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EXISTENCE OF POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF NONLINEAR BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract: In this work, we discuss the existence of positive solutions for a coupled system of nonlinear boundary value problems of fractional order with integral boundary conditions.

The existence result is obtained by means of Krasnosel'skii fixed-point theorem in a cone.

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1. Introduction

In this paper, we study the existence of positive solutions to the following coupled system of nonlinear boundary value problems of fractional order

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$$\left.\begin{aligned} D^{\alpha}u(t)+w(t)f(t,v(t))=0,\;\; t\in(0,1),\\ D^{\alpha}v(t)+z(t)g(t,u(t))=0,\;\; t\in(0,1),\\ u(0)=0,\;\; \gamma u(1)+\beta u'(1)=\int_0^1 p(t)u(t)dt,\\ v(0)=0,\;\; \gamma v(1)+\beta v'(1)=\int_0^1 p(t)v(t)dt, \end{aligned}\right\} \tag{1}$$

where $1<\alpha\leq 2$, D^{α} is the standard Riemann-Liouville fractional derivative, $w,z\in C([0,1],\ I\!R^{+})$, $p\in\mathcal{L}^{1}[0,1]$ is non-negative, $\gamma,\ \beta\in(0,1)$ and $f,g\in C([0,1]\times I\!R^{+},I\!R^{+})$.

In recent years, the study of positive solutions for nonlinear fractional differential equations and coupled systems has received much attention from many authors, see [1 - 2, 4, 6 - 9, 11 - 14, 16 - 17, 21 - 28, 30 - 33] and the references cited therein for details.

In [23], the authors studied the existence and uniqueness of positive solutions to the following nonzero boundary value problem (BVP for short) for a coupled system of nonlinear fractional differential equations:

$$\left.\begin{aligned} D^{\alpha}u(t)+f(t,v(t))=0,\;\; t\in(0,1),\\ D^{\beta}v(t)+f(t,u(t))=0,\;\; t\in(0,1),\\ u(0)=0,\;\; u(1)=au(\xi),\\ v(0)=0,\;\; v(1)=bv(\xi), \end{aligned}\right\} \tag{2}$$

where $1<\alpha<2$, $1<\beta<2$, $0\leq a,b\leq 1$, $1<\xi<2$, D^{α} and D^{β} are the standard Riemann-Liouville fractional derivatives.

In [14], the authors established the existence of a positive solution for the singular fractional differential equations with integral boundary conditions

$$\left.\begin{aligned} &{}^cD^p u(t) = \lambda h(t) f(t, u(t)), \quad t \in (0,1), \\ &u(0) - au(1) = \int_0^1 g_0(s)u(s)ds, \\ &u'(0) - b\,{}^cD^q u(1) = \int_0^1 g_1(s)u(s)ds, \\ &u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0, \end{aligned}\right\} \tag{3}$$

where ${}^cD^p$ is the Caputo fractional derivative, $n \geq 3$ is an integer, $p \in (n-1, n)$, $0 < q < 1$, $0 < a < 1$, $0 < b < \Gamma(2-q)$ are real numbers.

Further, the authors in [10] studied the existence of positive solutions to the following integral boundary value problem of nonlinear fractional differential equation

$$\left.\begin{aligned} D^\alpha x(t) + g(t) f(t, x) = 0, \quad t \in (0,1), \\ x(0) = 0, \; x'(1) = \int_0^1 h(t)x(t)dt, \end{aligned}\right\} \tag{4}$$

where $1 < \alpha \leq 2$, D^α is the standard Riemann-Liouville fractional derivative and $h \in \mathcal{L}^1[0,1]$ is non-negative.

Inspired by the works in [10, 14, 23], we consider in this paper the existence of positive solutions to the BVP (1). Many papers have dealt with coupled systems of nonlinear fractional differential equation with integral boundary conditions. However, to the best of our knowledge, the existence of positive solutions to the BVP (1) has not been discussed. In this paper, we consider an integral boundary condition which is very different from those in [1 - 4, 10 - 12, 16, 21, 23, 26, 29 - 33]. Our approach is based on compact integral operator and the application of Krasnosel'skii fixed-point theorem in a cone.

For this work, we make the following assumptions:

C_1. $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ and $g : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ are continuous.

C_2. $w : [0,1] \longrightarrow [0,\infty)$ and $z : [0,1] \longrightarrow [0,\infty)$ are continuous and $w(t) \neq 0$, $z(t) \neq 0$ on any subinterval of $[0,1]$.

The paper is organized as follows: In Section 2, some basic definitions, preliminary results and properties of the Green function used are presented. Finally, our existence result is stated and proved in Section 3.

2. Preliminary Results

In this section, we give some basic definitions and lemmas from the theory of fractional calculus which will be needed in the sequel. Moreover, we give the expression of Green's function associated with the BVP (1).

Definition 2.1. ([5, 19, 20]) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given continuous function $f : (0, \infty) \longrightarrow I\!R$ is defined to be

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. ([5, 19, 20]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a given continuous function $f : (0, \infty) \longrightarrow I\!R$ is defined to be

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

$n - 1 < \alpha \leq n$, provided the right side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of the number α.

Lemma 2.3. ([18]) *If $u \in C(0,1) \cap \mathcal{L}(0,1)$, then*

$$D^{\alpha} I^{\alpha} u(t) = u(t).$$

Lemma 2.4. ([5, 20]) *Let $\alpha > 0$ and $u \in C(0,1) \cap \mathcal{L}(0,1)$. Then the unique solution of $D^{\alpha} u(t) = 0$ is given by*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{5}$$

for $c_i \in I\!R$ and $i = 1, 2, \ldots, n$.

Lemma 2.5. ([5, 20]) *Let $\alpha > 0$ and $u, D^{\alpha} u \in C(0,1) \cap \mathcal{L}(0,1)$. Then*

$$\left.\begin{aligned} I^{\alpha} D^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \\ \text{for } c_i \in I\!R \text{ and } i = 1, 2, \ldots, n, \quad n \geq \alpha. \end{aligned}\right\} \tag{6}$$

Lemma 2.6. *Let* $h \in C[0,1]$ *and* $\eta = \dfrac{1}{a_0}\displaystyle\int_0^1 p(t)t^{\alpha-1}dt$. *Then the unique solution of the BVP*

$$\left.\begin{aligned} D^{\alpha}u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad \gamma u(1) + \beta u'(1) = \int_0^1 p(t)u(t)dt \end{aligned}\right\} \tag{7}$$

is given by

$$u(t) = \int_0^1 G^{\star}(t,s)h(s)ds,$$

where

$$G^{\star}(t,s) = G_1(t,s) + G_2(t,s), \tag{8}$$

$$G_1(t,s) = \begin{cases} \dfrac{1}{a_0\Gamma(\alpha)}\left[\beta(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-2} + \gamma t^{\alpha-1}(1-s)^{\alpha-1} \right. \\ \qquad\qquad \left. - a_0(t-s)^{\alpha-1}\right], \ s \le t, \\ \dfrac{1}{a_0\Gamma(\alpha)}\left[\beta(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-2} + \gamma t^{\alpha-1}(1-s)^{\alpha-1}\right], \\ \qquad\qquad t \le s, \end{cases} \tag{9}$$

and

$$G_2(t,s) = \lambda_0 G_1(t,s), \quad \lambda_0 = \left(\frac{\eta}{1-\eta}\right), \quad 0 < \eta < 1. \tag{10}$$

Proof. By Lemma 2.5, the BVP (7) can be reduced to an equivalent integral equation

$$\begin{aligned} u(t) &= -\,I^{\alpha}h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \\ &= -\,\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \end{aligned} \tag{11}$$

By $u(0) = 0$, we have $c_2 = 0$ and

$$u(t) = -\,\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1 t^{\alpha-1}, \tag{12}$$

$$u'(t) = \frac{-(\alpha-1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}h(s)ds + c_1(\alpha-1)t^{\alpha-2}, \tag{13}$$

$$\beta u'(1) = -\frac{\beta(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}h(s)ds + \beta(\alpha-1)c_1,$$

$$\gamma u(1) = -\frac{\gamma}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds + \gamma c_1.$$

Using the boundary condition $\gamma u(1) + \beta u'(1) = \int_0^1 p(t)u(t)dt$, we have

$$[\beta(\alpha-1)+\gamma]c_1 - \frac{\gamma}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds - \frac{\beta(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}h(s)ds$$

$$= \int_0^1 p(t)u(t)dt,$$

$$\Longrightarrow c_1 = \frac{\gamma}{a_0\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds + \frac{\beta(\alpha-1)}{a_o\Gamma\alpha}\int_0^1 (1-s)^{\alpha-2}h(s)ds$$

$$+ \frac{1}{a_0}\int_0^1 p(t)u(t)dt, \tag{14}$$

where $a_0 = [\beta(\alpha-1)+\gamma] > 0$.

Putting (14) into (12), we have the unique solution of the BVP (7) to be:

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{\gamma}{a_0\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds$$

$$+ \frac{\beta(\alpha-1)}{a_0\Gamma(\alpha0}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}h(s)ds + \frac{t^{\alpha-1}}{a_0}\int_0^1 p(t)u(t)dt,$$

$$= -\frac{1}{a_0\Gamma(\alpha)}\int_0^t a_0(t-s)^{\alpha-1}h(s)ds + \frac{\gamma}{a_0\Gamma(\alpha)}\int_0^t t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds$$

$$+ \frac{\gamma}{a_0\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds + \frac{\beta(\alpha-1)}{a_0\Gamma(\alpha)}\int_0^t t^{\alpha-1}(1-s)^{\alpha-2}h(s)ds$$

$$+ \frac{\beta(\alpha-1)}{a_0\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}h(s)ds + \frac{t^{\alpha-1}}{a_0}\int_0^1 p(t)u(t)dt$$

$$\Longrightarrow u(t) = \int_0^1 G_1(t,s)h(s)ds + \frac{t^{\alpha-1}}{a_0}\int_0^1 p(t)u(t)dt, \tag{15}$$

where $G_1(t,s)$ is defined by (9).

Multiplying (15) by $p(t)$ and integrating from 0 to 1, we have

$$\begin{aligned}\int_0^1 p(t)u(t)dt &= \int_0^1 p(t)\int_0^1 G_1(t,s)h(s)dsdt \\ &\quad + \frac{1}{a_0}\int_0^1 p(t)t^{\alpha-1}dt \cdot \int_0^1 p(t)u(t)dt\end{aligned}$$

$$\Longrightarrow \int_0^1 p(t)u(t)dt - \eta\int_0^1 p(t)u(t)dt = \int_0^1 p(t)\int_0^1 G_1(t,s)h(s)dsdt,$$

where

$$\eta = \frac{1}{a_0}\int_0^1 p(t)t^{\alpha-1}dt$$

$$\Longrightarrow (1-\eta)\int_0^1 p(t)u(t)dt = \int_0^1 p(t)\int_0^1 G_1(t,s)h(s)dsdt$$

$$\Longrightarrow \int_0^1 p(t)u(t)dt = \frac{1}{1-\eta}\int_0^1 p(t)\int_0^1 G_1(t,s)h(s)dsdt. \tag{16}$$

Put (16) into (15) and simplifying, we have

$$\begin{aligned}u(t) &= \int_0^1 G_1(t,s)h(s)ds + \frac{1}{a_0}\int_0^1 p(t)t^{\alpha-1}dt \cdot \frac{1}{1-\eta}\int_0^1 G_1(t,s)h(s)ds \\ &= \int_0^1 G_1(t,s)h(s)ds + \frac{\eta}{1-\eta}\int_0^1 G_1(t,s)h(s)ds \\ &= \int_0^1 \left[G_1(t,s) + \frac{\eta}{1-\eta}G_1(t,s)\right]h(s)ds \\ &\Longrightarrow u(t) = \int_0^1 G^\star(t,s)h(s)ds,\end{aligned}$$

where

$$G^\star(t,s) = G_1(t,s) + G_2(t,s) \ \text{ and } \ G_1(t,s),\ G_2(t,s)$$

are defined by (9) and (10) respectively. This completes the proof. □

Lemma 2.7. *The function $G_1(t,s)$ defined by (9) is continuous and satisfies the following conditions:*

(i) $G_1(t,s) \geq 0 \ \ \forall\, t,s \in [0,1]$ *and* $G_1(t,s) > 0, \ \forall\, t,s \in (0,1)$;

(ii) $G_1(t,s) \leq G_1(s,s) = \dfrac{s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{a_0\Gamma(\alpha)}$, *for all* $t,s \in [0,1]$;

(iii) $\min\limits_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge m(s) \max\limits_{0\le t\le 1} G_1(t,s) = m(s)G_1(s,s),$
for $\frac{1}{4} \le t \le \frac{3}{4},\ s \in (0,1)$ *and* $0 < m(s) < 1,$
where $m(s) \in C((0,1),\ I\!R^+)$ *and*

$$m(s) = \begin{cases} \dfrac{(\frac{3}{4})^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}] - a_o(\frac{3}{4}-s)^{\alpha-1}}{s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]}, & s \in (0, \frac{3}{4}], \\[2ex] \dfrac{1}{(4s)^{\alpha-1}}, \ s \in [\dfrac{3}{4}, 1). \end{cases}$$

Proof.
(i) It is standard and omitted.
(ii) For $t \le s$, we have

$$\begin{aligned} \frac{G_1(t,s)}{G_1(s,s)} &= \frac{\beta(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-2} + \gamma t^{\alpha-1}(1-s)^{\alpha-1}}{\beta(\alpha-1)s^{\alpha-1}(1-s)^{\alpha-2} + \gamma s^{\alpha-1}(1-s)^{\alpha-1}} \\ &= \frac{t^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]} \\ &= \frac{t^{\alpha-1}}{s^{\alpha-1}} \le 1. \end{aligned}$$

Similarly, for $s \le t$, we have $\dfrac{G_1(t,s)}{G_1(s,s)} \le 1.$

Hence, $G_1(t,s) \le G_1(s,s) = \dfrac{s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{a_0\Gamma(\alpha)},$
for all $t,s \in [0,1]$.
(iii) The proof of (iii) is similar to that of Lemma 2.4 in [5] and so omitted. □

Lemma 2.8. *Suppose* $0 < \eta < 1$. *Then* $G_2(t,s)$ *defined by (10) is continuous and satisfies the following conditions:*

(i) $G_2(t,s) \ge 0$ *for all* $t,s \in [0,1]$ *and* $G_2(t,s) > 0$ *for all* $t,s \in (0,1)$.

(ii) $G_2(t,s) \le \lambda_0 G_1(s,s)$
$= \dfrac{\lambda_0 s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{a_0\Gamma(\alpha)},$
$\forall\ t,s \in [0,1]$, *where* $\lambda_0 = \left(\dfrac{\eta}{1-\eta}\right) > 0.$

Lemma 2.9. *The Green function $G^{\star}(t,s)$ defined by (8) is continuous and satisfies the following conditions:*

(i) $G^{\star}(t,s) \geq 0$ *for all* $t,s \in [0,1]$ *and* $G^{\star}(t,s) > 0$ *for all* $t,s \in (0,1)$;

(ii) $$G^{\star}(t,s) \leq \sigma G_1(s,s) = \frac{\sigma s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{a_0\Gamma(\alpha)},$$ $\forall\ t,s \in [0,1]$, *where* $\sigma = (1+\lambda_0) > 0$; (17).

(iii) $$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G^{\star}(t,s) \geq \sigma m(s) G_1(s,s), \text{ for } \tfrac{1}{4} \leq t \leq \tfrac{3}{4},\ \ s \in (0,1). \tag{18}$$

Proof. (i) It is standard and omitted.

(ii) In view of Lemmas 2.7, 2.8 and equation (8), we have

$$\begin{aligned} G^{\star}(t,s) &= G_1(t,s) + G_2(t,s) \\ &= G_1(t,s) + \lambda_0 G_1(t,s) \\ &\leq G_1(s,s) + \lambda_0 G_1(s,s) \\ &\leq (1+\lambda_0)\, G_1(s,s) \\ &\leq \frac{\sigma s^{\alpha-1}[\beta(\alpha-1)(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{a_0\Gamma(\alpha)}, \end{aligned}$$

where $\sigma = (1+\lambda_0) > 0$.

(iii) By the standard argument of [5], we have

$$\max_{0\leq t\leq 1} G_1(t,s) = G_1(s,s) \text{ and } \max_{0\leq t\leq 1} G_2(t,s) = \lambda_0 G_1(s,s).$$

Also,

$$\begin{aligned} \max_{0\leq t\leq 1} G^{\star}(t,s) &= \max_{0\leq t\leq 1} \left[G_1(t,s) + G_2(t,s)\right], \\ &= \max_{0\leq t\leq 1} G_1(t,s) + \max_{0\leq t\leq 1} G_2(t,s), \\ &= G_1(s,s) + \lambda_0 G_1(s,s), \\ &= (1+\lambda_0) G_1(s,s) = \sigma G_1(s,s). \\ \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G^{\star}(t,s) &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \left[G_1(t,s) + G_2(t,s)\right], \\ &\geq \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \left[G_1(t,s) + \lambda_0 G_1(t,s)\right], \\ &\geq (1+\lambda_0) \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G_1(t,s), \\ &\geq \sigma m(s) G_1(s,s). \end{aligned}$$

□

In view of Lemma 2.6, $(u,v) \in C[0,1] \cap \mathcal{L}^1[0,1]$ is a solution of the BVP (1) if and only if (u,v) solves the system of integral equations

$$\left.\begin{aligned} u(t) &= \int_0^1 G^\star(t,s)w(s)f(s,v(s))ds, \\ v(t) &= \int_0^1 G^\star(t,s)z(s)g(s,u(s))ds, \end{aligned}\right\} \tag{19}$$

where $G^\star(t,s)$ is defined by (8).

The system of integral equations (19) can be written as

$$u(t) = \int_0^1 G^\star(t,s)w(s)f\left(s, \int_0^1 G^\star(s,r)z(r)g(r,u(r))dr\right)ds, \quad t \in (0,1). \tag{20}$$

Let $B^* = C[0,1]$ be a Banach space with norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Define a cone $K_* \subset B^*$ by

$$K_* = \left\{ u \in B^* : u(t) \geq 0 \text{ and } \min_{\frac{1}{4}\le t\le \frac{3}{4}} u(t) \geq m(s)\|u\|. \right\}.$$

Define an integral operator $A : K_* \longrightarrow B^*$ by

$$Au(t) = \int_0^1 G^\star(t,s)w(s)f\left(s, \int_0^1 G^\star(s,r)z(r)g(r,u(r))dr\right)ds, \tag{21}$$

for $u \in K_*$.

Lemma 2.10. ([3]) *Let the operator A be defined as in (21). Then $A : K_* \longrightarrow K_*$ is completely continuous.*

In view of the fixed point theory, the existence of positive solutions to the BVP (1) is equivalent to the existence of positive fixed points of the operator A in the cone K_*. We state the Krasnosel'skii fixed-point theorem, as follows.

Theorem 2.11. ([15, 29]) *Let B^* be a Banach Space and $K_* \subset B^*$ be a cone in B^*. Assume Ω_1, Ω_2 are open subsets of B^* such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $A : K_* \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \longrightarrow K_*$ is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$, $u \in K_* \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $u \in K_* \cap \partial\Omega_2$, or

(ii) $\|Au\| \geq \|u\|$, $u \in K_* \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$, $u \in K_* \cap \partial\Omega_2$,

then A has a fixed point in $K_* \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence Result

In this section, we apply the fixed point Theorem 2.11 to establish the existence of positive solutions to the BVP (1).

Theorem 3.1. *Assume conditions C_1, C_2 and the following hypotheses hold:*

(H_1) *There exist positive constants k_1, k_2 and e_1 such that $f(t,u) \leq k_1 u$ and $g(t,u) \leq k_2 u$, for $(t,u) \in ([0,1] \times [0,e_1])$, where k_1 and k_2 satisfy*

$$\sigma k_1 \int_0^1 G_1(s,s)w(s)ds \leq 1 \text{ and } \sigma k_2 \int_0^1 G_1(r,r)z(r)dr \leq 1,$$

respectively.

(H_2) *There exist positive constants λ_1, λ_2 and e_2 such that $f(t,u) \geq \lambda_1 u$ and $g(t,u) \geq \lambda_2 u$, for $(t,u) \in \left([\frac{1}{4}, \frac{3}{4}] \times (0,\ e_2]\right)$ with $e_1 > e_2$, where λ_1 and λ_2 satisfy*

$$\lambda_1 \int_{1/4}^{3/4} \sigma G_1(\frac{1}{2}, s)w(s)ds \geq 1 \text{ and } \lambda_2 \int_{1/4}^{3/4} \sigma m(r) G_1(r,r)z(r)dr \geq 1,$$

respectively.

Then the BVP (1) has at least one positive solution $u(t)$ in the cone K_*.

Proof. Let $u \in K_*$ with $\|u\| = e_1$. By hypothesis (H_1), we have $0 \leq u \leq e_1$ and

$$Au(t) = \int_0^1 G^\star(t,s)w(s)f\left(s, \int_0^1 G^\star(s,r)z(r)g(r,u(r))dr\right)ds.$$

$$\begin{aligned}
\|Au\| &\leq \int_0^1 G^\star(t,s)w(s)ds \cdot k_1 \int_0^1 G^\star(s,r)z(r)g(r,u(r))dr \\
&\leq \sigma \int_0^1 G_1(s,s)w(s)ds \cdot \sigma k_1 \int_0^1 G_1(r,r)z(r)g(r,u(r))dr \\
&\leq \sigma \int_0^1 G_1(s,s)w(s)ds \cdot \sigma k_1 \int_0^1 G_1(r,r)z(r) \cdot k_2 u dr \\
&\leq \sigma k_1 \int_0^1 G_1(s,s)w(s)ds \cdot \sigma k_2 \int_0^1 G_1(r,r)z(r) \cdot u dr \\
&\leq \sigma k_1 \int_0^1 G_1(s,s)w(s)ds \cdot \sigma k_2 \int_0^1 G_1(r,r)z(r) \cdot c_1 dr \\
&\leq \sigma k_1 \int_0^1 G_1(s,s)w(s)ds \cdot \sigma k_2 \int_0^1 G_1(r,r)z(r)\|u\|dr.
\end{aligned}$$

$$\implies \|Au\| \leq \|u\|.$$

If we set $\Omega_1 = \{u \in B^* : \|u\| < \rho_1\}$, then $\|Au\| \leq \|u\|$, for $u \in (K_* \cap \partial\Omega_1)$.

Next, let $u \in K_*$ with $\|u\| = e_2$. Then for $\frac{1}{4} \leq t \leq \frac{3}{4}$, we have

$$u(t) \geq \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq m(s)\|u\| = m(s)e_2,$$

whenever $m(s)e_2 \leq u \leq e_2$.

By hypothesis (H_2), we have:

$$\begin{aligned}
Au(\frac{1}{2}) &= \int_0^1 G^\star(\frac{1}{2},s)w(s)f\left(s, \int_0^1 G^\star(s,r)z(r)g(r,u(r))dr\right) ds \\
&\geq \int_{1/4}^{3/4} \sigma G_1(\frac{1}{2},s)w(s)ds \cdot \lambda_1 \int_{1/4}^{3/4} \sigma G_1(r,r)z(r)g(r,u(r))dr \\
&\geq \int_{1/4}^{3/4} \sigma G_1(\frac{1}{2},s)w(s)ds \cdot \lambda_1 \int_{1/4}^{3/4} \sigma G_1(r,r)z(r) \cdot \lambda_2 u dr \\
&\geq \lambda_1 \int_{1/4}^{3/4} \sigma G_1(\frac{1}{2},s)w(s)ds \cdot \lambda_2 \int_{1/4}^{3/4} \sigma G_1(r,r)z(r) \cdot m(r)\|u\|dr
\end{aligned}$$

$$\geq \lambda_1 \int_{1/4}^{3/4} \sigma G_1(\frac{1}{2}, s) w(s) ds \cdot \lambda_2 \int_{1/4}^{3/4} \sigma m(r) G_1(r,r) z(r) \|u\| dr$$
$$\Longrightarrow \|Au(\frac{1}{2})\| \geq \|u\|.$$

Thus

$$\|Au\| \geq \left| Au\left(\frac{1}{2}\right)\right| \geq \|u\|.$$

Setting $\Omega_2 = \{u \in B^* : \|u\| < \rho_2\}$, then $\|Au\| \geq \|u\|$, for $u \in (K_* \cap \partial\Omega_2)$.

By the application of part (i) of Theorem 2.11, the operator A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This means that the BVP (1) has a positive solution, say $u(t)$, with $e_2 \leq \|u(t)\| \leq e_1$. This completes the proof. □

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