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Variability of permeability with diameter of conduit

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Abstract. An entry length is always observed before laminar flow is achieved in fluid flowing in a conduit. This depends on the Reynolds number of the flow and the degree of smoothness of the conduit. This work examined this region and the point where laminar flow commences in the context of flow through conduit packed with porous material like beads, of known porosity. Using some theoretical assumptions, it is demonstrated that permeability varies from zero at wall–fluid boundary to maximum at mid-stream, creating a permeability profile similar to the velocity profile. An equation was obtained to establish this. We also found that peak values of permeability increase with increasing porosity, and therefore entry length increases with increasing porosity with all other parameters kept constant. A plot of peak permeability versus porosity revealed that they are linearly related.

Keywords. Laminar flow; Reynolds number; porosity; entry length; permeability.

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1. Introduction

At higher Reynolds numbers, the Poiseuille flow theory applies only after some distance down the pipe. The fluid is unlikely to enter the conduit with the appropriate parabolic velocity profile [1]. If the flow enters the pipe from a reservoir through a well-rounded entrance, the velocity at first is almost uniform over the cross-section. The wall shear stress (as the velocity must be zero at the wall) will slow down the fluid near the wall. As a consequence of continuity, the velocity must then increase in the central region. The transition/entry length for the characteristic parabolic velocity distribution to develop is a function of the Reynolds number [2]. Consequently, there is an entry length in which the flow tends towards the parabolic profile. At low Reynolds numbers, this is so short that it can be ignored [3]. But it is found both experimentally and theoretically that as the Reynolds number is increased, this is no longer true [3]. The details of the entry length depend of course on the actual velocity profile at entry, which in turn depends on the detailed geometry of the reservoir and its connection to the conduit. However, an important case which this work is focused on is the situation in which the fluid enters with uniform speed over the whole cross-section, such that there exist zero flow at the fluid–wall boundary and the velocity increases across the mid-stream with the distance of flow. Because of the no-slip condition, the fluid next to the wall must immediately be slowed down. This retardation spreads inward, whilst fluid at the centre must move faster, so that the average speed remains the same and mass is conserved. The fluid flows over a certain length of the pipe before a complete parabolic curve is formed. One thus gets a sequence of velocity profiles. Ultimately, the parabolic profile is approached and from there onwards the Poiseuille flow theory applies.

If the conduit is now packed/filled with porous material, say, beads or sand, we assert that there exists a parabolic profile but this time a permeability parabolic profile. It is necessary to say that Hagen–Poiseuille equation only applies to flow in conduit that is not filled with porous material. But, by introducing the dimensionless parameter Φ, i.e. porosity, to the Hagen–Poiseuille equation, it can then be applied to a situation where the conduit is filled with porous material. With this modification we can now make an assumption that Hagen–Poiseuille equation is equivalent to Darcy equation.

If the velocity changes followed a parabolic profile across a unit cross-sectional area, it is reasonable to think that the permeability k, of the porous medium should follow the same parabolic profile, i.e. the value should not be linear across the cross-section. If a porous system is conceived to be a bundle of capillary tubes of equal radii and length [4], the permeability k is expected to increase from zero from the wall–fluid boundary towards the centre of the flow.

2. Theoretical background

Limiting Navier–Stokes equations to incompressible fluids, we get

$$-\frac{1}{\rho}\frac{\partial}{\partial x}(p+\gamma h)+v\nabla^2 u=\frac{\mathrm{d}u}{\mathrm{d}t}, \quad (1)$$

$$-\frac{1}{\rho}\frac{\partial}{\partial y}(p+\gamma h)+v\nabla^2 v=\frac{\mathrm{d}v}{\mathrm{d}t}, \quad (2)$$

$$-\frac{1}{\rho}\frac{\partial}{\partial z}(p+\gamma h)+v\nabla^2 w=\frac{\mathrm{d}w}{\mathrm{d}t}, \quad (3)$$

in which $\gamma = \rho g$ and v is the kinematic viscosity ($v = \mu/\rho$), assumed to be a constant, $\mathrm{d}/\mathrm{d}t$ is the differentiation with respect to the motion [5]

$$\frac{\mathrm{d}}{\mathrm{d}t}=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}. \quad (4)$$

But for a nonviscous fluid, the Navier–Stokes equations (i.e. eqs (1), (2) and (3)) reduce to the Euler equations of motion in three dimensions [5], given by

$$-\frac{1}{\rho}\frac{\partial}{\partial x}(p+\gamma h)=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}, \quad (5)$$

$$-\frac{1}{\rho}\frac{\partial}{\partial y}(p+\gamma h) = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}, \tag{6}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial z}(p+\gamma h) = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}. \tag{7}$$

The first three terms on the right-hand sides of the equations are convective-acceleration terms, depending upon changes of velocity with space. The last term is the local acceleration, depending upon velocity change with time at a point.

For one-dimensional flow of a real fluid in the l direction (figure 1) with h vertically upward and y normal to l ($v = 0$, $w = 0$, $\partial u/\partial l = 0$), the Navier–Stokes equations reduce to

$$-\frac{1}{\rho}\frac{\partial}{\partial l}(p+\gamma h) + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}, \tag{8}$$

i.e.

$$\frac{\partial}{\partial y}(p+\gamma h) = 0 \quad \text{and} \quad \frac{\partial}{\partial z}(p+\gamma h) = 0$$

for steady flow

$$\frac{\partial}{\partial l}(p+\gamma h) = \frac{\partial^2 u}{\partial y^2} \tag{9}$$

and $p+\gamma h$ is a function of l only. Since u is a function of y only, $\tau = \mu\ \mathrm{d}u/\mathrm{d}y$ for one-dimensional, laminar, incompressible, steady flow between parallel plates

$$\frac{\mathrm{d}\tau}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}l}(p+\gamma h). \tag{10}$$

Since u is a function of y only, $\partial\tau/\partial y = \mathrm{d}\tau/\mathrm{d}y$, and since $p+\gamma h$ does not change value in the y direction (no acceleration), $p+\gamma h$ is a function of l only. Hence,

$$\frac{\mathrm{d}\tau}{\mathrm{d}y} = \mu\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{\mathrm{d}}{\mathrm{d}l}(p+\gamma h). \tag{11}$$

Integrating eq. (11) with respect to y yields

$$\mu\frac{\mathrm{d}u}{\mathrm{d}y} = y\frac{\mathrm{d}}{\mathrm{d}l}(p+\gamma h) + A. \tag{12}$$

Integrating again with respect to y gives

$$u = \frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}l}(p+\gamma h)y^2 + \frac{A}{\mu}y + B \tag{13}$$

in which A and B are constants of integration. To evaluate them, take $y = 0, u = 0$ and $y = a, u = U$ and obtain

$$B = 0; \quad U = \frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}l}(p + \gamma h)a^2 + \frac{Aa}{\mu} + B. \tag{14}$$

Eliminating A and B results in

$$u = \frac{Uy}{a} - \frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}l}(p + \gamma h)(ay - y^2). \tag{15}$$

For horizontal plates, $h = C$; for no gradient due to pressure or elevation, i.e., hydrostatic pressure distribution, $p + \gamma h = C$ and the velocity has a straight-line distribution. For fixed plates, $U = 0$, and the velocity distribution is parabolic. Then [6]

$$u = -\frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}l}(p + \gamma h)(ay - y^2). \tag{16}$$

3. Model

Equation (16) describes a paraboloid velocity profile. We are proposing that it can also be used to describe flow through a porous medium which is bounded on opposite sides (a slab) by impermeable layers (or flow through a cylindrical system) by introducing ϕ into it (eq. (16)). When such a slab is filled with a porous material (say beads) with porosity ϕ and is saturated with flowing fluid, Darcy's law comes to play.

Then Darcy's law is

$$U = -\frac{k}{\mu}\frac{\mathrm{d}}{\mathrm{d}l}(p + \gamma h), \tag{17}$$

where k is the permeability of the medium [7–10]. To equate eqs (16) and (17) they must dimensionally be equal and a constant or a normalizing factor is to be introduced, that is ϕ.

Then

$$-\frac{1}{2\mu}\phi\frac{\mathrm{d}}{\mathrm{d}l}(p + \gamma h)(ay - y^2) = -\frac{k}{\mu}\frac{\mathrm{d}}{\mathrm{d}l}(p + \gamma h), \tag{18}$$

$$k = \frac{\phi}{2}(ay - y^2), \tag{19}$$

where ϕ is the porosity of the medium.

Equation (19) simply shows that for a particular medium of porosity ϕ and thickness a, the permeability k is not uniform throughout the cross-section of flow.

3.1 *Application of model*

Case 1. When $y = 0$; $k = 0$ (i.e. fluid–slab boundary).

This condition establishes the no-slip condition which means that there is actually no movement at the point of contact of the fluid and the inner surface.

Case 2. U is the maximum at the position half way in the velocity profile, that is

$$y = \frac{a}{2}.$$

Thus

$$k_{\max} = \frac{\phi}{2}\left(a\left(\frac{a}{2}\right) - \left(\frac{a}{2}\right)^2\right),$$

$$k_{\max} = \frac{\phi}{2}\left(\frac{a^2}{2} - \frac{a^2}{4}\right),$$

$$k_{\max} = \phi\frac{a^2}{8}, \tag{20}$$

where a is the radius of the pipe

$$\boxed{k = \phi\frac{a^2}{8}}. \tag{21}$$

This is in agreement with

$$\bar{k} = \phi\frac{r^2}{8} \tag{22}$$

which is the average value for k for a system comprising of a bundle of capillary tubes of the same radii and length. r is the radius of each capillary tube. If a porous system is conceived to be a bundle of capillary tubes, then it can be shown that the permeability of the medium depends on the pore-size distribution and porosity. A flow network of tubes would be similar to layers of different permeabilities in parallel such that the average permeability could be calculated by adapting

$$\bar{k} = \frac{\sum_{j=1}^{n} k_j h_j}{\sum_{j=1}^{n} h_j}, \tag{23}$$

$$\bar{k} = \frac{\sum_{j=1}^{m} k_j A_j}{\sum_{j=1}^{m} A_j}, \tag{24}$$

where k_j is the permeability of one capillary tube and A_j is the area of flow represented by a bundle of tubes of permeability k_j.

Table 1. Variation of porosity with permeability from wall–fluid boundary to the mid-stream.

s/n	Porosity, ϕ	Δy	y	k (m^2)
1	0.361	0.00005	0	1.35×10^{-9}
2	0.375	0.00005	0.00005	1.40×10^{-9}
3	0.417	0.00005	...	1.55×10^{-9}
4	0.448	0.00005	...	1.67×10^{-9}
5	0.467	0.00005	0.1725	1.74×10^{-9}

The quantities k_j and A_j can be defined in terms of the radius of capillary tubes.

$$A_j = \pi n_j r_j^2, \quad k_j = \frac{r_j^2}{8},$$

where n_j is the number of tubes of radius r_j and

$$\sum_{j=1}^{m} A_j = \phi A_\mathrm{t},$$

where ϕ is the porosity of the flow network and A_t is the total cross-sectional area of flow network.

By substitution, eq. (22) reduces to

$$\bar{k} = \frac{\phi}{8}\frac{\sum_{j=1}^{m} n_j r_j^4}{\sum_{j=1}^{m} n_j r_j^2}. \tag{25}$$

It should be noted that the permeability of a bundle of tubes is a function not only of the pore size but of the arrangement or porosity of the system.

Hence for a system comprising of a bundle of capillary tubes of the same radii and length, k, the permeability, may be written from eq. (25) as [4]

$$\bar{k} = \phi\frac{r^2}{8}. \tag{26}$$

There is an agreement between eqs (21) and (23).

3.2 *An example*

To set our boundary conditions: Let us consider a cylindrical pipe filled with a porous material consisting of beads or sand from a river-bed which has been sieved into grades and mixed in proportions to produce samples of various porosities. Let the diameter of the cylindrical pipe be 1.725×10^{-2} m, i.e. $a = 0.001725$ m. If the fluid is assumed to be water of viscosity $\mu_\mathrm{w} = 10^{-3}$ Pas, at water–pipe boundary $y = 0$; Δy increases inwardly and is taken in steps of 5.0×10^{-5} m. Using eq. (19) the values of k at each step Δy inwardly can be generated using a FORTRAN programme.

Table 1 is a summary of the boundary conditions used.

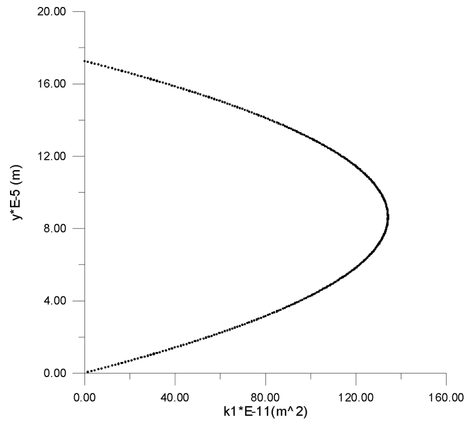

Figure 1. Permeability profile when $\phi = 0.361$.

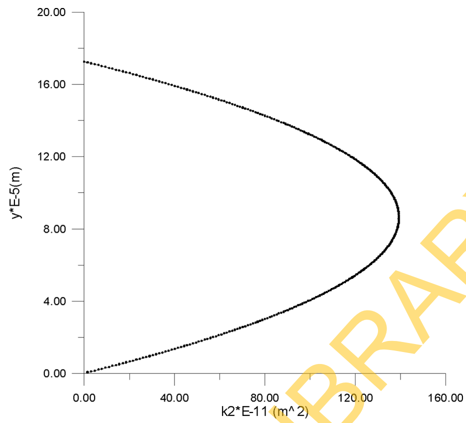

Figure 2. Permeability profile when $\phi = 0.375$.

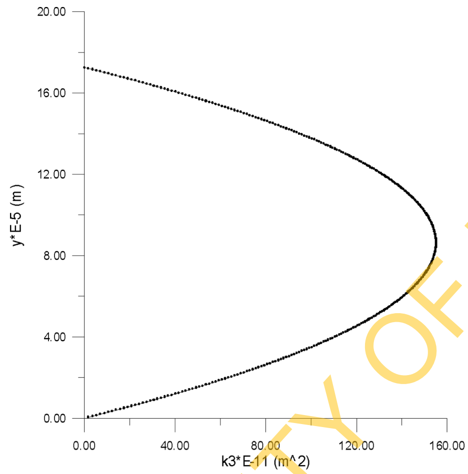

Figure 3. Permeability profile when $\phi = 0.417$.

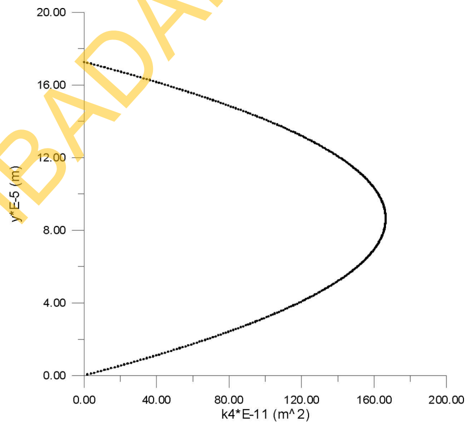

Figure 4. Permeability profile when $\phi = 0.448$.

4. Discussion and conclusions

Graphs of y was plotted against permeability k and a parabolic curve was obtained for various values of porosity indicating that there is actually no flow at the boundaries (confirming the no-slip condition) and at the mid-stream the value of the permeability assumed its maximum. A permeability profile is therefore established which is synonymous with the velocity profile. It must be stated that there is non-uniform variation in the value of permeability down the flow length in a streamline in the region of the entry length where laminar flow is yet to be attained. Once the parabolic curve is completely formed, i.e. the entry length is exceeded, the permeability becomes uniform down the flow length along a streamline of interest,

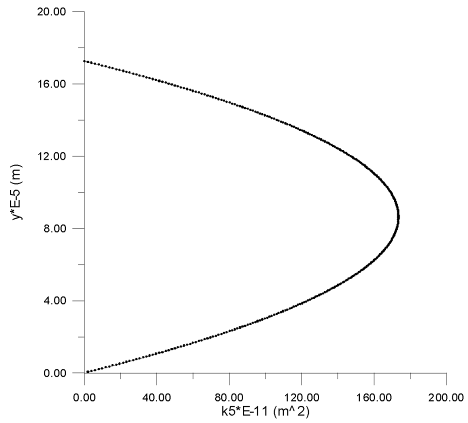

Figure 5. Permeability profile when $\phi = 0.467$.

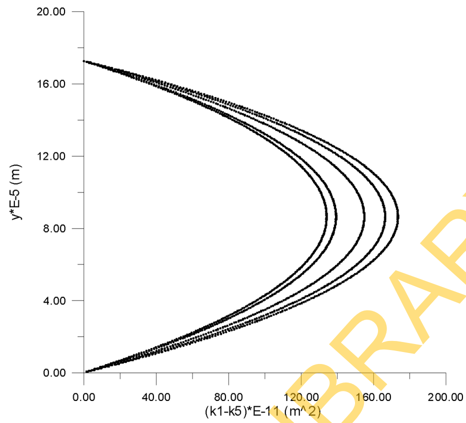

Figure 6. Permeability profile for all values of porosity ϕ.

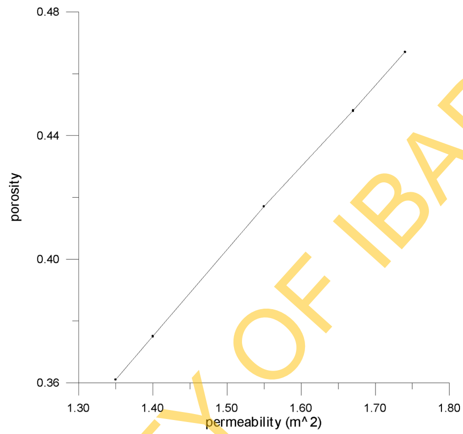

Figure 7. Graph of porosity against permeability.

but still varies across the cross-section. It is worthy to note that the entry length varies with porosity. It is shortest for the least porous medium while it is longest for the more porous medium. The maximum value of permeability which coincided with the point at which transition length is maximum varies with porosity of the medium. The less the porosity of a medium the shorter is the transition/entry length and vice-versa. Specifically, for porous media with porosities 0.361, 0.375, 0.417, 0.448 and 0.467, the entry length was attained when the values of the permeability were 1.35×10^{-9}, 1.40×10^{-9}, 1.55×10^{-9}, 1.67×10^{-9}, 1.74×10^{-9} m^2, that is k_1, k_2, k_3, k_4 and k_5 respectively (figures 1–6).

The graph of y (m) against porosity k (m^2) for porous media with porosity values 0.361, 0.375, 0.417, 0.448 and 0.467 are shown in figures 1–5. Figure 6 shows the combination of graphs 1–5. The innermost curve represents the medium of porosity 0.361 and the next curve is for 0.375 consecutively and the outmost curve is for the medium of porosity 0.467. A graph of porosity against the maximum value of permeability shows that they are linearly related (figure 7).

Equation (19) is said to be valid if the application of it confirmed or conformed with the existing theories. The application of the proposed model as shown in Case 1 where $y = 0$ and as a result $k = 0$ indicated the no-slip condition between fluid and the conduit. Case 2 as seen in eq. (22) confirmed with an already existing equation.

It can be concluded that the introduction of the dimensionless Φ into Hagen–Poiseuille equation is reasonable and will permit this equation to be applicable to flow in porous media (eq. (18)). By the introduction of the dimensionless parameter Φ, to the Hagen–Poiseuille equation, it can then be applied to a situation where the conduit is filled with a porous material.

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