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# Demonstration of Chaos in Selected Chaotic Systems

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**Abstract**
Chaos is one of the most important behavioural modes exhibited by dynamical systems and refers to the unpredictable, seemingly random, motion of trajectories of a dynamical system. In recent time, non-linear dynamics and chaos have become familiar in the technical vocabulary of most sciences and technology due to the various applications of chaos in these areas. Chaotic systems display chaotic behaviours only for specific range of values of their parameters. It is therefore important to understand the chaotic features as well as parameter values resulting in its emergence in systems. In this paper, the emergence of chaos in six chaotic systems was demonstrated using Simulink with specific initial conditions and parameter values. The simulation results obtained in form of time series illustrated noise-like waveforms which were unpredictable even after a long interval of time, while the phase portraits were irregular and sponge-like in appearance. The noise-like and unpredictable trajectories as well strange attractors observed clearly demonstrated the emergence of chaos the chosen parameter values.

**Keywords:** dynamical systems, chaos, non-linear dynamics, simulation, attractors



## INTRODUCTION

Non-linear dynamics and chaos have become familiar in the technical vocabulary of most sciences and technology(Barnerjee, 2011). Dynamical systems can exhibit a great variety of behavioural modes, some of which are of complex nature(Parker and Chua, 1987). Chaos refers to the unpredictable, seemingly random, motion of trajectories of a dynamical system which was first observed in the two-dimensional Van Der Pol equation with a forcing term(Kennedy and Chua, 1986).Chaos is mathematically defined as randomness generated by a simple deterministic system (Pecora and Carroll, 1990). This randomness is a result of the sensitivity of chaotic system to initial conditions. If any two identical chaotic systems have slightly different initial conditions and parameter variations, they will diverge from each other to give uncorrelated outputs. However, because the systems are deterministic, chaos sometimes has some orderliness inherent in it(Chua, et al., 1995). Two identical chaotic systems driven by the same signal will produce the same output, even though the output is unpredictable. Thus chaos can essentially be thought of as “deterministic noise”(Pecora and Carroll, 1990).

Chaotic dynamics have interested chemical engineers with respect to fluids applications and biotechnology engineers as regards periodic beat patterns in a rabbit heart (Liao and Huang, 1997).However, chaos, most of the time, has undesirable effect in mechanical and electrical engineering design and must be averted. Certain effects blamed on noise in electrical systems are examples of chaotic behaviour of a completely deterministic nature (Freire, et al., 1984). Chaotic systems, however, display chaos for certain range of values of their parameters. Due to wide application areas of chaos especially in the fields of science and engineering, it is important to understand its features as well as the parameter values that lead to its emergence in systems.

In this paper six chaotic systems were modelled and simulated to demonstrate the emergence and features of chaos.

## THEORY

Six chaotic systems studied in this paper are the Lorenz system, Chua’s circuit, Rossler system, the Logistic map, Henon map and Ikeda map.

### The Lorenz System

One of the earliest indications of chaotic behaviour was developed by Edward N. Lorenz in the 60’s(Lorenz, 1963). The Lorenz system was published as a model of two-dimensional convection in a horizontal layer of fluid heated from below. The original equations for this 3$^{rd}$–order non-linear system are(Gauthier, 1998):

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y - xz \quad (1)$$
$$\dot{z} = -bz + xy$$

Where x, y and z are the variables and σ, r and b are dimensionless parameters usually assumed positive. Varying the values of the parameters leads to series of bifurcation and eventually chaos. Typical parameter values are σ=10, b=8/3 and r=20. Cuomo et al observed that a direct implementation of equation (1) with an electronic circuit is difficult because the state variables occupy a wide dynamic range with values that exceed reasonable power supply limits (Cuomo and Oppenheim, 1993). However, this difficulty can be eliminated by a simple transformation of variables; specifically, for the coefficients σ, r, and b used, an appropriate transformation is u=x/10, =y/10, and z=w/10. With this scaling, the Lorenz equations are transformed to:

$$\dot{u} = \sigma(v - u)$$
$$\dot{v} = ru - v - 20uw \qquad (2)$$
$$\dot{w} = 5uv - bw$$

**The Chua's Circuit**

Chua's oscillator has served as the primary reference for studying and generating chaos in electronic systems(Elwakil and Kennedy, 2000; Kamil and Fakolujo, 2010). The circuit exhibits rich dynamics that demonstrate most well-known routes to chaos and has therefore been studied extensively(Chua, et al., 1993). Chua's oscillator is a 3rd order autonomous dissipative electrical circuit. As shown in Fig 1, it consists of a linear inductor L, two capacitors $C_1$ and $C_2$, a linear resistor R and a non-linear resistor $N_R$ often referred to as the Chua's diode(Chua, et al., 1996). The state equations for the Chua's circuit are given by:

$$C_1 \frac{dv_{c1}}{dt} = \frac{1}{R}(v_{c2} - v_{c1}) - h(v_{c1})$$
$$C_2 = \frac{dv_{c2}}{dt} = \frac{1}{R}(v_{c1} - v_{c2}) + i_L \qquad (3)$$
$$L\frac{di_L}{dt} = -v_{c2}$$

Where $v_{c1}$, $v_{c2}$ and $i_L$ are the voltages across $C_1$, the voltage across $C_2$ and the current through L, respectively, $h(v_{c1})$ is the piece-wise linear v-i characteristic of the Chua's diode illustrated in Fig. 2 and is given by :

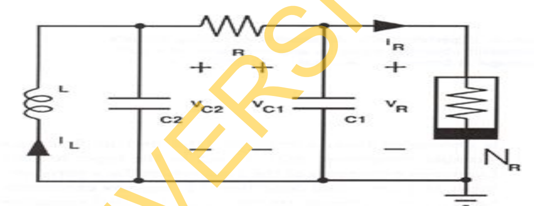



Fig. 1: The Chua's Circuit

Where $B_p$ is the breakpoint voltage of the Chua's diode and $G_a$ and $G_b$ are the slopes of the inner and outer regions respectively. Typical values of the Chua's circuit parameters are: L=18mH;$C_1$=10nF;$C_2$=100nF; and R=1.655kΩ;

**Rössler System**

The Rössler system- a system of three non-linear ordinary differential equations, define a continuous-time dynamical system that exhibits chaotic dynamics associated with the fractal properties of the Rössler attractor. The original Rössler paper says the Rössler attractor was intended to behave similarly to the Lorenz attractor, but also be easier to analyze qualitatively (Rossler, 1976). The state equations describing the system are given as:

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay \quad \textbf{(5)}$$
$$\dot{z} = b + z(x - c)$$

**Logistic Map**

The Logistic Map or Logistic Difference Equation is a model often used to introduce chaos(Clayton, 1997). Although it is simple, it displays the major chaotic concepts. As an example of the logistic map, consider a Limited Growth (Verhulst) Model which was used in 1845 by Verhulst to predict American population from census data (Harrel-II, 1992). This model is expressed as:

$$x_{n+1} = rx_n(1 - x_n) \qquad (6)$$

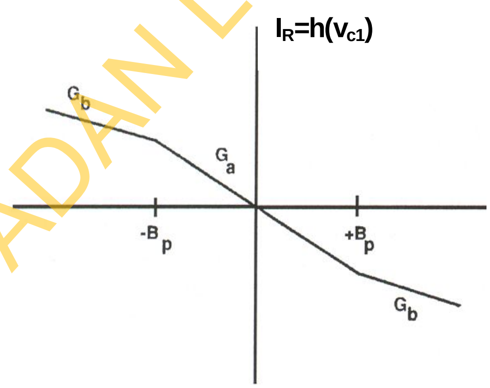



Fig. 2: The linear piece-wise characteristics of the Chua's diode

Where $x_{n+1}$ is the new population, $x_n$ is the old population, r is the rate of population growth and n is the time interval. In Eqn. 2.19, $x_{n+1}$ *and* $x_n$ are the variables while r is the parameter.

**Hénon Map**

The Hénon map, which was introduced by Michel Hénon, is a discrete-time dynamical system. It is one of the most studied examples of dynamical systems that exhibit chaotic behavior. The state equations for the system are (Henon, 1976):

$$x(n+1) = 1 - ax^2(n) + y(n)$$
$$y(n+1) - bx(n) \qquad (7)$$

The map depends on two parameters, *a* and *b*, which for the *canonical Hénon map* have values of *a* = 1.4 and *b* = 0.3. For the canonical values, the Hénon map is chaotic. For other values of *a* and *b* the map may be chaotic, intermittent, or converge to a periodic orbit.

**Ikeda Map**

The Ikeda map,which was introduced by Kensuke N. Ikeda, is also a discrete-time dynamical system described by (Galias, 2002):

$$x_{n+1} = 1 + u(x_n \cos t_n - y_n \sin t_n)$$
$$y_{n+1} = u(x_n \sin t_n + y_n \cos t_n) \qquad (8)$$

where$u$ is a parameter and$t_n = 0.4 - \frac{6}{1+x_n^2+y_n^2}$

The system is chaotic with a *u* value of 0.95

**MATERIALS AND METHOD**

Equations representing the six chaotic systems were modelled using Simulink function blocksas illustrated in Figs. 3 to 8, and simulations were carried out with specified initial conditions and parameter values.

The Logistic Map was simulated with the initial condition of x(0)=0.5and parameter r value of 3.99; the Hénon Map withinitial conditions of x(0)=y(0)=0.5 and parameter values of a=1.4, b=0.3; the Ikeda Map with initial conditions of x(0)=y(0)=0 and parameter value of u=0.95;the Rössler system with initial conditions x(0)=y(0)=z(0)=0.2 and parameter values of a=0.2, b=0.2, and c=5.7; the Lorenz system with initial conditions of u(0)=v(0)=w(0)=1 and parameter values of σ=16, r=45.6, and b=4; and the Chua's circuit with initial conditions of $v_{c1}(0) = 0.001$, $v_{c2}(0) = -0.05$ and $i_L(0) = -0.02$ and parameter values of $C_1 = 10nF$, $C_2 = 100nF, L = 18mH$, $G = \frac{1}{1655} S G_a = -756mS, G_b = -409mS$, and $B_p = 1.08V$.

The trajectory of each chaotic system in time as obtained from simulations was displayed on the Scope block of each model. Also the phase responses of the variables were sent to Matlab workspace and plotted as phase portraits.

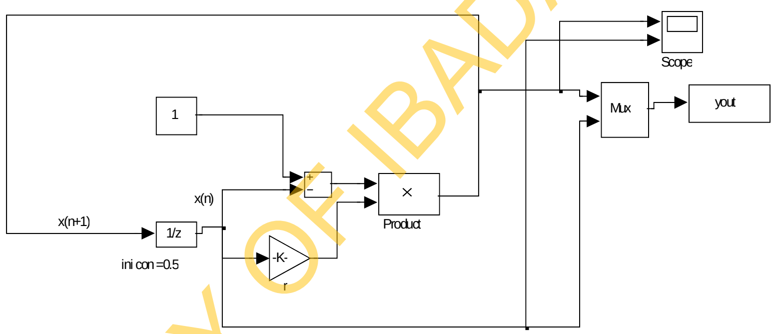



Fig. 3: Logistic Map Simulink Model

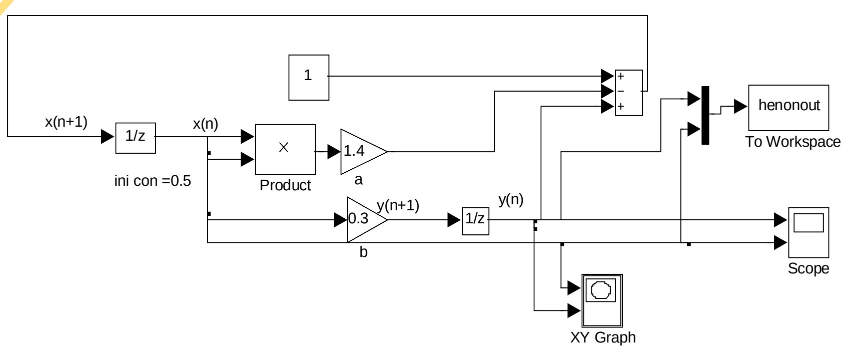



Fig. 4: Hénon Map Simulink Model

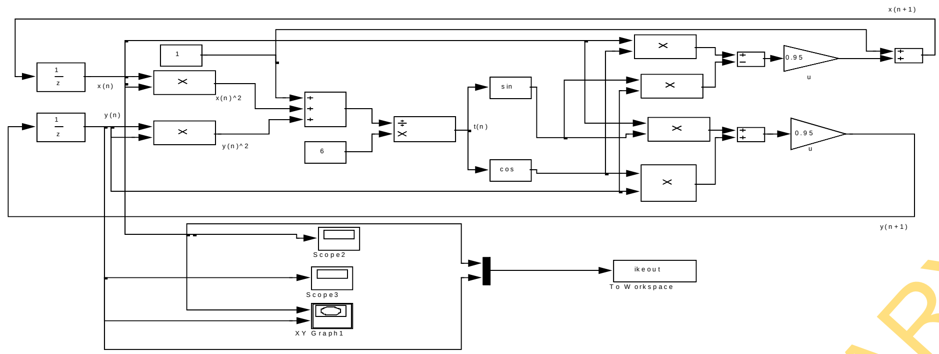



Fig. 5: Ikeda Map Simulink Model

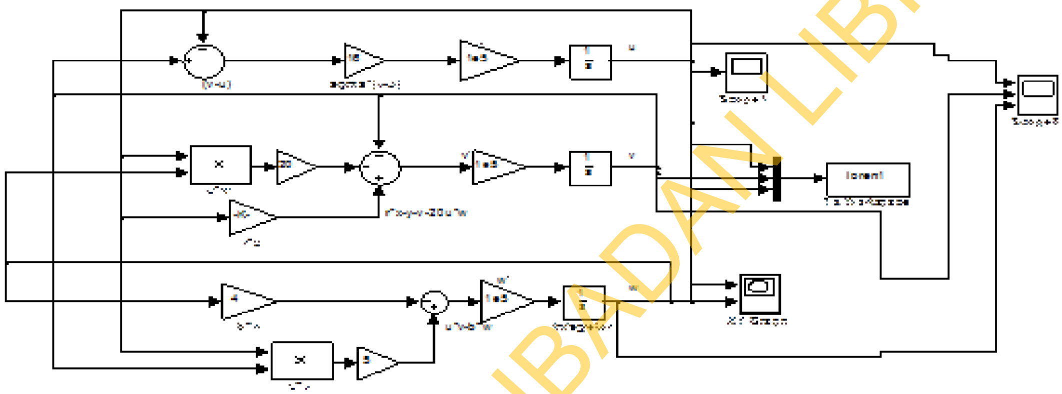



Fig. 6: Lorenz System Simulink Model

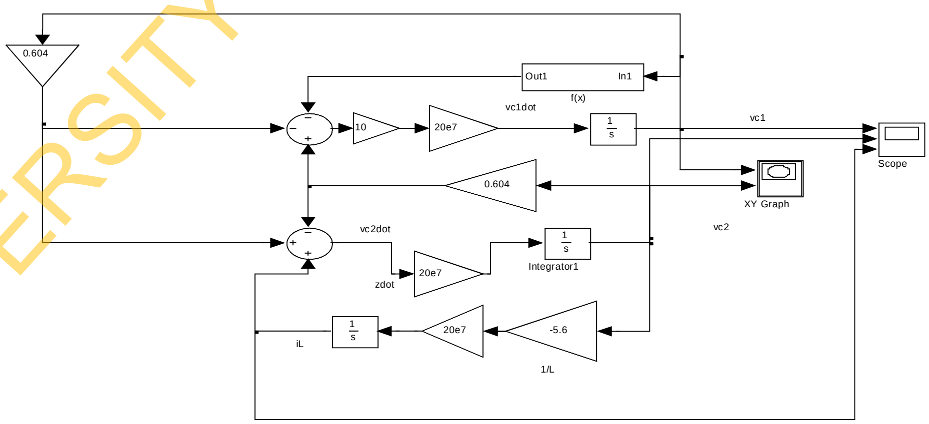



Fig. 7: Chua’s Circuit Simulink Model

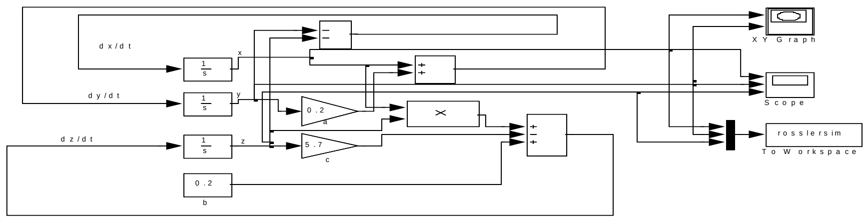
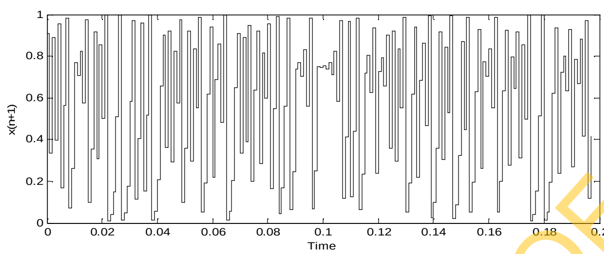



Fig. 8: Rössler System Simulink Model

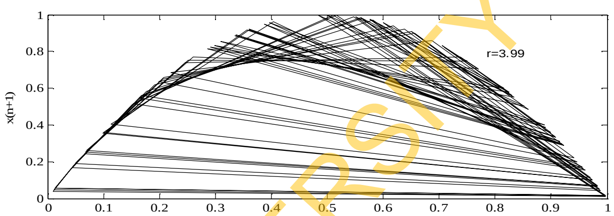
**RESULTS AND DISCUSSION**

The time series and phase portraits obtained from simulation of each of the six chaotic systems using Simulink function blocks are shown in Figs. 9 to 14.

In all the six chaotic systems simulated, noise-like trajectories and strange attractors were observed which are very good indicators of chaos. The emergence of chaos for certain parameter values in six chaotic systems was demonstrated. The six systems were modelled using Simulink. Double scroll attractors widely reported in literature were observed for Chua's Circuit and Lorenz System.

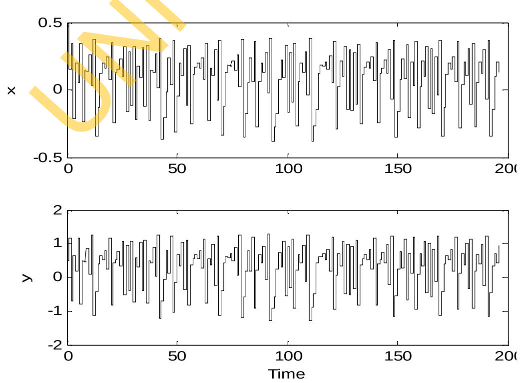



(a)

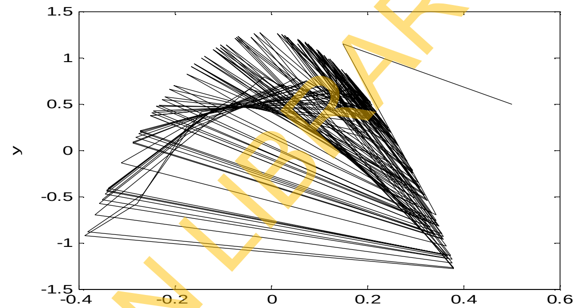



(b)

Fig. 9: Logistic Map (a) time series (b) phase portrait

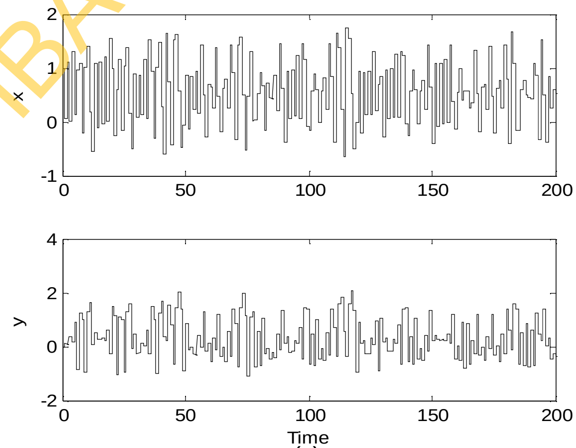



(a)

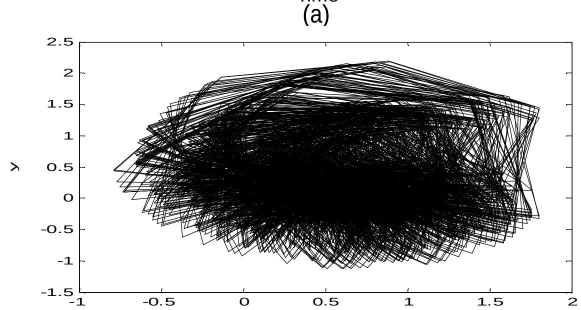



(b)

Fig. 10 Hénon Map (a) time series (b) phase portrait




(a)




(b)

Fig. 11:Ikeda Map (a) time series (b) phase portrait

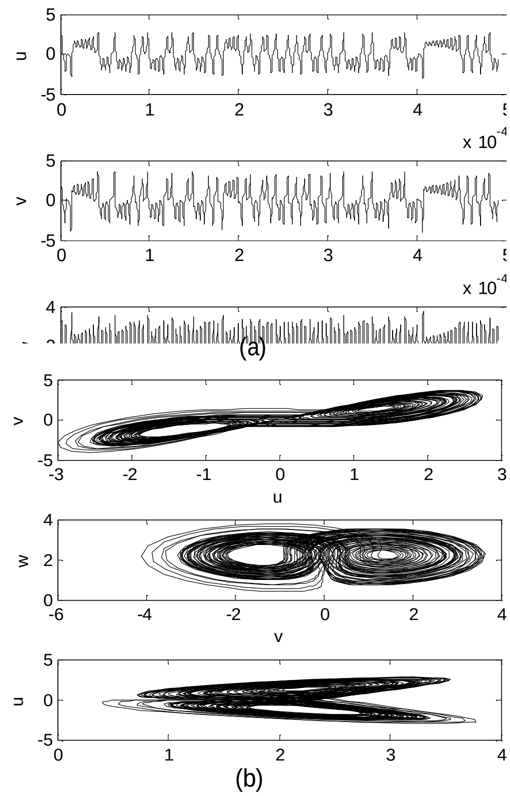



Fig. 12: Lorenz System (a) time series (b) phase portrait

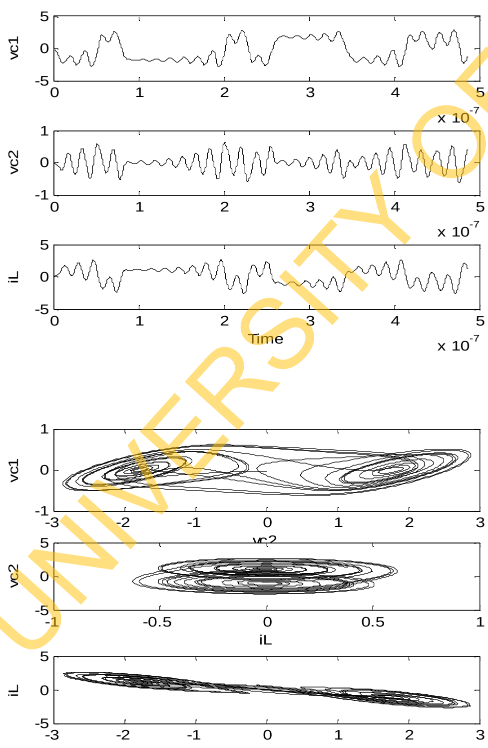



Fig. 13:Chua's Circuit (a) time series (b) phase portrait

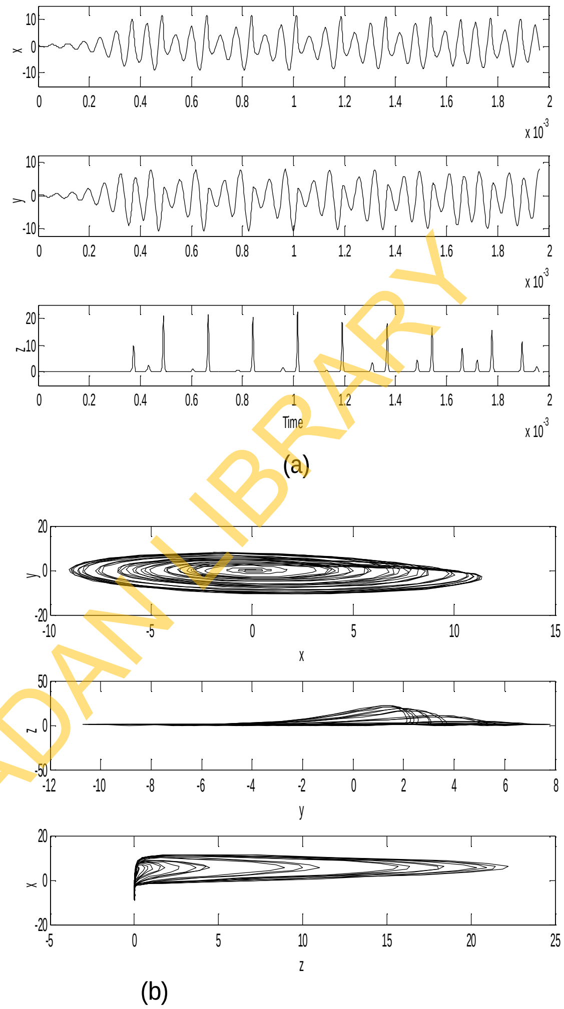



Fig. 14: Rössler System (a) time series (b) phase portrait

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