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Energy decay for a viscoelastic wave equation with space-time potential in $\mathbb{R}^n$

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we consider the following viscoelastic wave equation

$$\begin{cases} u_{tt} - \left(\Delta u - \int\limits_0^t g(t-s)\Delta u(s)ds\right) + b(t,x)u_t + |u|^{p-1}u = 0, & t > 0,\ x \in \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n \end{cases}$$

with space-time dependent potential and where the initial data $u_0(x)$, $u_1(x)$ have compact support. Under suitable assumptions on the relaxation function g and the potential b, we obtain a more general energy decay result using the perturbed energy method.

1. Introduction

In this paper, we are concerned with the energy decay of solutions to viscoelastic wave equations of the form

$$\begin{cases} u_{tt} - \left(\Delta u - \int\limits_0^t g(t-s)\Delta u(s)ds\right) + b(t,x)u_t + |u|^{p-1}u = 0, & t > 0,\quad x \in \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

having space-time dependent potential $b(t,x)$ and a power-type nonlinearity $|u|^{p-1}u$ with

$$1 < p < +\infty \quad (n = 2) \quad \text{and} \quad 2 < p + 1 < \frac{2n}{n-2} \quad (n \geq 3),$$

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where the initial data $u_0(x)$ and $u_1(x)$ belong to appropriate spaces and $u = u(t,x)$. In the case of energy decay in bounded domains, there is an extensive literature on initial boundary value problems of the form (1.1). Messaoudi [13] established a general decay result for the wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in} \quad \Omega \times (0,\infty)$$

which is not necessarily of exponential or polynomial type. Cavalcanti et al. [4] considered a damped wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \quad in \quad \Omega \times (0,\infty) \tag{1.2}$$

for $a : \Omega \to \mathbb{R}^+$ with $a(x) \geq a_0 > 0$ and established exponential decay result when the relaxation function $g(t)$ decays exponentially. Song et al. [18] also considered (1.2) under certain suitable assumptions on a, g and γ and proved energy decay results similar to that of [4,13] using a new perturbed energy technique. See [2,3,11] for related results.

In the case of unbounded domains, where the source and relaxation terms are absent, there is an extensive literature concerning total energy decay to the scalar valued wave equation

$$u_{tt} - \Delta u + b(t,x)u_t = 0 \quad (t,x) \in [0,\infty) \times \mathbb{R}^n. \tag{1.3}$$

Ikehata et al. [10], considered the linear wave equation (1.3) where (u_0, u_1) are compactly supported initial data in the energy space. They obtained polynomial energy decay under suitable assumptions on the potential $b(t,x)$. The result shows that for a potential of the form $V(x) \approx (1+|x|)^{-\alpha}$, $\alpha = 1$ is critical. The reader is referred to [5,6,12,14,15] for related results.

In the presence of an internal source term, Todorova and Yordanov [19] considered the problem

$$u_{tt} - \Delta u + b(t,x)u_t + |u|^{p-1}u = 0 \tag{1.4}$$

where $b(t,x) = b(x) \equiv b_0(1+|x|)^{-\alpha}$ with $\alpha \in [0,1)$ (the subcritical potential case) and obtained total energy decay rates which are almost optimal. By modifying the technique due to Todorova and Yordanov [19], Ikehata and Inoue [8] considered the wave problem (1.4) and obtained total energy decay results in the case when $\alpha = 1$.

In [14], Mochizuki considered the wave problem (1.4) and showed non-decay results for the energy function $E_u(t)$ in the case $b(t,x) \leq b_0(1+|x|)^{-1-\alpha}$ where $\alpha > 0$ (the supercritical potential case). For other related results, see [7,9,12,15,20] and for time dependent potential $b(t,x) \equiv b_0(1+t)^{-1}$ see [17,21,22].

Motivated by the results in the literature, we consider the viscoelastic wave problem (1.1) under suitable conditions on the relaxation function g and the damping potential b. We establish more general decay estimates where the initial data u_0, u_1 are assumed to have compact support in a ball $B(L)$ of radius L about the origin, where L satisfies the condition $\text{supp}\{u_0(x), u_1(x)\} \subset \{|x| \leq L\}$ and where the solution satisfy the finite speed of propagation property;

$$\text{supp } u(t,x) \in B(L+t), \qquad t \in (0,\infty).$$

We achieve these results by introducing weighted functions in order to compensate for the noncompactness that arises from the unboundedness of the domain. Our result improves on the perturbed energy technique for unbounded domains.

2. Preliminaries

In this section, we state some basic assumptions used in this paper. First, we introduce the following notations. $L^q(\mathbb{R}^n)$, $1 \le q \le \infty$, the Lebesgue space with norm $\|\cdot\|_q$ and $W^{i,q}(\mathbb{R}^n)$ the Banach space of functions in $L^q(\mathbb{R}^n)$ with i ($i \in \mathbb{N}$) generalized derivatives and $H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$. Also, we denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbb{R}^n)$.

Lemma 2.1. *(Sobolev, Gagliardo, Nirenberg [1]) Suppose that $1 \le q < n$. If $u \in W^{1,q}(\mathbb{R}^n)$, then $u \in L^{q^*}(\mathbb{R}^n)$ with*

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$$

Moreover, there is a constant $k = k(n,q)$ such that

$$\|u\|_{q^*} \le k\|\nabla u\|_q \qquad \forall u \in W^{1,q}(\mathbb{R}^n).$$

Lemma 2.2. *Let $u(t,x)$ be the solution of (1.1) for $n \ge 3$, then there exists a positive constant K such that*

$$\int |u(t,x)|^2 dx \le K^2(L+t)^2 \int |\nabla u(t,x)|^2 dx. \tag{2.1}$$

Proof. This follows directly from Holder's inequality, Lemma (2.1) and the finite speed of propagation property. Thus, we have

$$\int\limits_{\mathbb{R}^n} |u(t,x)|^2 dx \le \Big[\int\limits_{\mathbb{R}^n} |u(t,x)|^{\frac{2n}{n-2}} dx\Big]^{\frac{n-2}{n}} \Big[\int\limits_{B(t+L)} dx\Big]^{\frac{2}{n}} \le k^2[\omega_n(L+t)^n]^{2/n}\|\nabla u\|_2^2,$$

where $K = K(k,n,\omega_n)$ and ω_n is the volume of the unit sphere in $\mathbb{R}^n$. □

Definition 2.1. We define a weak solution of (1.1) as a function $u(t,x)$ satisfying the following

(i)

$$u \in L^\infty\big(0,T,H^1(\mathbb{R}^n)\big) \cap L^\infty\big(0,T,L^{p+1}(\mathbb{R}^n)\big),$$
$$u_t \in L^\infty\big(0,T,L^2(\mathbb{R}^n)\big), \quad u_{tt} \in L^2\big(0,T,H^{-1}(\mathbb{R}^n)\big)$$

(ii) we have

$$\int\limits_0^t \Big[\langle u_{tt}(s),v\rangle + \langle[\nabla u(s) - \int\limits_0^s g(s-r)\nabla u(r)dr], \nabla v\rangle + \langle b(t,x)u_t(s),v\rangle + \langle |u(s)|^{p-1}u(s),v\rangle\Big] ds = 0$$

for $v \in C_0^\infty([0,T]\times\mathbb{R}^n)$ and *a.e.* $t \in [0,T]$ such that

(iii)

$$u(0) = u_0 \in H^1(\mathbb{R}^n) \quad \text{and} \quad u_t(0) = u_1 \in L^2(\mathbb{R}^n)$$

For the potential $b(t,x)$ and the relaxation function $g(t)$, we state the following assumptions:

(A_1) $\int_{B(L+t)} b(t,x)^{\frac{n}{2}} dx \in L^\infty_{loc}(J_T)$. where $J_T = (0,\infty)$

(A_2) There exists a positive function $b_L(t)$ and a positive constant $c_{\alpha b}$ such that
$b(t,x) \geq b_L(t)$ for $x \in B(L+t)$ and $\alpha_L(t)b_L(t) \geq c_{\alpha b}$ where $\alpha_L(t) = \left[\int_{B(L+t)} b(t,x)^{\frac{n}{2}} dx\right]^{2/n}$

(A_3) g is a differentiable function satisfying

$$g(s) \geq 0, \qquad 1 - \int_0^\infty g(s)ds = \ell > 0 \quad \text{and} \quad g'(s) \leq 0 \qquad \text{for} \quad s \geq 0$$

(A_4) In addition, there exists a positive differentiable function μ satisfying the condition

$$g'(s) \leq -\mu(s)g(s), \quad \mu(s) \geq 0 \quad \text{and} \quad \mu'(s) \leq 0 \quad \text{for} \quad s \geq 0$$

We now define the modified energy functional $E(t)$ associated to problem (1.1) by

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\Big[1 - \int_0^t g(s)ds\Big]\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u) + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \tag{2.2}$$

where for easy representation, we shall use the following notation

$$(g \circ \nabla u) := \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds$$

and hence for the functional $E(t)$, we state the following lemma.

Lemma 2.3. *Suppose that the assumptions (A_1) to (A_4) hold. Let u be a solution of the problem (1.1), then for $t \geq 0$, the energy functional $E(t)$ satisfies*

$$E'(t) \leq -\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}(g' \circ \nabla u) \tag{2.3}$$

and we have

$$E(t) \leq E(0). \tag{2.4}$$

Proof. By multiplying (1.1) by u_t and integrating over $\mathbb{R}^n$, we obtain the estimate (2.3) for any regular solution. Thus by using density arguments, we get the desired result. □

3. Local existence

In this section, we shall discuss the existence of a weak solution to (1.1) in the maximal interval $[0, T]$ for $T < \infty$, using the Galerkin approximation technique.

Theorem 3.1. *Suppose that the assumptions (A_1) - (A_4) hold. Let $2 < p+1 \leq \frac{2n}{n-2}$ for $n \geq 3$, then there exists a unique solution*

$$u \in C([0,T]; H^1(\mathbb{R}^n)) \qquad \textit{and} \qquad u_t \in C([0,T]; L^2(\mathbb{R}^n))$$

with initial data $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ having compact support for T small enough.

The proof of the local existence result is similar to that of [16]. Hence, we will only give a sketch of the proof here.

Sketch of Proof. Assume the sequence $(w_j)_{j\in\mathbb{N}}$ is a basis in $H^1(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$ and consider a weak solution of the form

$$u^n(t) = \sum_{j=1}^{n} a_{jn}(t) w_j \tag{3.1}$$

satisfying the following approximate problem corresponding to (1.1)

$$\langle u^n_{tt}, w_j\rangle + \langle[\nabla u^n - \int_0^t g(t-s)\nabla u^n(s)ds], \nabla w_j\rangle + \langle b(t,x)u^n_t, w_j\rangle + \langle |u^n|^{p-1}u^n, w_j\rangle = 0 \tag{3.2}$$

for $w_j \in H^1(\mathbb{R}^n)$ with initial conditions

$$u^n(0) = u^n_0 \equiv \sum_{j=1}^{n} d_{jn} w_j \to u_0 \quad \text{strongly in} \quad H^1(\mathbb{R}^n) \quad \text{as} \quad n\to\infty \tag{3.3}$$

and

$$u^n_t(0) = u^n_1 \equiv \sum_{j=1}^{n} c_{jn} w_j \to u_1 \quad \text{strongly in} \quad L^2(\mathbb{R}^n) \quad \text{as} \quad n\to\infty, \tag{3.4}$$

where $a_{jn}(t) = \langle u^n(t), w_j\rangle$, $d_{jn} = \langle u^n_0, w_j\rangle$ and $c_{jn} = \langle u^n_1, w_j\rangle$.
We employ the following a priori bounds to obtain existence of solution to (3.2). Setting $w_j = u^n_t(t)$ in (3.2) and integrating over $\mathbb{R}^n$, we obtain the following estimate:

$$\begin{aligned}\frac{d}{dt}\Big[\frac{1}{2}\|u^n_t\|^2 + \frac{1}{2}\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^n\|_2^2 + \frac{1}{2}(g\circ\nabla u^n) + \frac{1}{p+1}\|u^n\|_{p+1}^{p+1}\Big] \\ \le -\int_{\mathbb{R}^n} b(t,x)|u^n_t|^2 dx - \frac{1}{2}g(t)\|\nabla u^n\|_2^2 + \frac{1}{2}(g'\circ\nabla u^n).\end{aligned} \tag{3.5}$$

From assumption (A_3) and (A_4), the second and third terms on the right hand side of (3.5) are negative. Hence, integrating over t for $t\in[0,T]$, gives

$$\begin{aligned}\frac{1}{2}\|u^n_t\|^2 + \frac{1}{2}\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^n\|_2^2 + \frac{1}{2}(g\circ\nabla u^n) + \frac{1}{p+1}\|u^n\|_{p+1}^{p+1} \\ + \int_0^t\int_{\mathbb{R}^n} b(t,x)|u^n_t|^2 dxdt \le \frac{1}{2}\|u^n_t(0)\|^2 + \frac{1}{2}\|\nabla u^n(0)\|_2^2 + \frac{1}{p+1}\|u^n(0)\|_{p+1}^{p+1},\end{aligned} \tag{3.6}$$

and there exists a positive constant C^* independent of n such that

$$\|u^n_t\|^2 \le C^*, \quad \|\nabla u^n\|_2^2 \le C^*, \quad \|u^n\|_{p+1}^{p+1} \le C^*, \quad \int_0^t\int_{\mathbb{R}^n} b(t,x)|u^n_t|^2 dxds \le C^*. \tag{3.7}$$

In the sequel, we denote by C_i^* (i=0, 1, 2) positive constants independent of n. Set $v = w_j$ in (3.2) to get

$$|\langle u_{tt}^n, v\rangle| \le |\langle[\nabla u^n - \int_0^t g(t-s)\nabla u^n(s)ds], \nabla v\rangle| + |\langle b(t,x)u_t^n, v\rangle| + |\langle |u^n|^{p-1}u^n, v\rangle|. \tag{3.8}$$

For the second term on the right hand side of (3.8), using Holder and Sobolev inequalities and assumption (A_1), we have the following estimate

$$\begin{aligned} |\langle b(t,x)u_t^n, v\rangle| &\le k\left[\Big(\int_{B(R+t)} |b(t,x)|^{\frac{n}{2}}dx\Big)^{\frac{2}{n}} \int_{\mathbb{R}^n} b(t,x)|u_t^n|^2 dx\right]^{\frac{1}{2}} \|\nabla v\|_2 \\ &\le C_0^*\Big[\int_{\mathbb{R}^n} b(t,x)|u_t^n|^2 dx\Big]^{\frac{1}{2}} \|\nabla v\|_2, \end{aligned} \tag{3.9}$$

and for the third term on the right hand side of (3.8), using Holder's inequality, we obtain

$$|\langle |u^n|^{p-1}u^n, v\rangle| \le C_1^*\|u^n\|_{p+1}^p \|\nabla v\|_2. \tag{3.10}$$

Substituting the estimates (3.9), (3.10) in (3.8) and using (3.7) together with Holder's inequality and assumption (A_3), it can be shown that

$$\int_0^t \|u_{tt}^n(s)\|_{H^{-1}}^2 \, ds \le C_2^*. \tag{3.11}$$

Therefore, for any $T > 0$, the nonlinear terms are uniformly bounded on $[0,T]$ and it follows that the solution $u^n(t)$ of (3.2) exists on $[0,T]$ for each n. The other details follow as in [16].

4. General decay

In this section, we consider the decay of the energy of solution to (1.1). To achieve this, we introduce the following functionals

$$M(t) := \int_{\mathbb{R}^n} uu_t \, dx$$

and

$$L(t) := \beta(t)E(t) + \nu_1\rho(t)M(t) \tag{4.1}$$

where ν_1 is a positive constant to be determined later and β, ρ are positive functions depending on the support radius L and satisfying the following:

(A_5) $0 < \beta(t)$, $\beta(t) \ge \rho(t)\alpha_L(t) \ge \rho(t)\alpha_L(0)$,
(A_6) There exist positive functions η_L and γ_L satisfying
- (i) $\eta_L(t)(L+t)^2 \le \alpha_L(t)$ and $[\eta_L(t)(L+t)^2]^{-1} \le c_\eta$
- (ii) $\gamma_L(t)(L+t)^2 \le \alpha_L(t)b_L(t)$ and $\frac{1}{\gamma_L(t)}\left[\frac{\beta(t)}{\rho(t)}\left|\left(\frac{\rho(t)}{\beta(t)}\right)'\right|\right]^2 \le c_{\rho\beta}^2$.

Lemma 4.1. *Suppose that the assumptions* $(A_5) - (A_6)$ *hold, then there exist positive constants* k_1 *and* k_2 *such that the relation*

$$k_1\beta(t)E(t) \leq L(t) \leq k_2\beta(t)E(t) \tag{4.2}$$

is satisfied.

Proof. Using Holder's, Sobolev's and Young's inequalities and the assumptions (A_5), (A_6)(i), we obtain the following estimate

$$\begin{aligned}
|L(t) - \beta(t)E(t)| \leq & \nu_1\rho(t)\int_{\mathbb{R}^n} |uu_t|dx \\
\leq & \frac{\nu_2}{2}\rho(t)\eta_L(t)(L+t)^2\|u_t\|^2 + \frac{\nu_1 k^2}{2\eta_L(t)}\rho(t)\|\nabla u\|^2 \\
\leq & \rho(t)\eta_L(t)(L+t)^2\Big[\frac{\nu_2}{2}\|u_t\|^2 + \frac{\nu_1 k^2}{2(L+t)^2\eta_L^2(t)}\|\nabla u\|^2\Big] \\
\leq & \rho(t)\eta_L(t)(L+t)^2\Big[\frac{\nu_2}{2}\|u_t\|^2 + \frac{\nu_1 k^2 c_\eta}{2}\|\nabla u\|^2\Big] \\
\leq & \frac{k^*\beta(t)}{2}\Big[\|u_t\|^2 + \ell\|\nabla u\|^2\Big] \leq k^*\beta(t)E(t)
\end{aligned}$$

where $k^* = \max\{\nu_2,\ \frac{\nu_1}{\ell}k^2c_\eta\}$ and $\nu_2 = \nu_2(\omega_n, \nu_1)$. $\square$

Now, we state the following lemma on the functional $M(t)$ above.

Lemma 4.2. *Let* u *be a solution of the problem (1.1), suppose that the assumptions* (A_2) *to* (A_3) *hold, then the functional* $M(t)$ *satisfies the following inequality*

$$\begin{aligned}
[\rho(t)M(t)]' \leq & \rho(t)\|u_t\|^2 - \rho(t)\Big(\ell - \frac{[1+k^2]}{2}\Big)\|\nabla u\|^2 + \frac{1}{2}\alpha_L(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx \\
& + \frac{1-\ell}{2}\rho(t)(g\circ\nabla u) - \rho(t)\|u\|_{p+1}^{p+1} + \rho'(t)M(t)
\end{aligned} \tag{4.3}$$

Proof. Differentiating $M(t)$ and using (1.1), we obtain

$$\begin{aligned}
M'(t) = & \|u_t\|^2 - \|\nabla u\|^2 + \int_0^t g(t-s)\int_{\mathbb{R}^n} \nabla u(s)\nabla u(t)dxds - \int_{\mathbb{R}^n} b(t,x)u_t u dx \\
& - \|u\|_{p+1}^{p+1}
\end{aligned} \tag{4.4}$$

and multiplying this by $\rho(t)$ gives

$$\begin{aligned}
[\rho(t)M(t)]' = & \rho(t)\|u_t\|^2 - \rho(t)\|\nabla u\|^2 + \rho(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} \nabla u(s)\nabla u(t)dxds \\
& - \rho(t)\int_{\mathbb{R}^n} b(t,x)u_t u dx - \rho(t)\|u\|_{p+1}^{p+1} + \rho'(t)M(t).
\end{aligned} \tag{4.5}$$

For the third term on the right hand side of (4.5), using Young's inequality, we obtain

$$\begin{aligned}
&\int_0^t g(t-s)\int_{\mathbb{R}^n} \nabla u(s)\nabla u(t)dxds \\
&\quad\leq \int_0^t g(t-s)\int_{\mathbb{R}^n} |\nabla u(s)-\nabla u(t)||\nabla u(t)|dxds + \int_0^t g(s)ds\|\nabla u\|^2 \\
&\quad\leq (\frac{1}{2}+\int_0^t g(s)ds)\|\nabla u\|^2 + \frac{1}{2}\int_0^t g(s)ds\big(g\circ\nabla u\big).
\end{aligned} \tag{4.6}$$

For the fourth term on the right hand side of (4.5), using (2.1), Holder's, Young's and Sobolev's inequalities, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} b(t,x)u_t u dx \\
&\leq \Big[\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx\Big]^{\frac{1}{2}}\left[\Big[\int_{\mathbb{R}^n}|u|^{\frac{2n}{n-2}}dx\Big]^{\frac{n-2}{n}}\Big[\int_{B(L+t)} b(t,x)^{\frac{n}{2}}dx\Big]^{\frac{2}{n}}\right]^{\frac{1}{2}} \\
&\leq \frac{1}{2}\alpha_L(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx + \frac{k^2}{2}\|\nabla u\|^2.
\end{aligned} \tag{4.7}$$

Substituting the estimates (4.6) -(4.7) into (4.5) gives the following

$$\begin{aligned}
[\rho(t)M(t)' \leq& \rho(t)\|u_t\|^2 - \rho(t)\Big(\ell - \frac{[1+k^2]}{2}\Big)\|\nabla u\|^2 + \frac{1}{2}\rho(t)\alpha_L(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx \\
&+\frac{(1-\ell)}{2}\rho(t)(g\circ\nabla u) - \rho(t)\|u\|_{p+1}^{p+1} + \rho'(t)M(t),
\end{aligned}$$

where $\int_0^t g(s)ds \leq \int_0^\infty g(s)ds = 1-\ell$. □

We now present the main result on decay of the energy.

Theorem 4.1. *Suppose that the assumptions (A_2) to (A_6) hold and let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. Then,*

(i) *if $\alpha_L(t)\mu(t) \geq c_\mu$, $c_\mu > 0$, the energy of solution to (1.1) satisfies*

$$E(t) \leq k_3E(0)exp\Big(-\frac{k^*}{k_2}\int_0^t \frac{\rho(s)}{\beta(s)}ds\Big) \qquad \forall t \geq 0 \tag{4.8}$$

for positive constants k^*, k_2 and k_3 to be determined later.

(ii) *if $\alpha_L(t)\mu(t) < c_\mu$, $c_\mu > 0$, the energy of solution to (1.1) satisfies*

$$E(t) \leq k_{10}E(0)exp\Big(-\frac{k^*}{k_8}\int_0^t \frac{\rho(s)\mu(s)}{\beta(s)}ds\Big) \qquad \forall t \geq 0 \tag{4.9}$$

for positive constants k^*, k_8 and k_{10} to be determined later.

Proof. Case (i): Multiplying the estimate in (2.3) by $\beta(t)$, we obtain

$$[\beta(t)E(t)]' \leq -\beta(t)\int_0^t b(t,x)|u_t|^2dx - \frac{\beta(t)}{2}g(t)\|\nabla u\|_2^2 + \frac{\beta(t)}{2}(g' \circ \nabla u) \tag{4.10}$$

Combining the estimates from (4.3) and (4.10) gives the following:

$$\begin{aligned} L'(t) \leq &- \Big[\beta(t) - \frac{\nu_1}{2}\rho(t)\alpha_L(t)\Big]\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx + \nu_1\rho(t)\|u_t\|^2 \\ &- \nu_1\rho(t)\Big[\ell - \frac{[1+k^2]}{2}\Big]\|\nabla u\|^2 - \frac{\beta(t)}{2}g(t)\|\nabla u\|_2^2 - \nu_1\rho(t)\|u\|_{p+1}^{p+1} \\ &+ \frac{\beta(t)}{2}(g' \circ \nabla u) + \frac{\nu_1(1-\ell)}{2}\rho(t)(g \circ \nabla u) + \beta'(t)E(t) + \nu_1\rho'(t)M(t). \end{aligned} \tag{4.11}$$

Estimating the last two terms on the right hand side of (4.11) in terms of $L(t)$ and $M(t)$, we get

$$\beta'(t)E(t) + \nu_1\rho'(t)M(t) = \frac{\beta'(t)}{\beta(t)}L(t) + \nu_1\beta(t)\Big[\frac{\rho(t)}{\beta(t)}\Big]'M(t) \tag{4.12}$$

and for the term $\nu_1\beta(t)\big[\frac{\rho(t)}{\beta(t)}\big]'M(t)$, using Holder's inequality, Young's inequality, Sobolev's inequality and assumption (A_6)(ii), we have the following estimate

$$\begin{aligned} &\nu_1\beta(t)\Big|\Big(\frac{\rho(t)}{\beta(t)}\Big)'\Big|\int_{\mathbb{R}^n}|u_tu|dx \\ &\leq \frac{\nu_2}{2}\rho(t)\gamma_L(t)(L+t)^2\|u_t\|^2 + \frac{\nu_1k^2}{2\gamma_L(t)}\Big[\frac{\beta(t)}{\rho(t)}\Big|\Big(\frac{\rho(t)}{\beta(t)}\Big)'\Big|\Big]^2\rho(t)\|\nabla u\|^2 \\ &\leq \frac{\nu_2}{2}\rho(t)\alpha_L(t)b_L(t)\|u_t\|^2 + \frac{\nu_1k^2c_{\rho\beta}^2}{2}\rho(t)\|\nabla u\|^2. \end{aligned} \tag{4.13}$$

Substituting (4.13) in (4.12) and the resulting estimate in (4.11), we employ assumption (A_5) together with assumption (A_4) to obtain

$$\begin{aligned} L'(t) \leq &- \rho(t)\Big[\big[1 - \frac{(\nu_1+\nu_2)}{2}\big]\alpha_L(t)b_L(t) - \nu_1\Big]\|u_t\|^2 \\ &- \nu_1\rho(t)\Big[\ell - \frac{[1+(1+c_{\rho\beta}^2)k^2]}{2}\Big]\|\nabla u\|^2 - \nu_1\rho(t)\|u\|_{p+1}^{p+1} \\ &- \rho(t)\Big[\frac{\alpha_L(t)\mu(t)}{2} - \frac{\nu_1(1-\ell)}{2}\Big](g \circ \nabla u) + \frac{\beta'(t)}{\beta(t)}L(t) \end{aligned} \tag{4.14}$$

Choosing ν_1, ν_2 small enough such that $\frac{(\nu_1+\nu_2)}{2} < 1$, and applying the condition $\alpha_L(t)\mu(t) \geq c_\mu$ together with assumption (A_2), then (4.14) reduces to

$$\begin{aligned} L'(t) \leq & -\rho(t)\Big[\big[1-\frac{(\nu_1+\nu_2)}{2}\big]c_{\alpha b}-\nu_1\Big]\|u_t\|^2 \\ & -\nu_1\rho(t)\Big[\ell-\frac{[1+(1+c_{\rho\beta}^2)k^2]}{2}\Big]\|\nabla u\|^2-\nu_1\rho(t)\|u\|_{p+1}^{p+1} \\ & -\rho(t)\Big[\frac{c_\mu}{2}-\frac{\nu_1(1-\ell)}{2}\Big](g\circ\nabla u)+\frac{\beta'(t)}{\beta(t)}L(t) \end{aligned} \tag{4.15}$$

Hence, there exists a positive constant k^* satisfying

$$\begin{aligned} &\big[1-\frac{(\nu_1+\nu_2)}{2}\big]c_{\alpha b}-\nu_1 \geq \frac{k^*}{2}, \quad \frac{c_\mu}{2}-\frac{\nu_1(1-\ell)}{2} \geq \frac{k^*}{2} \\ &\nu_1 \geq \frac{k^*}{p+1} \quad \text{and} \quad \nu_1\Big[\ell-\frac{[1+(1+c_{\rho\beta}^2)k^2]}{2}\Big] \geq \frac{\ell k^*}{2} \end{aligned}$$

such that (4.15) yields

$$\begin{aligned} L'(t) \leq & -k^*\rho(t)\Big(\frac{1}{2}\big[\|u_t\|^2+\ell\|\nabla u\|_2^2+(g\circ\nabla u)\big]+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big)+\frac{\beta'(t)}{\beta(t)}L(t) \\ \leq & -k^*\rho(t)E(t)+\frac{\beta'(t)}{\beta(t)}L(t). \end{aligned}$$

By the use of the integrating factor $\frac{1}{\beta(t)}$ and the estimate (4.2), we get

$$\Big[\frac{1}{\beta(t)}L(t)\Big]' \leq -\frac{k^*\rho(t)}{k_2[\beta(t)]^2}L(t). \tag{4.16}$$

Define G(t) by

$$G(t) := \frac{1}{\beta(t)}L(t), \tag{4.17}$$

then (4.16) reduces to

$$G'(t) \leq -\frac{k^*\rho(t)}{k_2\beta(t)}G(t) \tag{4.18}$$

and integrating (4.17) over $[0,t]$ gives

$$G(t) \leq G(0)exp\Big(-\frac{k^*}{k_2}\int_0^t \frac{\rho(s)}{\beta(s)}ds\Big). \tag{4.19}$$

From (4.17) and the estimate (4.2), we observe that $G(t)$ is equivalent to $E(t)$. Hence we have that the energy decay estimate is given by

$$E(t) \leq k_3E(0)exp\Big(-\frac{k^*}{k_2}\int_0^t \frac{\rho(s)}{\beta(s)}ds\Big) \qquad \forall t \geq 0. \quad \square$$

Proof. Case (ii): From (4.14), if we choose ν_1 and ν_2 small enough such that $\frac{(\nu_1+\nu_2)}{2} < 1$, then there exists a positive constant k^* satisfying

$$\Big[1-\frac{(\nu_1+\nu_2)}{2}\Big]c_{\alpha b}-\nu_1\geq\frac{k^*}{2},\quad \nu_1\geq\frac{k^*}{p+1}$$
$$\frac{\nu_1(1-\ell)}{2}\leq\frac{k^*}{2}\quad\text{and}\quad \nu_1\Big[\ell-\frac{[1+(1+c^2_{\rho\beta})k^2]}{2}\Big]\geq\frac{\ell k^*}{2}$$

such that (4.14) yields

$$\begin{aligned}L'(t)\leq&-k^*\rho(t)\Big[\frac{1}{2}\big[\|u_t\|^2+\ell\|\nabla u\|_2^2\big]+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big]\\&+\frac{k^*}{2}\rho(t)(g\circ\nabla u)+\frac{\beta'(t)}{\beta(t)}L(t).\end{aligned}\tag{4.20}$$

Using the energy estimate (2.2) in (4.20), we obtain

$$L'(t)\leq -k^*\rho(t)E(t)+k^*\rho(t)(g\circ\nabla u)+\frac{\beta'(t)}{\beta(t)}L(t)\tag{4.21}$$

and simplifying (4.21) using the integrating factor $\frac{1}{\beta(t)}$, we get

$$\Big[\frac{1}{\beta(t)}L(t)\Big]'\leq-\frac{k^*\rho(t)}{\beta(t)}E(t)+\frac{k^*\rho(t)}{\beta(t)}(g\circ\nabla u).\tag{4.22}$$

Now, multiplying (4.22) by $\mu(t)$ and using (2.3) together with assumption (A_4), gives

$$\Big[\frac{\mu(t)}{\beta(t)}L(t)\Big]'\leq-\frac{k^*\rho(t)\mu(t)}{\beta(t)}E(t)-\frac{k_7\rho(t)}{\beta(t)}E'(t).\tag{4.23}$$

Since $E'\leq 0$ and from assumption (A_5), we have $c_{\alpha_0}=c(\alpha_L(0))$ such that $\frac{\rho(t)}{\beta(t)}\leq c_{\alpha_0}$. By rearrangement (4.23) gives

$$\Big[\frac{\mu(t)}{\beta(t)}L(t)+c_{\alpha_0}k_7E(t)\Big]'\leq-\frac{k^*\rho(t)\mu(t)}{\beta(t)}E(t).\tag{4.24}$$

Define $F(t)$ by

$$F(t):=\frac{\mu(t)}{\beta(t)}L(t)+c_{\alpha_0}k_7E(t),\tag{4.25}$$

then from (4.25) and (4.2), we observe that

$$F(t)\leq k_2\mu(t)E(t)+c_{\alpha_0}k_7E(t)\leq k_8E(t),\tag{4.26}$$

where $k_8=\max\{k_2\mu(0),c_{\alpha_0}k_7\}$. Likewise,

$$F(t)\geq k_1\mu(t)E(t)+c_{\alpha_0}k_7E(t)\geq k_9E(t),\tag{4.27}$$

where $k_9=c_{\alpha_0}k_7$. Using the estimate (4.26), then (4.24) reduces to

$$F'(t)\leq-\frac{k^*\rho(t)\mu(t)}{k_8\beta(t)}F(t).\tag{4.28}$$

Integrating (4.28) over $(0,t)$, we obtain the following estimate

$$F(t) \leq F(0)exp\Big(-\frac{k^*}{k_8}\int_0^t \frac{\rho(s)\mu(s)}{\beta(s)}ds\Big) \tag{4.29}$$

and from (4.26) and (4.27), we obtain

$$E(t) \leq \frac{k_8}{k_9}E(0)exp\Big(-\frac{k^*}{k_8}\int_0^t \frac{\rho(s)\mu(s)}{\beta(s)}ds\Big) \leq k_{10}E(0)exp\Big(-\frac{k^*}{k_8}\int_0^t \frac{\rho(s)\mu(s)}{\beta(s)}ds\Big) \qquad \forall t \geq 0. \quad \square \tag{4.30}$$

5. Application

Assume that the function $b(t,x)$ is of the form $b(t,x) \approx b_0(1+t)^{-\beta_1}(1+|x|)^{-\alpha_1}$ for $0 \leq \alpha_1 + \beta_1 \leq 1$. Then, we have

$$\alpha_L(t) \approx C_1(L+t)^{2-(\alpha_1+\beta_1)}. \tag{5.1}$$

From assumption (A_2), for $x \in B(L+t)$, we have $b_L(t) \approx C_2(L+t)^{-(\alpha_1+\beta_1)}$ and

$$\alpha_L(t)b_L(t) \approx C_3(1+t)^{2-2(\alpha_1+\beta_1)} \geq C_3(L)^{2-2(\alpha_1+\beta_1)} = c_{\alpha b} \tag{5.2}$$

From assumption (A_5),

$$\frac{\rho(t)}{\beta(t)} \approx C_4(L+t)^{-2+(\alpha_1+\beta_1)} \tag{5.3}$$

and $\frac{\beta(t)}{\rho(t)} \approx C_4^0(L+t)^{2-(\alpha_1+\beta_1)} \geq C_4^0 L^{2-(\alpha_1+\beta_1)}$.
From assumption (A_6), $\eta_L(t) \approx (L+t)^{-\alpha_1-\beta_1}$ and so

$$\frac{1}{(L+t)^2\eta_L^2(t)} \approx (L+t)^{-2+2(\alpha_1+\beta_1)} \leq L^{-2+2(\alpha_1+\beta_1)}. \tag{5.4}$$

Since $\gamma_L(t)(L+t)^2 \leq \alpha_L(t)b_L(t)$ and $\Big[\frac{\beta(t)}{\rho(t)}\Big|\Big(\frac{\rho(t)}{\beta(t)}\Big)'\Big|\Big]^2 \approx (L+t)^{-2}$, we have

$$\gamma_L(t) \approx (1+t)^{-2(\alpha_1+\beta_1)} \quad \text{and} \quad \frac{1}{\gamma_L(t)}\Big[\frac{\beta(t)}{\rho(t)}\Big|\Big(\frac{\rho(t)}{\beta(t)}\Big)'\Big|\Big]^2 \approx (L+t)^{-2+2(\alpha_1+\beta_1)} \leq c_{\rho\beta}^2 \tag{5.5}$$

In the case of polynomial growth of g, for $g(t) \approx (1+t)^{-a}$, we have from assumption (A_4) that $\mu(t) \approx C_1^0(1+t)^{-1}$ and the condition

$$\alpha_L(t)\mu(t) \approx C_1(1+t)^{1-(\alpha_1+\beta_1)} \geq c_\mu. \tag{5.6}$$

Consequently, we have from Theorem 4.1(i) and (5.3) that

$$E(t) \leq K_5E(0)exp\big(-C_5(1+s)^{-1+\alpha_1+\beta_1}\big) \qquad \forall t \geq 0, \tag{5.7}$$

which yields a polynomial decay in the case $\alpha_1 + \beta_1 = 1$.
Observe that the assumptions (A_2) and (A_6) are satisfied for $\alpha_1 + \beta_1 \leq 1$ but fail for $\alpha_1 + \beta_1 > 1$.

In the case $g(t) \approx \delta_1(1+t)^{-(1+\epsilon)}e^{\delta_*(1+t)^{-\epsilon}}$ with δ_1, δ^* small enough such that assumption (A_3) is satisfied, we have from assumption (A_4) that $\mu(t) \approx C_6(1+t)^{-1-\epsilon}$ and so $\alpha_L(t)\mu(t) \approx C_6(1+t)^{1-(\alpha_1+\beta_1+\epsilon)}$. If in addition to $0 \leq \alpha_1 + \beta_1 \leq 1$, we have $\alpha_1 + \beta_1 + \epsilon > 1$, then there exists a positive constant c_μ such that $\alpha_L(t)\mu(t) < c_\mu$. Consequently, from Theorem 4.1(ii), the energy decay takes the form:

$$E(t) \leq K_6 E(t_0) exp\big(-C_6^0 (1+s)^{-2-\epsilon+\alpha_1+\beta_1}\big) \qquad \forall t \geq 0. \tag{5.8}$$

Remark 5.1. Let $b(t,x)$ be as defined earlier, then

(i) within the ball $\{|x| \leq L\}$, $\alpha_L(t) \approx C_7$, so from Theorem 4.1(i) we have

$$E(t) \leq K_8 E(0) exp\big(-C_8 t\big) \qquad \forall t \geq 0$$

and in the case of polynomial growth of g $(\mu(t) \approx \delta_2 (1+t)^{-1})$, a consequence of Theorem 4.1(ii) is that

$$E(t) \leq K_9 E(0) exp\Big(-C_9 \int_0^t \mu(s) ds\Big) \qquad \forall t \geq 0.$$

(ii) within the ball $\{|x| \leq (L+t)^{\frac{1+\beta_1}{2-\alpha_1}}\}$, we observe that the solution has a polynomial energy decay for $\alpha_1 + \beta_1 \in [0,1]$ and $\mu \approx constant$, since $\alpha_L(t) \approx C_1(L+t)$. In addition, when $\mu(t) \approx (1+t)^{-1}$ the condition $\alpha_L(t)\mu(t) \geq c_\mu$ is satisfied and the energy decay remains in the polynomial form.

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