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OSCILLATION CRITERIA FOR THREE DIMENSIONAL NONLINEAR CONFORMABLE FRACTIONAL DELAY DIFFERENTIAL SYSTEM WITH FORCING TERMS

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Abstract. In this paper, we study the oscillation of three dimensional nonlinear conformable delay differential system with forcing terms. By using generalized Riccati transformation, conformable derivatives and some inequality based techniques, we obtain several oscillation criteria for the system. Furthermore, an example is given to authenticate our results.

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1. Introduction

Research on the qualitative properties of solutions of differential equations which includes the problems of oscillation and non-oscillation of solutions dated back to the time of C. Sturm in 1836. Since that period, researchers have continued to study the oscillation of differential equations [[5],[8],[24],[25],[27]] using different approaches.

The theory of fractional calculus attracted many researchers in the last few decades due to the applicability of fractional differential equations in science and engineering [[1], [9]-[12]]; thus, researchers have developed interest in the study of oscillation of the Caputo, Riemann-Liouville, modified Riemann-Liouville, and conformable fractional differential equations [[2]-[4],[6],[7],[19]].

In [[17],[18]], oscillation and non-oscillation of two-dimensional differential systems were studied. Ogunbanjo and Arawomo also investigated the oscillation criteria for a nonlinear conformable fractional differential system with a forcing term [[20]]. Some authors have worked on the oscillation of three-dimensional differential systems [[16],[23],[26]] using different methods.

However, to the best of our knowledge, little or no work has been done on the oscillation of three-dimensional nonlinear fractional delay differential system using

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conformable fractional differential system with two or more forcing terms. Motivated by these observations, we study the following fractional differential system:

$$\left.\begin{aligned} D^{\alpha}(x(t)) &= p(t)g(y(\sigma(t))) \\ D^{\alpha}(y(t)) &= -q(t)v(z(t)) + \phi(t) \\ D^{\alpha}(z(t)) &= r(t)f(x(\tau(t))) + \psi(t,x(t)) \\ t \geq t_0;& \quad 0<\alpha<1 \end{aligned}\right\} \tag{1}$$

where D^{α} denotes the conformable fractional derivative of order α w.r.t t.

Now, we state some conditions that will be useful throughout this paper:

Λ_1 - $p(t) \in C^{2\alpha}([t_0,\infty),\Re^+)$, $q(t) \in C^{\alpha}([t_0,\infty),\Re^+)$, $r(t) \in C([t_0,\infty),\Re^+)$; $p(t), q(t)$ and $r(t)$ are not identically zero on any interval of the form $[T_0,\infty)$, where $T_0 \geq t_0$, $q(t)$ and $r(t)$ are positive and decreasing;

Λ_2 - $g \in C^{\alpha}(\Re,\Re)$, $yg(y) > 0$, $D^{\alpha}g(y) \geq l' > 0$, $v \in C^{\alpha}(\Re,\Re)$, $zv(z) > 0$, $D^{\alpha}v(z) \geq m' > 0$, $f \in C^{\alpha}(\Re,\Re)$ and $xf(x) > 0$, for $x \neq 0$

Λ_3 - $\sigma(t) \leq t$, $\tau(t) \leq t$ with $D^{\alpha}\sigma(t) \geq l > 0$

Λ_4- $\int_{t_0}^{\infty} s^{\alpha-1}\frac{1}{b(s)}ds = \infty$, $\int_{t_0}^{\infty} s^{\alpha-1}\frac{1}{a(s)}ds = \infty$, where $b(t) = \frac{1}{q(t)}$, $a(t) = \frac{1}{p(t)}$ and $c(t) = ll'm'r(t)$; $a(t), b(t)$ and $c(t)$ are positive real valued continuous functions with $(tz)^{\alpha-1} \geq 1$

Λ_5 - $\frac{\phi(t)}{q(t)} \leq \xi(t)$, $\frac{f(x)}{x(t)} \leq k$, $\frac{\psi(t,x(t))}{x(t)} \leq \gamma(t)$. $\xi(t) \in C^{\alpha}([t_0,\infty),\Re^+)$, $\xi_*'(t), \gamma(t) \in C([t_0,\infty),\Re^+)$ and k is a constant. $\psi(t,x(t))$ is a continuous function on $[t_0,\infty) \times \Re$.

For a solution of system (1), we mean that it is a vector valued function $\chi(t) = (x(t), y(t), z(t))$ with $T_1 = \min\{\tau(t_1),\sigma(t_1)\}$ for some $t_1 \geq t_0$ which has the property that $b(t)D^{\alpha}(a(t)D^{\alpha}x(t)) \in C^{\alpha}([T_1,\infty),\Re)$ and the system (1) on $[T_1,\infty)$.

The solution $(x(t), y(t), z(t))$ of system (1) will be called oscillatory if all the components are oscillatory, otherwise it will be called nonoscillatory. The system (1) is called oscillatory if all its solutions are oscillatory.

2. Preliminaries

In this section, we are supposed to explain the basic concept of conformable fractional derivative, but we refer the readers who are not familiar with the concept of conformable fractional derivatives and its properties to see [[13]-[15]].

3. Main Results

Here, we are concerned with the oscillation of system (1). Before this, we first state and establish the following lemmas needed in proving our theorems.

Lemma 1. Suppose $\psi'(t) \leq M$ and let

$$t^{1-\alpha}x(t) \leq (t-T)^3\psi'(t)a(t)D^{\alpha}x(t)$$

Then,

$$\frac{x(t)}{a(t)D^{\alpha}x(t)} \leq \frac{(t-T)^3 M}{t^{1-\alpha}}$$

Lemma 2. Suppose $p(t) \leq 0$. Then, the first component $x(t)$ of a nonoscillatory solution $(x(t), y(t), z(t))$ of (1) is also nonoscillatory.

Proof. The proof follows from Lemma 7.2.1[[21]].

Lemma 3. Suppose that (Λ_1) and (Λ_4) holds. Then, there exists a $t_1 \geq t_0 \ni$ either

(Λ_a)- $x(t) > 0, D^{\alpha}x(t) > 0, D^{\alpha}(a(t)D^{\alpha}x(t)) > 0$ for $t \geq t_1$.

or
(Λ_b)- $x(t) > 0, D^\alpha x(t) < 0, D^\alpha(a(t)D^\alpha x(t)) > 0$ for $t \geq t_1$ holds.
Proof. Let $x(t)$ be an eventually positive solution of (1) on $[t_0, \infty)$. From (1) and conditions Λ_2 - Λ_5, we arrived at

$$D^\alpha[b(t)D^\alpha[a(t)D^\alpha x(t)]] + c(t)f(x) + k_1\psi(t, x(t))) - k_2\xi'(t) \leq 0 \qquad (2)$$

where $c(t) = ll'm'r(t)$, $k_1(t) = ll'm'$ and $k_2(t) = ll'$. $k_1(t)$ and $k_2(t)$ are functions. From (2), we get

$$D^\alpha[b(t)D^\alpha[a(t)D^\alpha x(t)]] \leq 0 \;\; for \;\; t \geq t_0$$

The rest of the proof follows from Lemma 3.2 [[16]].
Lemma 4. Suppose that the conditions (Λ_1) - (Λ_5) hold. Assume also that Case (Λ_b) of lemma 3 holds. If

$$\int_{t_0}^{\infty} \frac{\theta^{\alpha-1}}{a(\theta)}\Big(\int_{\theta}^{\infty} \frac{\eta^{\alpha-1}}{b(\eta)} \int_{\eta}^{\infty} s^{\alpha-1}V(s)dsd\eta\Big)d\theta = \infty \qquad (3)$$

Then, $\lim_{t\to\infty} x(t) = 0$
Proof. Considering Case (Λ_b) of lemma 3. Since $x(t)$ is positive and decreasing, there exists a $\lim_{t\to\infty} x(t) = \mu \geq 0$. If $\mu > 0$, and from (2), we have

$$\frac{D^\alpha[b(t)D^\alpha[a(t)D^\alpha x(t)]}{x(t)} \leq -\frac{c(t)f(x)}{x(t)} - \frac{k_1\psi(t, x(t)))}{x(t)} + \frac{k_2\xi'(t)}{x(t)}$$

which implies that

$$D^\alpha[b(t)D^\alpha[a(t)D^\alpha x(t)]] \leq -x(t)V(t)$$

where

$$\frac{\xi^{'}(t)}{x(t)} = \xi_*^{'}(t) \;\; and \;\; V(t) = kc(t) + k_1\gamma(t) - k_2\xi_*^{'}(t)$$

integrating the above inequality from t to ∞ twice w.r.t $d_\alpha s$, we have

$$-D^\alpha x(t) \geq \frac{\mu}{a(t)} \int_t^{\infty} \frac{\eta^{\alpha-1}}{b(\eta)} \int_{\eta}^{\infty} s^{\alpha-1}V(s)dsd\eta$$

integrating the above inequality once again from t_0 to ∞ w.r.t $d_\alpha s$, we have

$$x(t_0) \geq \mu \int_{t_0}^{\infty} \frac{\theta^{\alpha-1}}{a(\theta)}\Big(\int_{\theta}^{\infty} \frac{\eta^{\alpha-1}}{b(\eta)} \int_{\eta}^{\infty} s^{\alpha-1}V(s)dsd\eta\Big)d\theta$$

which contradicts (3). Whence $\mu = 0$ i.e $x(t) \to 0$ as $t \to \infty$

Now, we state and prove our main results.
Theorem 1. Suppose that the conditions (Λ_1)-(Λ_5) hold and $\exists$ a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[\frac{(s-T)^3\rho(s)\Phi(s)M}{s^{2(1-\alpha)}} - \frac{s^{1-\alpha}(\rho'(s))^2 b(s)}{4\rho(s)}\Big]ds = \infty \qquad (4)$$

then every solution of (1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.
Proof. Suppose that (1) has a nonoscillatory solution $(x(t), y(t), z(t))$ on $[t_0, \infty)$. From lemma 2, $x(t)$ is always nonoscillatory. Without loss of generality, we shall assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T \geq t_0$. Suppose also that case (Λ_a) of lemma 3 holds for $t \geq t_1$.
Define

$$\omega(t) = \rho(t)\frac{b(t)D^\alpha[a(t)D^\alpha x(t)]}{a(t)D^\alpha x(t)} \qquad (5)$$

Thus, $\omega(t) > 0$, for $t \geq t_1$

$$D^\alpha\omega(t) = \frac{b(t)D^\alpha[a(t)D^\alpha x(t)]}{a(t)D^\alpha x(t)}D^\alpha\rho(t)+\rho(t)\frac{D^\alpha[b(t)D^\alpha[a(t)D^\alpha x(t)]]}{a(t)D^\alpha x(t)}-b(t)[\frac{D^\alpha[a(t)D^\alpha x(t)]}{a(t)D^\alpha x(t)}]^2 \tag{6}$$

using (2), (5), assumption Λ_5 and lemma 1 respectively in (6), we have

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\rho(t)(t-T)^3}{t^{1-\alpha}}M\Phi(t) - \frac{\omega^2(t)}{t^{1-\alpha}\rho(t)b(t)} \tag{7}$$

where $\Phi(t) = k(t)c(t) - k_1(t)\gamma(t) - k_2(t)\xi_*'(t)$
simplifying the above inequality and later integrating from t to t_0, we get

$$\omega(t) \leq \omega(t_0) - \int_{t_0}^{t}\left[\frac{(s-T)^3\rho(s)}{s^{2(1-\alpha)}}M\Phi(s) - \frac{s^{1-\alpha}(\rho'(s))^2b(s)}{4\rho(s)}\right]ds$$

taking the lim sup as $t \to \infty$, we have

$$\limsup_{t\to\infty}\omega(t) \leq -\infty$$

which contradicts (4), the proof is complete.
Corollary 1. If conditions (Λ_1)-(Λ_5) hold such that (4) is replaced by

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{(s-T)^3\rho(s)}{s^{2(1-\alpha)}}M\Phi(s)ds = \infty \tag{8}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{s^{1-\alpha}(\rho'(s))^2b(s)}{4\rho(s)}ds < \infty \tag{9}$$

Then, every solution of (1) is either oscillatory or $\lim_{t\to\infty}x(t) = 0$.

In what follows, we introduce a class of functions G. Let

$$D_0 = \{(t,s) : t > s \geq t_0\}; D = \{(t,s) : t \geq s \geq t_0\}$$

The function $H \in C(D, \Re)$ is said to belong to the class G, if
Λ_6 - $H(t,t) = 0$ for $t \geq t_0$; $H(t,s) > 0$ for $(t,s) \in D_0$
Λ_7 - $H(t,s)$ has a continuous and non-positive partial derivative

$$\frac{\partial H(t,s)}{\partial s} \quad and \quad h(t,s) = \frac{\partial H(t,s)}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}$$

Theorem 2. Suppose that the conditions (Λ_1)-(Λ_7) hold and $\exists$ a differentiable function $\rho : [t_0,\infty) \to (0,\infty)$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{(s-T)^3H(t,s)\rho(s)\Phi(s)M}{s^{2(1-\alpha)}} - \frac{s^{1-\alpha}\rho(s)b(s)h^2(t,s)}{4H(t,s)}\right]ds = \infty \tag{10}$$

Then every solution of (1) is either oscillatory or $\lim_{t\to\infty}x(t) = 0$.
Proof. Suppose that (1) has a nonoscillatory solution $(x(t), y(t), z(t))$ on $[t_0,\infty)$. Following the proof of Theorem 1, we obtain (7). Multiplying (7) by $H(t,s)$ and integrate from t_0 to t, we get

$$\int_{t_0}^{t}H(t,s)\omega'(s)ds \leq \int_{t_0}^{t}H(t,s)\frac{\rho'(s)}{\rho(s)}\omega(s)ds - \int_{t_0}^{t}H(t,s)\frac{\rho(s)(s-T)^3}{s^{1-\alpha}}M\Phi(s)ds$$
$$-\int_{t_0}^{t}H(t,s)\frac{\omega^2(s)}{s^{1-\alpha}\rho(s)b(s)}ds$$

simplifying the above inequality, we arrive at

$$\int_{t_0}^{t}\left[\frac{(s-T)^3H(t,s)\rho(s)\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}\rho(s)b(s)h^2(t,s)}{4H(t,s)}\right]ds\leq H(t,t_0)\omega(t_0)$$

dividing the inequality above by $H(t,t_0)$ and taking the lim sup as $t\to\infty$, we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{(s-T)^3H(t,s)\rho(s)\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}\rho(s)b(s)h^2(t,s)}{4H(t,s)}\right]ds\leq\omega(t_0)<\infty$$

which contradicts (10), the proof is complete.

Corollary 2. If conditions (Λ_1)-(Λ_7) hold such that (10) is replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{(s-T)^3H(t,s)\rho(s)\Phi(s)M}{s^{2(1-\alpha)}}ds=\infty \tag{11}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{s^{1-\alpha}\rho(s)b(s)h^2(t,s)}{4H(t,s)}ds<\infty \tag{12}$$

Then, every solution of (1) is either oscillatory or $\lim_{t\to\infty}x(t)=0$.

Let $H(t,s)=(E(t)-E(s))^\beta$, such that

$$E(t)=\int_{t_0}^{t}\frac{ds}{e(s)}\ \textit{and}\ \limsup_{t\to\infty}E(t)=\infty$$

for a positive constant $\beta>1$. Then we have the following result.

Theorem 3. Suppose that the conditions (Λ_1)-(Λ_7) hold and $\exists$ a differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ such that

$$\limsup_{t\to\infty}\frac{1}{(E(t)-E(t_0))^\beta}\int_{t_0}^{t}(E(t)-E(s))^\beta\rho(s)\times$$
$$\left[\frac{(s-T)^3\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}b(s)}{4}\left(\frac{\beta}{e(s)(E(t)-E(s))}-\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds=\infty \tag{13}$$

Then every solution of (1) is either oscillatory or $\lim_{t\to\infty}x(t)=0$.

Proof. Following the proof of Theorem 1, we obtain (7). Multiplying (7) by $(E(t)-E(s))^\beta$ and integrate from t_0 to t, we get

$$\int_{t_0}^{t}(E(t)-E(s))^\beta\rho(s)\left[\frac{(s-T)^3\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}b(s)}{4}\left(\frac{\beta}{e(s)(E(t)-E(s))}-\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds$$
$$\leq(E(t)-E(t_0))^\beta\omega(t_0)$$

dividing the above inequality by $(E(t)-E(t_0))^\beta$ and taking the lim sup as $t\to\infty$, we arrive at

$$\limsup_{t\to\infty}\frac{1}{(E(t)-E(t_0))^\beta}\int_{t_0}^{t}(E(t)-E(s))^\beta\rho(s)\times$$
$$\left[\frac{(s-T)^3\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}b(s)}{4}\left(\frac{\beta}{e(s)(E(t)-E(s))}-\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds\leq\omega(t_0)<\infty$$

which contradicts (13), the proof is complete.

Corollary 3. If conditions (Λ_1)-(Λ_7) hold such that (13) is replaced by

$$\limsup_{t\to\infty}\frac{1}{(E(t)-E(t_0))^\beta}\int_{t_0}^{t}(E(t)-E(s))^\beta\rho(s)\frac{(s-T)^3\Phi(s)M}{s^{2(1-\alpha)}}ds=\infty \tag{14}$$

and

$$\limsup_{t\to\infty} \frac{1}{(E(t)-E(t_0))^\beta}\int_{t_0}^{t}(E(t)-E(s))^\beta \rho(s)\frac{s^{1-\alpha}b(s)}{4}\left(\frac{\beta}{e(s)(E(t)-E(s))}-\frac{\rho'(s)}{\rho(s)}\right)^2 ds<\infty \tag{15}$$

Then, every solution of (1) is either oscillatory or $\lim_{t\to\infty} x(t)=0$.

Example

In this section, we give example to show the relevance of our results.

Consider the system of fractional differential equations

$$\left.\begin{aligned} D^{3/5}(x(t)) &= 2\sqrt{t}(y^2(\tfrac{3t}{2})+1)\\ D^{3/5}(y(t)) &= -t^{1/5}z(t)\exp(z(t))+t^{1/4}\\ D^{3/5}(z(t)) &= \tfrac{6}{\sqrt{t}}(\tfrac{x(t)}{x^2(t)+1})+\tfrac{5x^2(t)+3}{t}\\ 0<\alpha &= \tfrac{3}{5}<1,\ \ t_0=1 \end{aligned}\right\} \tag{16}$$

By comparing (1) with (16), we deduce that

$$\left.\begin{aligned} \alpha=3/5,\ \ p(t)=2\sqrt{t},\quad q(t)=t^{1/5},\quad \sigma(t)=\tfrac{3t}{2},\\ r(t)=\tfrac{6}{\sqrt{t}},\quad h(z(t))=z(t)\exp(z(t)),\ \ g(y(\delta(t)))=y^2(\tfrac{3t}{2})+1,\\ \phi(t)=t^{1/4},\ \ \psi(t,x(t))=\tfrac{5x^2(t)+3}{t},\quad f(x(t))=\tfrac{x(t)}{x^2(t)+1} \end{aligned}\right\} \tag{17}$$

Also, we set

$$\left.\begin{aligned} \rho(t)=1,\ \ x(t)=t^2\\ H(t,s)=(t-s)^2,\ \ M=7,\quad z(t)=t \end{aligned}\right\} \tag{18}$$

From (17) and (18), we get

$$\left.\begin{aligned} D^\alpha[h(z(t))]\geq 5t^{2/5}=m'\\ D^\alpha[g(y(\sigma(t)))]\geq 2t^{2/5}=l',\quad \tfrac{\partial H(t,s)}{\partial s}=-2(t-s)\\ D^\alpha[\sigma(t))]\geq 1.5t^{2/5}=l,\quad \rho'(t)=0,\quad a(t)=1/(2\sqrt{t})\\ b(t)=1/(t^{1/5}),\quad \tfrac{\psi(t,x)}{x(t)}\leq 8t=\gamma(t)\\ \tfrac{\phi(t)}{q(t)}<t=\xi(t),\quad \xi'(t)=1,\quad \tfrac{\xi'(t)}{x(t)}=1/t^2\leq t=\xi_*'(t)\\ \tfrac{f(x(t))}{x(t)}\leq \tfrac{t^3}{2}=k(t) \end{aligned}\right\} \tag{19}$$

This implies that

$$\left.\begin{aligned} c(t)=ll'm'r(t)=90t^{-109/250},\quad k_1(t)=ll'm'=15t^{\frac{8}{125}},\quad k_2=ll'=3t^{4/25}\\ V(t)=\Phi(t)=45t^{2.56}+120t^{1.06}-3t^{1.2},\ \ h(t,s)=-2(t-s),\quad h^2(t,s)=4(t-s)^2 \end{aligned}\right\} \tag{20}$$

using condition Λ_4,

$$\int_{t_0}^{\infty}\frac{s^{\alpha-1}}{a(s)}ds=\int_1^\infty s^{-2/5}\times 2s^{1/2}ds=\int_1^\infty 2s^{1/10}ds=\infty$$

$$\int_{t_0}^{\infty}\frac{s^{\alpha-1}}{b(s)}ds=\int_1^\infty s^{-2/5}\times s^{1/5}ds=\int_1^\infty s^{-1/5}ds=\infty$$

By substituting (19) and (20) into (3), we have

$$\int_{t_0}^{\infty}\frac{\theta^{\alpha-1}}{a(\theta)}\left(\int_\theta^\infty\frac{\eta^{\alpha-1}}{b(\eta)}\int_\eta^\infty s^{\alpha-1}V(s)dsd\eta\right)d\theta$$

$$=\int_1^\infty\frac{2\theta^{-2/5}}{1/(\theta^{1/2})}\left(\int_\theta^\infty\frac{\eta^{-2/5}}{1/(\eta^{1/5})}\int_\eta^\infty s^{-2/5}(45s^{2.56}+120s^{1.06}-3s^{1.2})dsd\eta\right)d\theta=\infty$$

because

$$\int_{\eta}^{\infty} s^{-2/5}(45s^{2.56}+120s^{1.06}-3s^{1.2})ds=\infty$$

for $\eta \geq 1$, and so equation (3) holds.

Also substituting (18) - (20) into the left hand side of (4), we have

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{(s-T)^3\rho(s)\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}{\rho'}^2(s)b(s)}{4\rho(s)}\right]ds$$

$$=\limsup_{t\to\infty}\int_{1}^{\infty}7(s-T)^3(s^{-4/5})(45s^{2.56}+120s^{1.06}-3s^{1.2})ds$$

$$=\limsup_{t\to\infty}\int_{1}^{\infty}7(s-T)^3(45s^{1.76}+120s^{0.26}-3s^{0.4})ds=\infty$$

In corollary 1, (8) gives infinity and (9) gives zero.

In the same way, we substitute (17) - (20) into the left hand side of (10), we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{(s-T)^3H(t,s)\rho(s)\Phi(s)M}{s^{2(1-\alpha)}}-\frac{s^{1-\alpha}\rho(s)b(s)h^2(t,s)}{4H(t,s)}\right]ds=\limsup_{t\to\infty}\frac{1}{(t-1)^2}$$

$$\times\int_{1}^{\infty}[7(s-T)^3(t-s)^2(45s^{1.76}+120s^{0.26}-3s^{0.4})-s^{1/5}]ds=\infty$$

In corollary 2, (11) gives infinity and (12) is zero. These show that Theorem 1 and Theorem 2 with their corollaries are satisfied. Hence, every solution of (16) is either oscillatory or tends to zero since (3) holds.

Suppose

$$\beta=2 \quad and \quad e(s)=1, \;\; then \;\; E(t)=t. \tag{21}$$

Substitute (18) - (21) into the left hand side of (13), we arrived at infinity. Similarly in corollary 3, (14) gives infinity and (15) gives zero. These shows that Theorem 3 and its corollary are satisfied. Whence, every solution of (16) is either oscillatory or tends to zero.

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