists a constant $H_1 > 0$ such that $f(u) \geq (f_0 - \varepsilon)u$ and $g(u) \geq (g_0 - \varepsilon)u$, for $0 < u \leq H_1$.
For $u \in K$ with $\|u\| = H_1$, we have

$$\begin{aligned}
Au(\tfrac{1}{2}) &= \lambda \int_0^1 G(\frac{1}{2}, s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right) ds. \\
&\geq \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)(f_0 - \varepsilon)\mu \int_{1/4}^{3/4} G(s,r)b(r)g(u(r))drds. \\
&\geq \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)(f_0 - \varepsilon)ds \cdot \mu M \int_{1/4}^{3/4} G(r,r)b(r)(g_0 - \varepsilon)udr. \\
&\geq \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)(f_0 - \varepsilon)ds \cdot \mu M^2 \int_{1/4}^{3/4} G(r,r)b(r)(g_0 - \varepsilon)\|u\|dr. \\
&\geq \lambda M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)(f_0 - \varepsilon)ds \cdot \mu M \int_{1/4}^{3/4} G(r,r)b(r)(g_0 - \varepsilon)\|u\|dr.
\end{aligned}$$

Using (14) and (15), we have

$$\left|Au\left(\frac{1}{2}\right)\right| \geq \|u\|.$$

Thus, $\|Au\| \geq \left|Au\left(\frac{1}{2}\right)\right| \geq \|u\| \Longrightarrow \|Au\| \geq \|u\|$.

If we set $\Omega_1 = \{u \in B : \|u\| < H_1\}$, we have $\|Au\| \geq \|u\|$, for $u \in (K \cap \partial\Omega_1)$.

Next, consider f_∞ and g_∞: Then there exists a constant $H_{2*} > 0$ such that $f(u) \leq (f_\infty + \varepsilon)u$ and $g(u) \leq (g_\infty + \varepsilon)u$, for all $u \geq H_{2*}$.
There are two cases:

Case 1: Suppose f and g are bounded. Then there exists a constant $N > 0$, $N_0 > 0$ such that $f(u) \leq N$ and $g(u) \leq N_0$, for $0 < u < \infty$.
Let $H_2 = \max.\left\{2H_1, \ \lambda N \int_0^1 G(s,s)a(s)ds\right\}$. Then for $u \in K$ and $\|u\| = H_2$,

we have

$$\begin{aligned}
Au(t) &= \lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.\\
\|Au\| &\le \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.\\
&\le \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)\cdot N_0 dr\right)ds.\\
&\le \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)\cdot N_0 dr\right)ds.\\
&\le \lambda\int_0^1 G(s,s)a(s)\cdot N ds\\
&\le \lambda N\int_0^1 G(s,s)a(s)ds.\\
&\le H_2 = \|u\|.
\end{aligned}$$

$\Longrightarrow \|Au\| \le \|u\|$.
If we set $\Omega_2 = \{u \in B : \|u\| < H_2\}$, then $\|Au\| \le \|u\|$, for $u \in (K \cap \partial\Omega_2)$.

Case 2: Suppose f and g are not bounded and let $H_2 \ge \max\{2H_1,\ H_{2*}\}$ be chosen such that $H_{2*} \le u \le H_2$. Then for $u \in K$ with $\|u\| = H_2$, we have

$$Au(t) = \lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.$$

$$\begin{aligned}
\|Au\| &\le \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.\\
&\le \lambda\int_0^1 G(s,s)a(s)\left(f_\infty+\varepsilon\right)\mu\int_0^1 G(s,r)b(r)g(u(r))drds.\\
&\le \lambda\int_0^1 G(s,s)a(s)\left(f_\infty+\varepsilon\right)ds\cdot\mu\int_0^1 G(r,r)b(r)(g_\infty+\varepsilon)u\cdot dr.\\
&\le \lambda\int_0^1 G(s,s)a(s)\left(f_\infty+\varepsilon\right)ds\cdot\mu\int_0^1 G(r,r)b(r)(g_\infty+\varepsilon)\|u\|dr.
\end{aligned}$$

Using (14) and (15), we have $\|Au\| \le \|u\|$.
If we set $\Omega_2 = \{u \in B : \|u\| < H_2\}$, then $\|Au\| \le \|u\|$ for $u \in (K \cap \partial\Omega_2)$.
Therefore, in either case,

$$\|Au\| \le \|u\|, \ \text{ for } \ u \in (K \cap \partial\Omega_2).$$

By the second part of Theorem 3.1, the operator A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. ☐

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