MELLIN TRANSFORM METHOD FOR THE VALUATION OF AMERICAN POWER PUT OPTION

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Certification

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Dedication

This work is dedicated to God Almighty, Who is able to do exceeding abundantly above all that we ask or think, according to the power that works in us.

Abstract

American Power Put Option (APPO) is a financial contract with a non-linear payoff that can be applied at any time on or before its expiration date and offers flexibility to investors. Analytical approximations and numerical techniques have been proposed for the valuation of Plain American Put Option (PAPO) but there is no known closed-form solution for the price of APPO. Mellin transform is a useful method for dealing with unstable mathematical systems. This study was designed to derive a closed-form solution for APPO by means of the Mellin transform method that enables option equations to be solved directly in terms of market prices and to investigate the efficiency and robustness of the method.

The Ito's lemma under the geometric Brownian motion was used to derive a non-homogeneous Partial Differential Equation (PDE) for the price of APPO. The Mellin transform with its shifting and derivative properties were used to solve the non-homogeneous PDE. The Mellin inversion formula and the value-matching condition were used to recover the integral representations for the price and the free boundary of APPO, respectively. The convolution theorem for the Mellin transform was used to prove the equivalence of the integral representation for the price of APPO, for $n = 1$. The integral representation was transformed into a form that permits the use of the Gauss-Laguerre quadrature method to obtain the closed-form solution for the price of APPO. By varying the volatility (σ), strike price (K) and time to expiry (T), numerical experiments were performed to compare the results of the Mellin transform method for the price of APPO for $n = 1$ with accelerated binomial model, binomial model, finite difference and recursive methods.

A non-homogeneous Black-Scholes-Merton-like PDE for the price of APPO was obtained. The integral representations for the price and the free bound-

ary of APPO were obtained respectively as:

$$A_p^n(S_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega$$
$$+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T (S_t^n)^{-\omega}\frac{(\bar{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$
$$- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T (S_t^n)^{-\omega}\frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$

and

$$K - \bar{S}_t^n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(\bar{S}_t^n)^{-\omega}d\omega$$
$$+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T (\bar{S}_t^n)^{-\omega}\frac{(\bar{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$
$$- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T (\bar{S}_t^n)^{-\omega}\frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$

for $(S_t^n,t) \in \{(0,\infty)\times[0,T]\}, c \in (0,\infty), \{\omega \in \mathbb{C}|0 < \Re(\omega) < \infty\}$, $\alpha_1^* = \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right)$, $\alpha_2 = \frac{2r}{n^2\sigma^2}$, where c, $\mathbb{C}$, $A_p^n(S_t^n,t)$, n, $\bar{S}_t^n$, S_t^n, t, $\Re(\omega)$, q, r and ω are the constant, set of complex numbers, price of the option, power of the option, free boundary, underlying asset price, current time, real part, dividend yield, risk-free interest rate and complex number, respectively. The integral representation for the price of APPO, for $n = 1$ was proved to be equivalent to the Kim integral equation for PAPO. With the Gauss-Laguerre quadrature method, the closed-form solution of the price of APPO was also obtained. The numerical results showed that the Mellin transform method was efficient and more accurate for higher values of volatility and time to expiry when compared with the other methods.

Mellin transform method has been used to derive a closed-form solution for the price of American power put option which was computationally efficient and robust at $n = 1$.

Keywords: American power put option, Geometric Brownian motion, Ito's lemma, Non-homogeneous Black-Scholes-Merton-like equation, Value-matching condition.

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May God in His unlimited mercy reward you all abundantly.

List of Acronymns

ABM	Accelerated Binomial Model
APPO	American Power Put Option
BM	Binomial Model
BS	Black-Scholes Model
BSM	Black-Scholes-Merton Model
CBOE	Chicago Board of Options Exchange
FFT	Fast Fourier Transform
FDM	Finite Difference Method
FMT	Fourier-Mellin Transform
FT	Fourier Transform
GBM	Geometric Brownian Motion
LT	Laplace Transform
MCM	Monte Carlo Method
MTM	Mellin Transform Method
OTC	Over-The-Counter
PAPO	Plain American Put Option
PDE	Partial Differential Equation
RM	Recursive Method
SDE	Stochastic Differential Equation

List of Notations

μ	Expected return of the underlying asset
σ	Volatility of the underlying asset
τ	Reversed time
$A_p^n(S_t^n, t)$	Price of American power put options with dividend yield
$A_p(S_t, t)$	Price of American put options with dividend yield
$A_p(\mathbf{S}, t)$	Price of the American put option on a basket of multi-dividend paying stocks
$A_\infty^n(S_t^n, t)$	Price of perpetual American power put options with dividend yield
$E_p^n(S_t^n, t)$	Price of European power put options with dividend yield
$E_p(S_t, t)$	Price of European put options with dividend yield
$E_p(\mathbf{S}, t)$	Price of the European put option on a basket of multi-dividend paying stocks
$f(S_t^n, t)$	Early exercise function for the case of non-dividend yield
$f^*(S_t^n, t)$	Early exercise function for the case of dividend yield
$\mathcal{F}(.)$	The Fourier transform
$H(.)$	Heaviside step function
K	Strike price
$\mathcal{L}(.)$	The Laplace transform
$\mathcal{M}(.)$	The Mellin transform
n	Power of option
$\mathcal{N}(.)$	Normal distribution function
$P_c^n(S_T^n, T)$	Payoff for the power call option
$P_p^n(S_T^n, T)$	Payoff for the power put option

$P_A^n(S_t^n, t)$	Price of American power put options with non-dividend yield
$P_A(S_t, t)$	Price of American put options with non-dividend yield
$P_E^n(S_t^n, t)$	Price of European power put options with non-dividend yield
$P_E(S_t, t)$	Price of European put options with non-dividend yield
$P_\infty^n(S_t^n, t)$	Price of perpetual American power put options with non-dividend yield
q	Dividend paying stock
r	Risk-free interest rate
S_t	Underlying asset price/Stock price
$\bar{S}_t^n$	Free boundary of American power put option with dividend yield
$\hat{S}_t^n$	Free boundary of American power put option with non-dividend yield
$\bar{S}_\infty^n$	Free boundary of perpetual American power put option with dividend yield
$\hat{S}_\infty^n$	Free boundary of perpetual American power put option with non-dividend yield
t	Current time
T	Time to expiry/Maturity date
W_t	Brownian motion
X_t	Stochastic process

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Chapter 1

Introduction

1.1 Background of the Study

The derivative market have become extremely popular, this popularity exceeds that of the stock exchange. Many problems in mathematical finance entail the computation of a particular integral. In many cases these integrals can be solved analytically and in some cases they can be solved using numerical integration.

The history of options extends back to several centuries, it was not clear until 1973 that the trading of options was formalized by the establishment of the Chicago Board of Options Exchange (CBOE) with more than one million contracts per day. This same year was also a turning point for research in option valuation. Black and Scholes (1973) published their work on option pricing in which they described a mathematical frame work for finding the fair price of a European call option. In the recent years, the complexity of numerical computation in financial theory and practice has increased greatly, putting more demands on computation speed and efficiency.

Securities are paper assets which are issued by a government or company in order to acquire capital financing; examples of securities include bonds, bills of exchange, promissory notes, shares and financial derivatives. An option is defined as a contract that grants its holder the right, without obligation to buy or sell a specific underlying asset S_t on or before a given date in the future (expiry date, T) for an agreed price K, called the strike price. The underlying assets include stocks, foreign currencies, interest rates, stock indices and commodities. A call option gives the holder the right to buy the underlying asset, whereas a put option gives the right to sell (Hull (2002)). Power option is a financial contract in which the payoff at expiry date is related to the n^{th} power of the underlying asset price; thus the payoff is a nonlinear function of the underlying. Power option is appropriate for hedging non-linear price risks. The difference between the American and the European power options is that the European power option can only be exercised at the maturity or expiry date while the American power option can be exercised by its holder at any time on or before the expiry date. Most of the Over-The-Counter (OTC) traded options are of the American power type. The early exercise feature makes the valuation of the American power option mathematically challenging. Analytical approximations and numerical techniques have been proposed for the valuation of plain American option but no known closed-form solution for the price of American power option has been derived.

Nowadays, investment companies use options for their risk management through hedging against possible fluctuations of the underlying asset price. Hence the valuation of options is an important field in financial research (Zhang (2007)).

The subject of numerical methods in the area of options valuation and hedging is very broad. A wide range of different types of contracts are available and in many cases there are several candidate models for the stochastic evolution of the underlying state variables.

1.2 Aim and Objectives of the Study

This work is concerned with financial mathematics in continuous time. The aim of this work is the study of the applicability of the Mellin transform method in the field of American power put option valuation.
The objectives of the study are as follows:

(i) To use the Mellin transform method to solve the partial differential equations for the price of power put options namely European and American power put options with non-dividend and dividend yields, respectively.

(ii) To obtain the integral representations for the price of the European power put option which pays both non-dividend and dividend yields, respectively.

(iii) To use the convolution property of the Mellin transform method to obtain the fundamental option valuation formulae known as "The Black-Scholes-like model" and "The Black-Scholes-Merton-like model" for the cases of non-dividend and dividend yields, respectively.

(iv) To obtain the integral representations for the price and the optimal exercise boundary (called the free boundary) of the American power put options with non-dividend and dividend yields, respectively.

(v) To extend the integral representations for the price of the American power put option for the cases of non-dividend and dividend yields, respectively to obtain the optimal exercise boundary and the analytic valuation formula for perpetual American power put option.

(vi) To obtain a closed-form solution for the price of American power put option with dividend yield.

(vii) To extend the Mellin transform method in higher dimensions for the valuation of put options on a basket of multi-dividend paying stocks.

1.3 Motivation

Methods for the valuation of vanilla and path dependent options analytically are often derived by solving partial differential equations. Since these backward-in-time equations are parabolic in nature, they must be solved with payoff-specific boundary conditions. Although a solution can be derived directly in some cases, many contracts have corresponding partial differential equations that are too complex to allow for a standard solution. Examples are the European and the American options in stochastic rate models and stochastic volatility. For the European options, the resulting equations become two or higher dimensional depending on the number of state variables. The American options have partial differential equations of free boundary type. The main difficulty in valuing American style options analytically is the presence of the early exercise optimally prior to expiry. The optimal exercise boundary is not known and must be determined simultaneously as part of the underlying valuation problem. This feature makes the valuation and hedging of American options mathematically challenging and created great field of research throughout the last three decades. In both cases of the options, advanced method based on the integral transforms used in theoretical and applied mathematics are needed to provide an accurate approximation of solution and to tackle the complexity of the options by reducing the dimensionality existing in the valuation problem.

The history of integral transforms began with D'Alembert in 1747. D'Alembert proposed using a superposition of sine functions to describe the oscillations of a violin string (D'Alembert (1747a)). Examples of integral transforms are; the Mellin transforms, the Laplace transforms, the Fourier transforms and the Hilbert transforms. These integral transforms are used to solve differential and integral equations arising in engineering and applied mathematics. Among the integral transforms, the Mellin transform has gained great popularity in complex analysis and analytic number theory for its applications to problems related to the Gamma function, summation of infinite series and other Dirichlet series. The main difference between the Mellin transform and the Fourier transform is that the Mellin transform exists in vertical strips of the complex plane whereas the Fourier transform is defined in horizontal strips.

In mathematical finance, the Mellin transform enables option equations to be solved directly in terms of market prices rather than log-prices, providing a more natural setting to the valuation problem. Despite this, the Mellin transform's ascension into the realm of mathematical finance is only about one decade old.

In this thesis, the Mellin transform method was used for the valuation of American power put option with non-dividend and dividend yields, respectively under the geometric Brownian motion.

1.4 Structure of the Study

The structure of the thesis is organized as follows. Chapter One consists of introduction. Chapter Two presents the literature review. Chapter Three presents the concept of the Mellin transforms, some of its basic operational properties and its extension to the multidimensional case. The Laplace and Fourier transforms and their properties were presented. Stochastic calculus and basic principles of option valuation were discussed. In Chapter Four, it was shown that the stock dynamics of power options followed a lognormal distribution. The generalized fundamental valuation equation for the price of power options with non-dividend and dividend yields, respectively was derived. The valuation formula for power call option in the Black-Scholes model framework was obtained by means of risk-free probability measure. The Mellin transform method was applied to obtain the integral representations for the price (and the free boundary) of power put options on a single underlying stock with non-dividend and dividend yields, respectively. The integral representations for the price of the American power put option with non-dividend and dividend yields, respectively was used to obtain the optimal exercise boundary and the analytical valuation formula for the perpetual American power put option. A closed-form solution for the price of the American power put option with dividend yield was obtained. Basket option was described. The integral representations for put options on a basket of multi-dividend yields using the multidimensional Mellin transform method was obtained. Other related methods for options valuation were considered. Some numerical experiments and discussion of results were also presented. Chapter Five presents conclusions and recommendations.

Chapter 2

Literature Review

The revolution on derivative securities, both in exchange markets and in academic communities began in the early 1970's (Weber (2008)). In 1973, Black and Scholes published their paper on option valuation, in which a closed-form expression for the price of the European call option was derived. They used a no-arbitrage argument to describe a partial differential equation which governs the evolution of the option price with respect to the maturity time and the underlying asset price.

Moreover, in the same year, Merton (1973) extended the Black-Scholes model in several important ways. Since its invention, the Black-Scholes formula has been widely used by traders to determine the price of an option. However this famous formula has been questioned after the crash of the stock market in 1987 (Carlson (2006)). Following the Black and Scholes option pricing model in 1973, a number of other popular approaches were developed, such as Merton (1976), Brennan and Schwartz (1978), Cox et al. (1979) and Boyle et al. (1997) to price the derivative governed by solving the underlying partial differential equation.

In 2002, Cruz-Baéz and González-Rodríguez pioneered the method of using the Mellin transform to solve the associated Black-Scholes partial differential equation for a European call option. Esser (2003) investigated the valuation of power and powered options in the Black-Scholes model and used the technique of change of numéraire. The valuation of power options in the Black-Scholes model was investigated by Esser (2004), following similar arguments used in deriving the Black-Scholes formula of the valuation of plain vanilla European options.

Mellin transforms in option theory were also introduced by Panini and Srivastav (2004). They derived integral equations for the price of European and American basket put options using Mellin transform techniques. Panini and Srivastav (2005) derived the expression for the free boundary and price of an American perpetual put as the limit of a finite-lived option. Company et al. (2006) constructed an explicit solution of the Black-Scholes equation with a weak payoff function. By means of the Mellin transform of a class of weak functions, they obtained a candidate integral formula for the solution. Rodrigo and Mamon (2007) used the Mellin transform approach to prove the existence and uniqueness of the price of a European option under the framework of a Black-Scholes model with time-dependent coefficients. They also derived a maximum principle and used it to prove uniqueness of the option price.

Frontczak and Schöbel (2008) extended the results obtained in Panini and Srivastav (2005) and showed how the Mellin transform approach can be used to derive the valuation formula for perpetual American put options on dividend-paying stocks. Frontczak and Schöbel (2009) used a framework based on the Mellin transforms and showed how to modify the approach to value American call options on dividend paying stocks. Zieneb and Rokiah (2011) derived a closed form solution for a continuous arithmetic Asian option by means of partial differential equation. They also provided a new method for solving arithmetic Asian options using Mellin transforms in a stock price. The pricing of power options under generalized Black-Scholes model was considered by Wu and Xu (2011). Under the Heston model, pricing formulas for power options were derived analytically in Kim et al. (2012b). Kim (2014) considered the pricing of power options under the regime-switching model by means of the Laplace transforms.

Manuge and Kim (2015) derived the analytical pricing formulas and Greeks for European and American basket put options using the Mellin transform. They assumed that assets are driven by geometric Brownian motion which exhibit correlation and pay a continuous dividend rate. Xu (2015) derived a closed-form solution formulae for the pricing of powered options and capped powered options in the Black-Scholes-Merton environment. Closed-form pricing formula for exchange option with credit risk by means of the Mellin transform was derived by Kim and Koo (2016). Zhang et al. (2016) investigated the valuation of power option under the assumption that the underlying stock price is assumed to follow an uncertain differential equation.

Several approximations and numerical techniques that have been proposed for the valuation of plain American options can be found in Mc Kean (1965), Samuelson (1965), Merton (1973), Johnson (1983), Geske and Johnson (1984), Mc Millian (1986), Baron-Adesi and Whaley (1987), Breen (1989), Kim (1990), Jacka (1991), Carr et al. (1992), Carr and Faguet (1994), Wilmott et al. (1995), Balakarishna (1996), Broadie and Detemple (1996), Huang et al. (1996), Carr (1998), Ju (1998), Kuske and Keller (1998), Kwok (1998), Chiarella et al. (1999), Sullivan (2000), Ekström (2004), Panini (2004), Peskir (2005), Belomestny and Milstein (2006), Heider (2007), Chen et al. (2008), Li (2010b) and Kim et al. (2012a).

For mathematical backgrounds, other sporadic applications of transform methods in financial contexts (see Widder (1941), Spiegel (1965), Buser (1986), Beaglehoe (1992), Rogers and Shi (1992), Shimko (1992), Poularikas (1999), Geman and Yor (1993), Jodar et al (2002), Petrella and Kuo (2004), Cruz-Báez and González-Rodriguez (2005), Szymon et al. (2005), Company et al. (2007), Frontczak (2013), Zieneb and Rokiah (2013), AlAzemi et al. (2014), Manuge and Kim (2014) and Lee and Shin (2015)), just to mention a few.

Chapter 3

The Mellin Transforms and Foundations

In this chapter, the concept of the Mellin transforms, some of its operational properties and its extension to the multidimensional case were presented. Fundamental concepts of stochastic calculus used in continuous-time mathematical finance are also dealt with. Some terminologies and basic principles of option valuation were also presented.

3.1 The Mellin Transforms

The first occurrence of the Mellin transform was found in a memoir by Riemann in which he used it to study the famous Zeta function (Titchmarsh (1986)). However, Mellin (1854-1933) was the first to give a systematic formulation of the Mellin transformation and its inverse (Lindelöf and Mellin (1934)).[1] Working in the theory of special functions, he developed

[1]Robert Hjalmar Mellin (1854-1933) was a Finnish function-theorist who studied under Gösta Mittag-Leffler, Karl Weierstrass and Leopold Kronecker.

applications to the solution of hypergeometric differential equations and to the derivation of asymptotic expansions. The Mellin contribution gives a prominent place to the theory of analytic functions and relies essentially on Cauchy's theorem and the method of residues (Bertrand et al (2000)). The Laplace transform has been widely used in many engineering applications. It provides a useful method for solving some types of differential equations when certain initial conditions are given. A detailed presentation of the topic including proofs and examples can be found in Widder (1941), Reed (1944), Sneddon (1972), Titchmarsh (1986), Brychkov et al. (1992), Hai and Yakubovich (1992), Flajolet et al. (1995), Debnath and Bhatta (2007).

Definition 3.1.1

The Mellin transform is a complex valued function defined on a vertical strip in the ω-plane whose boundaries are determined by the asymptotic behaviour of $f(x)$ as $x \to 0^+$ and $x \to \infty$. The Mellin transform of the function $f(x)$ is denoted by $\mathcal{M}(f(x), \omega)$ and defined as

$$\mathcal{M}(f(x), \omega) := \tilde{f}(\omega) = \int_0^\infty f(x) x^{\omega-1} dx \tag{3.1}$$

where $f(x)$ is a locally Lebesgue integrable function. The Mellin transform variable ω is a complex number, $\omega = \Re(\omega) + i\Im(\omega)$, where i is the imaginary unit, and $\Re(.)$ and $\Im(.)$ are real and imaginary parts, respectively. However, the Mellin transform of a function does not always exist. The following result summarizes the conditions that ensure the existence of (3.1). The largest strip (a_1, a_2) in which the integral converges is called the fundamental strip.

Lemma 3.1.1 (Existence Theorem for Mellin Transform) (Flajolet et al. (1995))

Let $f(x)$ be a continuous function such that

$$f(x) = \begin{cases} O(x^a), & x \to 0^+ \\ O(x^b), & x \to \infty. \end{cases} \tag{3.2}$$

Then the Mellin transform $\tilde{f}(\omega)$ exists for any $\omega \in \mathbb{C}$ on $-a < \Re(\omega) < -b$.

Remark 3.1.1

(i) This interval, known as the strip of definition of the Mellin transform and often denoted by $(-a, -b)$ is the domain of analyticity of $\tilde{f}(\omega)$. To show this, consider the absolute bound of $f(x)$,

$$\left| \int_0^\infty f(x)x^{\omega-1}dx \right| \leq \int_0^1 |f(x)|x^{\Re(\omega)-1}dx + \int_1^\infty |f(x)|x^{\Re(\omega)-1}dx \tag{3.3}$$

$$\leq \hat{c}_1 \int_0^1 x^{\Re(\omega)+a-1}dx + \hat{c}_2 \int_1^\infty x^{\Re(\omega)+b-1}dx \tag{3.4}$$

where $\hat{c}_1, \hat{c}_2 \in \mathbb{R}^+ \cup \{0\}$. Since the first integral in (3.4) converges for $\Re(\omega) > -a$ and the second integral converges for $\Re(\omega) < -b$, it follows that $\tilde{f}(\omega)$ exists on $(-a, -b)$. Thus the existence is granted for locally integrable functions, whose exponent in the order at 0 is strictly greater than the exponent of the order at ∞.

(ii) Consider instead the scenario, the Mellin transform of a function is known and one wishes to recover the original function. For a function $\tilde{f} : \mathbb{C} \to \mathbb{C}$, it can be shown under general conditions that an inverse

$f(x) \in I\!R^+$ only exists, but is also unique (for a given fundamental strip) (Manuge (2013)).

Definition 3.1.2

If $f(x)$ is an integrable function with fundamental strip (a_1, a_2), then if c is such that $a_1 < c < a_2$ and $\{\tilde{f}(\omega) : \omega = c + it, c \in \Re(\omega)\}$ is integrable, the equality

$$\mathcal{M}^{-1}(\tilde{f}(\omega)) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\omega)x^{-\omega}d\omega \qquad (3.5)$$

holds almost everywhere. Moreover, if $f(x)$ is continuous, then the equality holds everywhere on $(0, \infty)$. Obviously, $\mathcal{M}$ and $\mathcal{M}^{-1}$ are linear integral operators. Equation (3.5) justifies the formal statement, which goes under the name of the Mellin inversion formula.

Three important examples of the Mellin transform were presented as follows:

(i) The function $f(x) = e^{-x}$ satisfies $e^{-x} = O(x^0)$ as $x \to 0^+$ and $e^{-x} = O(x^{-b})$ as $x \to \infty$ for any $b > 0$ so that its transform (the Gamma function)

$$\mathcal{M}(e^{-x}, \omega) = \tilde{f}(\omega) = \int_0^\infty e^{-x}x^{\omega-1}dx = \Gamma(\omega), \Re(\omega) > 0 \qquad (3.6)$$

is defined and analytic on $(0, \infty)$.

(ii) The function $f(x) = (e^x - 1)^{-1}$ satisfies $f(x) = O(x^0)$ as $x \to 0^+$ and $f(x) = O(x^{-b})$ for all $b > 0$ as $x \to \infty$. Hence $f(x)$ is analytic and defined on $(1, \infty)$. We find

$$\mathcal{M}((e^x - 1)^{-1}, \omega) = \tilde{f}(\omega) = \int_0^\infty (e^x - 1)^{-1}x^{\omega-1}dx \qquad (3.7)$$

But

$$\sum_{m=1}^{\infty} e^{-mx} = \frac{1}{(e^x - 1)} = \frac{e^{-x}}{(1 - e^{-x})} \tag{3.8}$$

and hence

$$\begin{aligned}
\mathcal{M}((e^x - 1)^{-1}, \omega) &= \sum_{m=1}^{\infty} \int_0^{\infty} e^{-mx} x^{\omega - 1} dx \\
&= \sum_{m=1}^{\infty} \frac{\Gamma(\omega)}{m^{\omega}} \qquad (3.9)\\
&= \Gamma(\omega)\zeta(\omega)
\end{aligned}$$

$$\mathcal{M}((e^x - 1)^{-1}, \omega) = \Gamma(\omega)\zeta(\omega), \quad \Re(\omega) > 1 \tag{3.10}$$

The function

$$\zeta(\omega) = \sum_{m=1}^{\infty} \frac{1}{m^{\omega}}, \quad \Re(\omega) > 1$$

is the famous Riemann Zeta function. It is required that $\Re(\omega) > 1$ for convergence of the Riemann Zeta function and it is clearly seen that this validates the strip $(1, \infty)$ on which $\tilde{f}(\omega)$ is defined and analytic.

(iii) The function $f(x) = (1 + x)^{-1}$ is $O(x^0)$ as $x \to 0^+$ and $O(x^{-1})$ as $x \to \infty$. Hence a guaranteed strip of existence for $\tilde{f}(\omega)$ is $(0, 1)$. Set $x = \frac{w}{1-w}$. Then

$$\begin{aligned}
\tilde{f}(\omega) &= \int_0^1 \left(\frac{w}{1-w}\right)^{\omega - 1} \frac{1}{1 + \frac{w}{1-w}} (1 - w)^{-2} dw \\
&= \int_0^1 \left(\frac{w}{1-w}\right)^{\omega - 1} (1 - w)^{-1} dw \\
&= \int_0^1 w^{\omega - 1} (1 - w)^{-\omega} dw \\
&= \Gamma(\omega)\Gamma(1 - \omega)
\end{aligned}$$

3.1.1 Relation to Laplace and Fourier Transforms

Mellin transform is closely related to an extended form of other popular transforms, particularly Laplace and Fourier. Both can be obtained through a change of variables. By setting

$$x = e^{-t}, \;\; dx = -e^{-t}dt \tag{3.11}$$

The Mellin transform (3.1) yields[2]

$$\mathcal{M}(f(x), \omega) = (f(e^{-t}), \omega) = \int_{-\infty}^{\infty} f(e^{-t})e^{-\omega t}dt = \mathcal{L}(f(e^{-t}), \omega) \tag{3.12}$$

After the change of function

$$g(t) \equiv f(e^{-t}) \tag{3.13}$$

The two sided Laplace transform of (3.13) is defined by

$$\mathcal{L}(g(t), \omega) = \int_{-\infty}^{\infty} g(t)e^{-\omega t} = \tilde{f}(\omega) \tag{3.14}$$

This can be written symbolically as;

$$\mathcal{M}(f(x), \omega) = \mathcal{L}(f(e^{-t}), \omega) \equiv \mathcal{L}(g(t), \omega) \tag{3.15}$$

The Laplace inversion formula is given by

$$\mathcal{L}^{-1}(\tilde{f}(\omega)) = f(e^{-t}) \equiv g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\omega)e^{\omega t}d\omega \tag{3.16}$$

[2]The occurrence of a strip of holomorphy for Mellin transform can be deduced directly from (3.12). The usual right-sided Laplace transform is analytic in a half-plane $\Re(\omega) > a_1$. In the same way, one can define a left-sided Laplace transform analytic in the region $\Re(\omega) > a_2$. If the two half-planes overlap, the region of holomorphy of the two sided transform is thus the strip $a_1 < \Re(\omega) < a_2$ obtained as their intersection.

To obtain Fourier's transform, let $\alpha, \beta \in I\!R$ and set $\omega = \alpha + 2\pi i\beta$ in (3.12). Then

$$\mathcal{M}(f(x), \omega) = \tilde{f}(\beta) = \int_{-\infty}^{\infty} f(e^{-t})e^{-(\alpha+2\pi i\beta)t}dt = \int_{-\infty}^{\infty} h(t)e^{-2\pi i\beta t}dt \quad (3.17)$$

The result becomes

$$\mathcal{M}(f(x), \omega) = \mathcal{M}(f(x), \alpha + 2\pi i\beta) = \tilde{f}(\alpha + 2\pi i\beta) = \mathcal{F}(h(t), \beta) \quad (3.18)$$

Equation (3.17) is called the Fourier transform of $h(t) = f(e^{-t})e^{-\alpha t}$. The Fourier inversion formula is obtained as

$$\mathcal{F}^{-1}(\tilde{f}(\beta)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\beta)e^{2\pi\beta it}d\beta \equiv h(t) \quad (3.19)$$

Remark 3.1.2

(i) A famous example of (3.5) follows from considering $\Gamma(\omega)$ with real $c > 0$. By means of Stirling's formula[3]

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\omega)x^{-\omega}d\omega$$

Practical inversion can sometimes pose a challenge due to the complex nature of the integral. When possible, this is often achieved by direct contour integration, conversion to polar coordinates, recasting the problem as a product of gamma functions, exploiting properties of the transform in conjunction with the inversion theorem, or by means of previously solved tables of transforms (Oberhettinger (1974)).

[3] $|\Gamma(a + ib)| \sim \sqrt{\pi}|b|^{a-0.5}e^{-0.5|b|\pi}$ when $|b| \to \infty$. See Poularikas (1999).

(ii) For a given value of $\Re(\omega) = \alpha$ belonging to the definition strip, the Mellin transform of a function can be expressed as a Fourier transform.

(iii) By means of a change of variables $x = e^{-t}, dx = -e^{-t}dt$, it is observed that the Mellin transform bears a striking resemblance to the Laplace and the Fourier transforms. In particular, if $\mathcal{L}(.)$ and $\mathcal{F}(.)$ denote the two-sided Laplace and Fourier transforms, respectively, then

$$\mathcal{M}(f(x), \omega) = \mathcal{L}(f(e^{-t}), \omega) = \mathcal{F}(f(e^{-t})e^{-ct}, \beta) \tag{3.20}$$

(iv) There are numerous applications where it has been established that it is more convenient to operate directly with the Mellin transform rather than the Laplace-Fourier version such as theory of analytic functions.

3.1.2 Operational Properties of the Mellin Transforms

The Mellin transform has the ability to reduce complicated functions by realization of its many properties. This section describes the effect of the Mellin transform $\mathcal{M}(f(x), \omega)$ of some special operations performed on $f(x)$. The resulting formulas are very useful for deducing new correspondences from a given one.

Let $\tilde{f}(\omega) = \mathcal{M}(f(x), \omega)$ be the Mellin transform of a distribution and denote $U_f = \omega : a_1 < \Re(\omega) < a_2$, then the following properties of the Mellin transform hold.

Scaling Property

$$\mathcal{M}(f(ax), \omega) = \int_0^\infty f(ax)x^{\omega-1}dx = a^{-\omega}\tilde{f}(\omega), \quad a > 0 \tag{3.21}$$

Shifting Property

$$\mathcal{M}(x^a f(x), \omega) = \int_0^\infty x^a f(x) x^{\omega-1} dx = \tilde{f}(a+\omega), \quad a > 0 \tag{3.22}$$

Mellin Transform of Derivatives

$$\mathcal{M}\left(\frac{d^k}{dx^k} f(x), \omega\right) = \int_0^\infty \frac{d^k}{dx^k} f(x) x^{\omega-1} dx = (-1)^k (\omega-1)_k \tilde{f}(\omega-k) \tag{3.23}$$

where

$$(\omega-k)_k = (\omega-k)(\omega-k+1)...(\omega-1) = \frac{(\omega-1)!}{(\omega-k-1)!} = \frac{\Gamma(\omega)}{\Gamma(\omega-k)} \tag{3.24}$$

for a positive integer k, provided that for $r = 0, 1, 2, ..., k-1$

$$\lim_{x\to 0^+} x^{\omega-r-1} f^{(k-r-1)}(x) = \lim_{x\to\infty} x^{\omega-r-1} f^{(k-r-1)}(x) = 0$$

For $k = 1$, (3.23) becomes

$$\mathcal{M}\left(\frac{d}{dx} f(x), \omega\right) = \int_0^\infty \frac{d}{dx} f(x) x^{\omega-1} dx = -(\omega-1)\tilde{f}(\omega-1)$$

provided

$$\lim_{x\to 0^+} x^{\omega-1} f(x) = \lim_{x\to\infty} x^{\omega-1} f(x) = 0$$

The statement is proved straightforwardly using integration by parts.

Derivative Multiplied by Independent Variable

$$\mathcal{M}\left(x^k \frac{d^k}{dx^k} f(x), \omega\right) = \int_0^\infty x^k \frac{d^k}{dx^k} f(x) x^{\omega-1} dx = (-1)^k \omega_k \tilde{f}(\omega)$$
$$= (-1)^k \frac{\Gamma(\omega+k)}{\Gamma(\omega)} \tilde{f}(\omega)_k \tag{3.25}$$

For example, if $k = 2$, using (3.25) yields

$$\mathcal{M}\left(x^2\frac{d^2}{dx^2}f(x), \omega\right) = \int_0^\infty x^2\frac{d^2}{dx^2}f(x)x^{\omega-1}dx = (\omega^2+\omega)\tilde{f}(\omega)$$

Mellin Transform of Integrals

$$\mathcal{M}\left(\left(\int_0^x f(x)dx\right), \omega\right) = \int_0^\infty \left(\int_0^x f(x)dx\right)x^{\omega-1}dx = \frac{-\tilde{f}(\omega+1)}{\omega} \tag{3.26}$$

Raising the Independent Variable to a Real Power

$$\mathcal{M}(f(x^a), \omega) = \int_0^\infty f(x^a)x^{\omega-1}dx$$

Let $x = t^{\frac{1}{a}}$, this implies that $dx = \frac{1}{a}t^{\left(\frac{1-a}{a}\right)}dt$. Therefore

$$\begin{aligned}\mathcal{M}(f(x^a), \omega) &= a^{-1}\int_0^\infty f(t)t^{\left(\frac{1-a}{a}\right)}t^{\left(\frac{\omega-1}{a}\right)}dt \\ &= a^{-1}\int_0^\infty f(t)t^{\left(\frac{\omega}{a}-1\right)}dt \\ &= a^{-1}\tilde{f}\left(\frac{\omega}{a}\right)\end{aligned} \tag{3.27}$$

where $a \geq 0$ is required for $\tilde{f}\left(\frac{\omega}{a}\right)$ to be analytic. By a similar method to (3.22) and (3.27) leads to a relation

$$\mathcal{M}(x^{-1}f(x^{-1}), \omega) = \tilde{f}(1-\omega) \tag{3.28}$$

Equation (3.28) is the property of the Mellin transform for inverse of independent variable.

Multiplication of the Original Function by $\ln x$

$$\mathcal{M}((\ln x)f(x), \omega) = \frac{d}{d\omega}\tilde{f}(\omega) \tag{3.29}$$

In general,

$$\mathcal{M}((\ln x)^k f(x), \omega) = \frac{d^k}{d\omega^k}\tilde{f}(\omega), \; k \in \mathbb{Z}^+ \tag{3.30}$$

Equation (3.30) is the multiplication of the original function by the power of $\ln x$.

Convolution Property

$$\mathcal{M}(f(x).g(x), \omega) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(z_0)\tilde{g}(\omega - z_0)dz_0 \tag{3.31}$$

Multiplicative Convolution Property

$$\mathcal{M}(f(x) * g(x), \omega) = \mathcal{M}\left(\int_0^\infty f(u)g\left(\frac{x}{u}\right)\frac{du}{u}, \omega\right) = \tilde{f}(\omega)\tilde{g}(\omega) \tag{3.32}$$

$$\mathcal{M}\left(\int_0^\infty f(x,u)g(u)du, \omega\right) = \tilde{f}(\omega)\tilde{g}(1-\omega) = \mathcal{M}(f(x)\circ g(x), \omega) \tag{3.33}$$

Parseval's Formula

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}(f(x), 1-\omega)\mathcal{M}(g(x), \omega)d\omega \tag{3.34}$$

Remark 3.1.3

(i) Equations (3.22) and (3.23) can be used in various ways to find the effect of linear combinations of differential operators such that $x^k\left(\frac{d}{dx}\right)^m$, k, m integers. The most remarkable result is

$$\mathcal{M}\left(\left(x\frac{d}{dx}\right)^k f(x), \omega\right) = (-1)^k\omega^k\tilde{f}(\omega) \tag{3.35}$$

Other combinations can be computed. For example

$$\mathcal{M}\left(\frac{d^k}{dx^k}x^k f(x), \omega\right) = (-1)^k(\omega - k)_k\tilde{f}(\omega) \tag{3.36}$$

These relations are easily verified on infinitely differentiable functions.

(ii) The properties presented above are merely a preview of the transform's applicability on a function of variable. A detailed approach can be found in Zemanian (1968), Sneddon (1972) and Fikioris (2007).

3.2 Multidimensional Mellin Transforms

For multidimensional problems one can extend the concept of the Mellin transforms to functions of several variables. The double Mellin transform was first introduced by Reed (1944), he proved the conditions for which the transform and its inverse exist. For instance the double Mellin transform of a function $f(x_1, x_2)$ is defined by

$$\begin{aligned}\mathcal{M}(f(x_1,x_2),\omega_1,\omega_2) &:= \tilde{f}(\omega_1,\omega_2)\\ &= \int_0^{\infty}\int_0^{\infty} f(x_1,x_2)x_1^{\omega_1-1}x_2^{\omega_2-1}dx_1dx_2\end{aligned} \tag{3.37}$$

for all functions f so that the double integral converges (Applebaum (2009) and Brychkov et al. (1992)). The inversion formula for the double Mellin transform is given by

$$\mathcal{M}^{-1}(\tilde{f}(\omega_1,\omega_2)) = \frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\tilde{f}(\omega_1,\omega_2)x_1^{\omega_1}x_2^{-\omega_2}d\omega_1d\omega_2 \tag{3.38}$$

provided that the integral exists. A convolution-type theorem similar to the one-dimensional case is of the form

$$\begin{aligned}\mathcal{M}(f(x_1,x_2)g(x_1,x_2),\omega_1,\omega_2) &= \mathcal{M}\left(\int_0^{\infty}\int_0^{\infty}\tilde{f}(u,\rho)\left(\frac{x_1}{u},\frac{x_2}{\rho}\right)\frac{1}{u\rho}dud\rho\right)\\ &= \tilde{f}(\omega_1,\omega_2)\tilde{g}(\omega_1,\omega_2)\end{aligned} \tag{3.39}$$

More details on the double Mellin transforms may be found in Reed (1944), Delavault (1961), Brychkov et al. (1992), Hai and Yakubovich (1992), Nguyen and Yakubovich (1992), Eltayeb and Kilicman (2007).

Remark 3.2.1

(i) The definition of the multidimensional Mellin transform and its inverse are given below (Brychkov et al. (1992)):

(a) Let $X = (x_1, x_2, ..., x_n)'$ and $W = (\omega_1, \omega_2, ..., \omega_n)'$. For a function $f(x) \in I\!R^n_+$, the Multidimensional Mellin transform is the complex function

$$\mathcal{M}(f(X), W) := \tilde{f}(W) = \int_{I\!R^n_+} f(X) X^{W-1} dX \tag{3.40}$$

Equation (3.40) can also be written as

$$\begin{aligned}\mathcal{M}(f((x_1, ...x_n), \omega_1, ..., \omega_n)) &:= \tilde{f}(\omega_1, ..., \omega_n) \\ &= \int_{I\!R^n_+} f(x_1, ...x_n) x_1^{\omega_1 - 1} ... x_n^{\omega_n - 1} dx_1 ... dx_n\end{aligned}$$

Therefore

$$\mathcal{M}(f((x_1, ...x_n), \omega_1, ..., \omega_n) = \int_{I\!R^n_+} f(x_1, ...x_n) \prod_{j=1}^{n} x_j^{\omega_j - 1} dx_j \tag{3.41}$$

Existence in the multidimensional case extends naturally from Lemma 3.1.1. Similar to Fourier and Laplace, an inversion theorem in the multidimensional case holds under suitable conditions (Brychkov et al. (1992) and Manuge (2013)).

(b) Let $X = (x_1, x_2, ..., x_n)^{'}$, $W = (\omega_1, \omega_2, ..., \omega_n)^{'}$ and $\tilde{f}(W)$ be analytic on $\vartheta = \times_{j=1}^{n} \vartheta_j$, where ϑ_j are strips in $\mathbb{C}$ defined by $\vartheta_j = \{a_j + ib_j : a_j \in I\!R, b_j = \pm\infty\}$ with $a_j \in \Re(\omega_j)$. Suppose $f(X) \in I\!R_+^n$ is a continuous function, then the inversion formula for the multidimensional Mellin transform is defined as:

$$\mathcal{M}^{-1}(f(W)) := f(X) = \frac{1}{(2\pi i)^n} \int_{\vartheta} \tilde{f}(W) x^{-W} dW \tag{3.42}$$

Equation (3.42) implies that

$$\begin{aligned} \mathcal{M}^{-1}(f(\omega_1, ..., \omega_n)) &= f(x_1, ..., x_n) \\ &= \frac{1}{(2\pi i)^n} \int_{\vartheta} \tilde{f}(\omega_1, ..., \omega_n) x_1^{-\omega_1} ... x_n^{-\omega_n} d\omega_1 ... d\omega_n \end{aligned}$$

Thus

$$\mathcal{M}^{-1}(f(\omega_1, ..., \omega_n)) = \frac{1}{(2\pi i)^2} \int_{\vartheta} \tilde{f}(\omega_1, ..., \omega_n) \prod_{j=1}^{n} x_j^{-\omega_j} d\omega_j \tag{3.43}$$

(ii) The properties of the Mellin transform for single function in subsection 3.1.2 can also be used to obtain solutions of the multidimensional Mellin transform. For example the property in (3.25) for univariate Mellin transform holds for the multidimensional Mellin transform.

$$\mathcal{M}\left(x_i x_j \frac{d^2}{dx_i dx_j} f(X), W \right) = \begin{cases} \omega_i(\omega_i - 1)\tilde{f}(W), & i = j \\ \omega_i \omega_j \tilde{f}(W), & i \neq j. \end{cases} \tag{3.44}$$

where $f(X) \in I\!R_+^n$ is twice differentiable w.r.t x_i and x_j and provided $\prod_{i=1}^{n} x_i^{\omega_i} f(X)$ vanishes as $x_i \to 0^+$ and $x_i \to +\infty$.

3.3 Elements of the Laplace Transforms

Definition 3.3.1

Let $f(x)$ be a piece-wise continuous function[4] on every closed interval $\{a \leq x \leq b\} \subset \{0 \leq x < \infty\}$ there exists $f : \{0 \leq x < \infty\} \to I\!R$, $f : x \to f(x)$ such that $s \in I\!R$. Then $F(s)$ is called the Laplace transform of $f(x)$ and is given by

$$\mathcal{L}(f(x))(s) := F(s) = \int_0^\infty f(x)e^{-sx}dx \tag{3.45}$$

whenever the integral exists. From (3.45), $\mathcal{L}(.)$ is called the Laplace transform and s is called Laplace transform variable.

Definition 3.3.2

Let $\mathcal{L}(f(x))(s) = F(s)$ in the transformed $s-$space, that is, $F(s)$ is the Laplace transform of the function $f(x)$. Then $f(x)$ is called the inverse Laplace transform of $F(s)$. In that case,

$$\mathcal{L}^{-1}(F(s)) := f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{sx}ds \tag{3.46}$$

3.3.1 Operational Properties of the Laplace Transforms

Some of the operational properties of the Laplace transform are presented below;

[4]Intuitively, a piece-wise continuous function is a function that has a finite number of breaks in it and does not blows up to ∞.

Linearity of the Laplace Transforms

$$\begin{aligned}\mathcal{L}(af(x)+bg(x))(s) &= \int_0^\infty (af(x)+bg(x))e^{-sx}dx \\ &= a\mathcal{L}(f(x))(s)+b\mathcal{L}(g(x))(s)\end{aligned} \quad (3.47)$$

Also, if $F(s)=\mathcal{L}(f(x))(s)$ and $G(s)=\mathcal{L}(g(x))(s)$, then

$$\mathcal{L}^{-1}(aF(s)+bG(s)) = af(x)+bg(x) \quad (3.48)$$

The above property is intermediate from the definition and the linearity of the definite integral.

Scaling Property

Let $f(x)$ be a piece-wise continuous function with the Laplace transform $F(s)$. Then for $a>0$. $\mathcal{L}(ax)(s)=\frac{1}{a}F\left(\frac{s}{a}\right)$. That is

$$\mathcal{L}(f(ax))(s) = \int_0^\infty e^{-sx}f(ax)dx = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)}f(z)dz = \frac{1}{a}F\left(\frac{s}{a}\right) \quad (3.49)$$

Commutativity Property

The Laplace transform is commutative. That is

$$F(s)*G(s) = \int_0^x f(x-\varsigma)g(\varsigma)d\varsigma = \int_0^x g(x-\varsigma)f(\varsigma)d\varsigma = G(s)*F(s) \quad (3.50)$$

Shifting Property

$$\mathcal{L}(e^{ax}f(x))(s) = \int_0^\infty e^{ax}e^{-sx}dx = \int_0^\infty e^{(a-s)x}dx = F(s-a) \quad (3.51)$$

The Laplace Transforms on Differentiation

Let $f(x)$, for $x>0$, be a differentiable function with the derivative $f'(x)$

being continuous. Suppose that there exist constant M and X such that $|f(x)| \leq Me^{\alpha x} \forall\ x \geq X$. If $\mathcal{L}(f(x))(s) = F(s)$, then

$$\mathcal{L}(f(x))(s) = \int_0^\infty e^{-sx} f'(x)dx = \lim_{b\to\infty} \int_0^b e^{-sx} f'(x)dx = sF(0) - f(0) \quad (3.52)$$

Note that the condition $|f(x)| \leq Me^{\alpha x}$, $\forall\ x \leq\ X \Rightarrow \lim_{b\to\infty} f(b)e^{-sb} = 0$ for $s > \alpha$.

Convolution Property

Let $F(s)$ and $G(s)$ denote the Laplace transforms of $f(x)$ and $g(x)$, respectively. Then the product given by $H(s) = F(s)G(s)$ is the Laplace transform of the convolution of f and g is denoted by $h(x) = (f * g)(x)$ and has the integral representation

$$h(x) = (f * g)(x) = \int_0^x f(\varsigma)g(x - \varsigma)d\varsigma \quad (3.53)$$

3.4 Elements of the Fourier Transforms

Definition 3.4.1

Suppose $f(x)$ is absolutely integrable in $(-\infty, \infty)$, that is, $\int_{-\infty}^{\infty} |f(x)|dx < \infty$, then the Fourier transform of $f(x)$ is defined as

$$\mathcal{F}(f(x), \theta) = \tilde{f}(\theta) = \int_{-\infty}^{\infty} f(x)e^{i\theta x}dx \quad (3.54)$$

Conversely the inverse Fourier transform of $\tilde{f}(k)$ is defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\theta)e^{-i\theta x}d\theta \quad (3.55)$$

3.4.1 Operational Properties of the Fourier Transforms

Let the Fourier transform of $f(x)$ be defined as $\mathcal{F}(f(x),\theta) = \tilde{f}(\theta)$ then the following properties hold as follows;

Scaling Property

$$\mathcal{F}(f(cx),\theta) = \int_{-\infty}^{\infty} f(cx)e^{i\theta x}dx = \frac{1}{|c|}\tilde{f}\left(\frac{\theta}{c}\right) \tag{3.56}$$

Translation Property

$$\mathcal{F}(f(x-x_0),\theta) = \int_{-\infty}^{\infty} f(x-x_0)e^{i\theta x}dx = e^{i\theta x_0}\tilde{f}(\theta) \tag{3.57}$$

Fourier Transform of Derivatives

$$\mathcal{F}\left(\frac{df(x)}{dx},\theta\right) = i\theta\tilde{f}(\theta) \tag{3.58}$$

This process can be iterated for the n^{th} derivative to yield

$$\mathcal{F}\left(\frac{d^n f(x)}{dx^n},\theta\right) = (i\theta)^n\tilde{f}(\theta) \tag{3.59}$$

Linearity Property

$$\mathcal{F}((af(x)+bg(x)),\theta) = \int_{-\infty}^{\infty} (af(x)+bg(x))e^{i\theta x}dx = a\tilde{f}(\theta) + b\tilde{g}(\theta) \tag{3.60}$$

Convolution Property

One of the most valuable properties of the Fourier transforms is that convolution in the x-domain reduces to multiplication in the θ-domain.

Let $f(x)$ and $g(x)$ be two functions whose Fourier transforms are given by

$\tilde{f}(\theta)$ and $\tilde{g}(\theta)$, respectively. The convolution of $f(x)$ and $g(x)$, denoted as $(f * g)(x)$ is then given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy \tag{3.61}$$

(Note that the order of convolution is immaterial, that is, $f * g = g * f$)

$$\begin{aligned}
\mathcal{F}((f * g)(x), \theta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\theta x} f(y)g(x-y)dxdy & (3.62)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{i\theta(x-y)}g(x-y)dx)e^{i\theta y} f(y)dy & (3.63)\\
&= \int_{-\infty}^{\infty} e^{i\theta y} f(y)dy \int_{-\infty}^{\infty} e^{i\theta(x-y)}g(x-y)dx & (3.64)\\
&= \tilde{f}(\theta)\tilde{g}(\theta) & (3.65)
\end{aligned}$$

3.5 Stochastic Calculus

Due to the underlying random nature of financial markets, stochastic calculus is an important tool for the modelling of financial processes. Even though assets are not traded continuously and asset prices change by discrete values, continuous-time and continuous variable processes are useful to model these prices. The theoretical concepts presented in this section are described on a more rigorous level in Wilmott et al. (1995), Karatzas and Shreve (1998), Oksendal (2003), Protter (2007), Applebaum (2009), Ekhaguere (2010).

Definition 3.5.1

A stochastic process X_t index $T \subseteq I\!R$ is a collection of $\{X_t : t \in T\}$ of random variable on a probability space $(\Omega, I\!B, \mathbf{P})$. That is, $\omega \to X(t, \omega) \in I\!R^d$,

$\omega \to X(t,\omega) = X_t(\omega)$. Now, this means that X_t is an $I\!R^d$-valued random variable for each $t \in T$, Ω is a sample space, $I\!B$ is a set of events and $\mathbf{P}$ is the measure that assigns probabilities to each event and $\omega \in \Omega$.

Definition 3.5.2

A random process $W_t, t \in [0,\infty]$ is a Brownian motion if

(i) W_t has both stationary and independent increments, that is, if $0 < t_1 < ... < t_n$, then the random variables $W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are stochastically independent.

(ii) W_t is a continuous function of time with $W_0 = 0$, almost surely.

(iii) For $0 \leq s \leq t, W_t - W_s$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2|t-s|$. This property indicates that $(W_t - W_s) \sim N(\mu(t-s), \sigma^2|t-s|)$, where μ and $\sigma \neq 0$ are real numbers.

Remark 3.5.1

(i) The $(0,1)$ Brownian motion is called the standard Brownian motion or a Wiener process.

(ii) A (μ,σ) Brownian motion is also called a generalized Wiener process or the Wiener Bachelier process.

Definition 3.5.3

If X_t is a Brownian motion with drift rate μ and variance rate σ^2, the process $\{Y_t = e^{X_t}, t \geq 0\}$ is called a geometric Brownian motion or expected

Brownian motion. The mean and the variance are given by $E[Y_t] = e^{\left(\mu+\frac{\sigma^2}{2}\right)t}$ and $Var[Y_t] = e^{(2\mu+\sigma^2)t}(e^{\sigma^2 t} - 1)$ respectively. Figure 3.1 below shows the behaviour of two sample paths of geometric Brownian motion with different parameters.

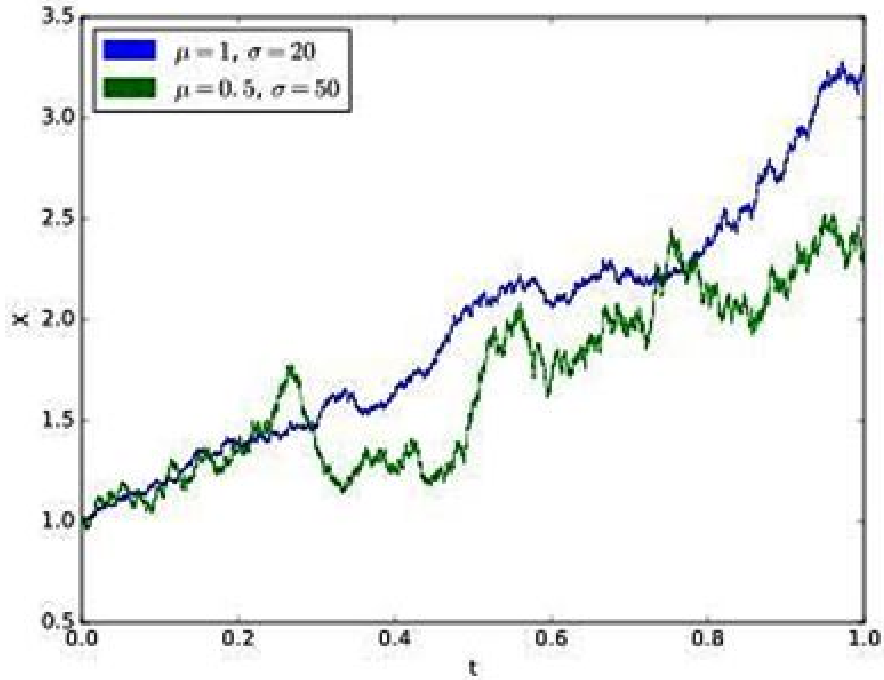

Figure 3.1: Two sample paths of geometric Brownian motion, with different parameters. The blue line has larger drift, the green line has larger variance.

Definition 3.5.4

Let $X : T \to L^0(\Omega, I\!R^d)$ be an adapted $I\!R$-valued stochastic process on a filtered probability space $(\Omega, I\!B, \mathbf{P}, \mathcal{F}(I\!B))$, where $\mathcal{F}(I\!B) = \{I\!B_t : t \in [0, \infty)\}$ is called filtration of $I\!B$. As usual assume that $\mathcal{F}(I\!B)$ satisfies the condition of right continuity.[5] Under this framework, the filtration represents an increasing set of observable that becomes known to market participants as time progresses. Then X is called a martingale if $E(X_t \setminus I\!B_s) = X_s$ (almost surely, whenever $t > s$).

3.5.1 Stochastic Differential Equation

A stochastic differential equation is a differential equation in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process. Stochastic processes under consideration will be defined in terms of their stochastic differential equations

$$dX_t = \mu(X_t, t)X_t dt + \sigma(X_t, t)X_t dW_t, \quad X(t_0) = x_0 \tag{3.66}$$

where $\mu(X_t, t)$ and $\sigma(X_t, t)$ are called the drift and diffusion functions, respectively from $I\!R \times [0, T]$ to $I\!R$. The sufficient conditions for a unique (path-by-path) solution are called the growth condition and the Lipschitz condition.

Growth Condition: There exists a constant $K > 0$ such that

$$\mu^2(x, t) + \sigma^2(x, t) \le K(1 + x^2), \quad (x, t) \in I\!R \times [0, T] \tag{3.67}$$

[5]No jump discontinuity occurs while approaching the limit from the right.

Lipschitz Condition: There exists a constant $L > 0$ such that

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le L|x-y|, \quad x, y \in I\!R, t \in [0,T] \quad (3.68)$$

For the proof of the above conditions (Karatzas and Shreve (1998), Øksendal (2003)).

3.5.2 It$\hat{o}$'s Calculus

Let $(\Omega, I\!B, \mu, \mathcal{F}(I\!B))$ be a filtered probability space and W_t is a Brownian motion defined on this space. Then the stochastic process $X = \{X_t, t \ge 0\}$ that solves

$$X_t = x_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s \quad (3.69)$$

is called an It$\hat{o}$'s process provided the functions $\mu(X_t, t)$ and $\sigma(X_t, t)$ satisfy the following conditions

$$P\left[\int_0^t |\mu(X_s, s)|ds < \infty, \ \forall \ t \ge 0\right] = 1 \quad (3.70)$$

$$P\left[\int_0^t |\sigma(X_s, s)|ds < \infty, \ \forall \ t \ge 0\right] = 1 \quad (3.71)$$

Remark 3.5.2

(i) The above conditions (3.70) and (3.71) required that the drift μ and diffusion σ parameters do not vary much over time.

(ii) Since (3.66) can be represented as a sum of a Lebesgue and It$\hat{o}$ integral, It$\hat{o}$'s lemma provides its solution.[6]

[6]Alternate forms of this theorem can be stated when the function is driven by a Lévy process or more general semimartingale of arbitrary dimension.

Lemma 3.5.1 (Itô's Lemma) (Proter (2004))

Let $u(x,t) \in I\!R^2$ be twice differentiable in x and once in t. Then (3.66) becomes

$$du(X_t,t) = \left(\frac{\partial u(X_t,t)}{\partial t} + \mu_t \frac{\partial u(X_t,t)}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 u(X_t,t)}{\partial x^2}\right)dt + \sigma_t \frac{\partial u(X_t,t)}{\partial x} dW_t \tag{3.72}$$

in P, almost surely [7].

Remark 3.5.3

(i) Equation (3.72) has been proved to be vital in mathematical modelling of derivative pricing. Then $u(X_t,t)$ follows an Itô's process with drift rate $\left(\frac{\partial u(X_t,t)}{\partial t} + \mu_t \frac{\partial u(X_t,t)}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 u(X_t,t)}{\partial x^2}\right)$ and the variance $\left(\sigma_t \frac{\partial u(X_t,t)}{\partial x}\right)^2$.

(ii) For $t \in [0,T]$, one-dimension Brownian motion becomes,

$$d\ln(X_t) = d\ln\left(\frac{X_t}{X_0}\right) = \sigma dW_t + (\mu - \frac{\sigma^2}{2})dt$$

and hence the solution is given by

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

(iii) The above result is crucial for solving stochastic differential equation in one-dimensional space and time. Arguably the best known application of Itô's lemma is for obtaining the solution to the Black-Scholes-Merton equation (Black and Scholes (1973)).

[7]Note the form of (3.72); the solution of an Itô drift-diffusion process is an Itô drift-diffusion process.

3.5.3 Underlying Asset Price Dynamics

It is assumed that the underlying asset price S_t follows a geometric Brownian motion with drift (expected return) μ and volatility σ. That is

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.73}$$

where W_t is a standard Brownian motion. Applying Itô's lemma (3.72) to $u(S_t, t) = \ln S_t$ yields

$$d(\ln S_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t \tag{3.74}$$

It follows that $\ln S_t$ is a Brownian motion with drift $\left(\mu - \frac{\sigma^2}{2}\right)$ and variance σ^2. Therefore,

$$\ln S_T - \ln S_t \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right) \tag{3.75}$$

where $\mathcal{N}$ is the normal distribution function. Therefore, the underlying asset price S_t is lognormally distributed random variable.

Remark 3.5.4

(i) One important consequence of this lognormal assumption is that the underlying asset price becomes zero at $t = 0$, then the asset remain worthless for any time $t \leq s$.

(ii) The explicit formula for the evolution of the underlying asset price at $t = 0$ is given by

$$S_T = S_0 \exp\left[\left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma Z\sqrt{T}\right] \tag{3.76}$$

where $Z \sim \mathcal{N}(0, 1)$.

(iii) The evolution of an underlying asset price in a geometric Brownian motion path using (3.76) is shown in Figure 3.2 below. Figure 3.2 gives a better understanding of the stochastic behaviour of the underlying assets and the assumption that stock returns are lognormally distributed.

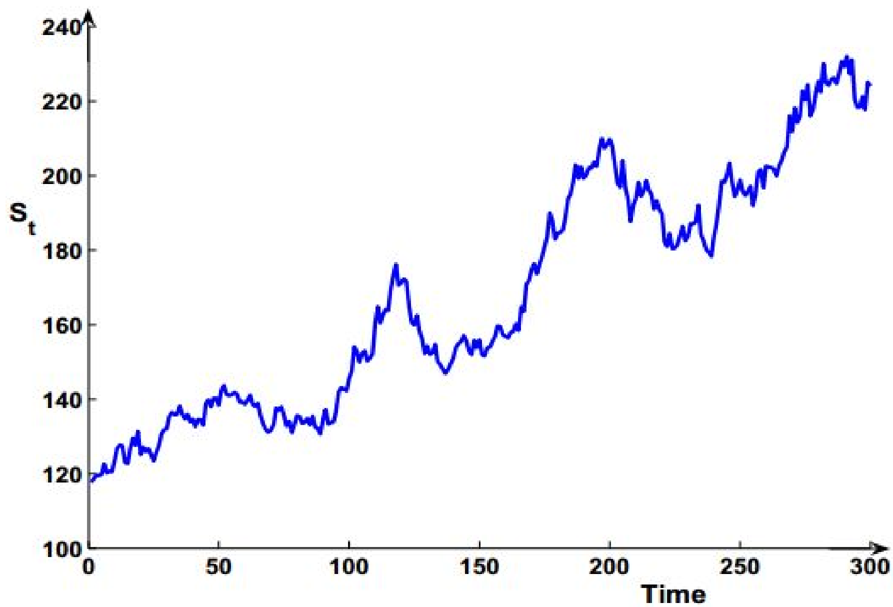

Figure 3.2: Simulation of a geometric Brownian motion path with the following parameters $S_0 = 120, \sigma = 0.30, \mu = 0.15, T = 1$ and $N = 300$ as samples drawn from the standard normal distribution.

3.6 Derivative Security

Derivative security is defined as a financial asset whose value is derived in part from the value and characteristics of some other underlying assets. This term is very broad due to the introduction of complex and varying derivatives in the markets. There are four (4) types of derivative securities namely: options, forward, futures and swaps.

Definition 3.6.1

Vanilla options are actively traded on organized exchanges. They are also subject to certain regularity and standardization conditions. Vanilla options can be classified according to their exercise features as European options and American options.

Definition 3.6.2

The European call(put) option gives the holder the right but not the obligation to buy(sell) the underlying asset S_t at a given expiry date T and for a fixed price K. European options are easier to study and can provide key insights into pricing issues. Let the European call(put) option be denoted by $E_c(E_p)$. The payoff of the European call option E_c at the expiry date T is given by $\text{Payoff}(E_c) = \max(S_T - K, 0) = (S_T - K)^+$. If $S_T < K$, the European will be worthless and the holder will not be able to exercise the right. The payoff of the European put option E_p at the expiry date T is given by $\text{Payoff}(E_p) = \max(K - S_T, 0) = (K - S_T)^+$. If $S_T > K$, then the European put option will be worthless and the holder will not exercise the right. The put-call parity is the relationship between the European call and put, given by

$$E_c + Ke^{-rt} = E_p + S_t \tag{3.77}$$

where r denotes the risk-free interest rate and S_t denotes the underlying asset price.

Remark 3.6.1

> Consider the holder of a European call or put option. If the future price of the underlying asset will be greater (call) or less (put) than the strike price declared at insurance, the holder may buy or sell the option for a positive return. Otherwise, the value of the option is zero as shown in the Figures 3.3 and 3.4 below.

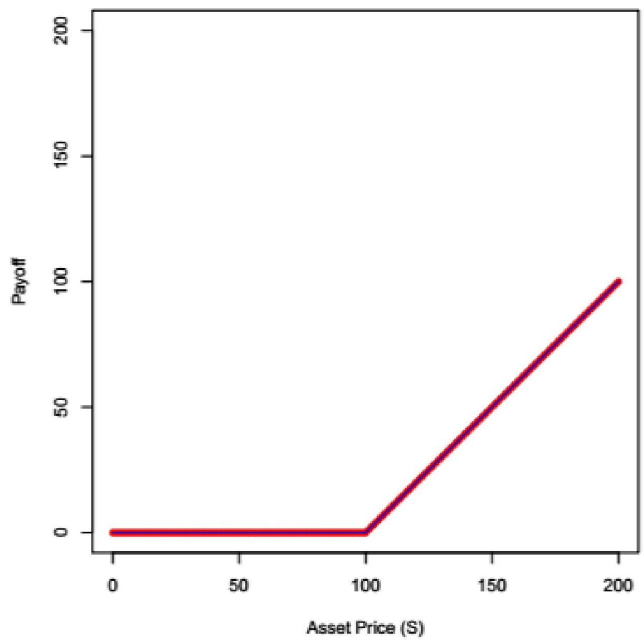

Figure 3.3: The payoff for a European call option for different values of the asset price S_t, given strike price $K = \$100$.

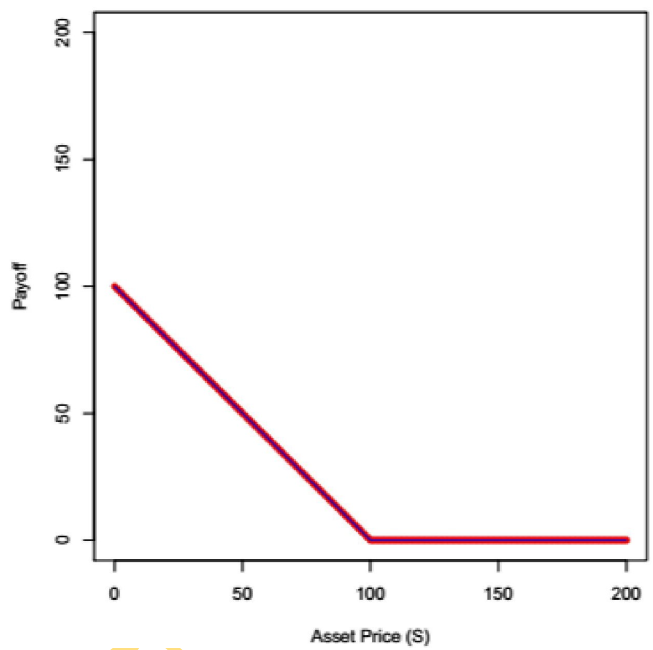

Figure 3.4: The payoff for a European put option for different values of the asset price S_t, given strike price $K = \$100$.

Definition 3.6.3

An American option gives a financial agent the right, but not obligation to buy (if it is a call option) or to sell (if it is a put option) an underlying assets on or prior to the expiry date T at the specified price called the strike price K. Most of the options traded on the exchanges are of the American type. Let the price of the American call(put) option be denoted by $A_c(A_p)$.
The payoff of the American call option A_c at the expiry date T is given by $\text{Payoff}(A_c) = \max(S_T - K, 0) = (S_T - K)^+$. The payoff of the American put option A_p at the expiry date T is given by $\text{Payoff}(A_p) = \max(K - S_T, 0) = (K - S_T)^+$. The price boundary and the put-call parity for the American option is given by

$$S_t - K \leq A_c - A_p \leq S_t - Ke^{-rt} \tag{3.78}$$

Definition 3.6.4

An exotic option is a derivative which has features making it more complex than commonly traded products such as vanilla options. Exotic options are generally traded over the counter. Some of these options include Asian options, where the payoff depends on the average stock price, barrier options that become worthless if the stock price goes above or below a prescribed value and others like power, one-touch, rainbow, forward start, chooser, look-back, contingent premium and quanto options.

3.6.1 Power Options

Power option is a financial derivative in which the payoff at time to expiry is related to the n^{th} power of the underlying asset price. Because of the non-linear characteristics of these options, they are appropriate for hedging non-linear price risks. Power options preserve volatility exposure better than plain vanilla options if the underlying moves significantly in the same direction. These options offer flexibility to investors and of practical interest since many OTC-traded options exhibit such a payoff structure. For example, an option whose payoff is a polynomial function of the Nikkei level at the expiry was issued in Tokyo (Heynen and Kat (1996)). Bankers Trust in Germany has issued capped foreign-exchange power options with power exponent two (Topper (1999), Zhang et al. (2016)). More examples can be found in Tompkins (1999) and Macovschi and Quittard-Pinon (2006). Power option comes in two forms namely power call option and power put option. A power call option is an option with non-linear payoff given by the difference between underlying asset price at expiry raised to a strictly positive power and the strike price. A power put option is an option with non-linear payoff given by the difference between the strike price and underlying asset price at expiry raised to a strictly positive power. For a power option on the underlying asset price S_T^n with strike price K and time to expiry T, the payoff for the power call option is given by

$$P_c^n(S_T^n, T) = \max(S_T^n - K, 0) = (S_T^n - K)^+ \tag{3.79}$$

and the payoff for the power put option is given by

$$P_p^n(S_T^n, T) = \max(K - S_T^n, 0) = (K - S_T^n)^+ \tag{3.80}$$

where n is some power $(n > 0)$

Remark 3.6.2

(i) For $n = 1$, (3.79) and (3.80) become the payoffs for plain vanilla call and put options given by $P_c(S_T, T) = (S_T - K)^+$ and $P_c(S_T, T) = (K - S_T)^+$ respectively.

(ii) For $n > 0$, power option allows parties to negotiate the underlying asset price, strike price, time to expiry and other features. It also gives investors the opportunity to trade on a large scale with expanded or eliminated position limit and is of practical interest since over-the-counter (OTC) traded options exhibit such a payoff structure.

(iii) For $n < 1$, the payoff curve for power call option becomes concave and thus the option can have negative time value. That is
$S_t^n < P_c^n(S_T^n, T)$.

(iv) For $n > 1$, the payoff curve for power put option becomes concave and thus the option can have negative time value. That is
$S_t^n > P_p^n(S_T^n, T)$.

(v) For power call option, the option value becomes very large as n increases.

(vi) For power put option, the option value becomes very large as n decreases.

More details on exotic and vanilla options can be found in Fisher (1993), Wilmott et al. (1995), Hull (1997), Taleb (1997), Kwok (1998), Zhang (1998) and Bellalah (2009).

3.7 Black-Scholes-Merton Model

Assume that the price of a risky asset S_t at current time t is given by

$$S_t = S_0 e^{X_t} \tag{3.81}$$

where X_t is the Brownian motion. Imposing general conditions on some function $f(S_t, t)$, a partial differential equation representing the option price can be obtained. Otherwise known as the Black-Scholes-Merton equation, it provides the price of European options when the appropriate boundary conditions are imposed. For geometric Brownian motion represented by (3.66), a continuous dividend rate q is included in the model by setting $\mu = (r - q)$. Then (3.73) becomes

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{3.82}$$

Once again, applying the Itô's lemma to a function $f(S_t, t)$ representing the option value with dividend yield q leads to

$$\begin{aligned} df(S_t, t) = &\left(\frac{\partial f(S_t, t)}{\partial t} + (r - q)S_t \frac{\partial f(S_t, t)}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f(S_t, t)}{\partial S_t^2} \right) dt \\ &+ \sigma S_t \frac{\partial f(S_t, t)}{\partial S_t} dW_t \end{aligned} \tag{3.83}$$

By constructing a self financing portfolio $\Pi = f(S_t, t) - \Delta S_t$ (with $\Delta = \frac{\partial f(S_t,t)}{\partial S_t}$) consisting of an option $f(S_t, t)$ and underlying asset S_t, therefore

$$d\Pi = \left(\frac{\partial f(S_t,t)}{\partial t} - qS_t\frac{\partial f(S_t,t)}{\partial S_t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2}\right) dt \tag{3.84}$$

Under no-arbitrage condition, the portfolio must earn risk-free rate of return such that $d\Pi = r\Pi dt$ (Wilmott (1995), Hull (2002), Øksendal (2003)). Hence,

$$d\Pi = r\left(f(S_t,t) - \frac{S_t \partial f(S_t,t)}{\partial S_t}\right) dt \tag{3.85}$$

By combining (3.84) and (3.85), then the Black-Scholes-Merton equation is obtained as

$$\frac{\partial f(S_t,t)}{\partial t} + (r-q)S_t\frac{\partial f(S_t,t)}{\partial S_t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2} - rf(S_t,t) = 0 \tag{3.86}$$

Remark 3.7.1

(i) The constant dividend yield q is most suitable for options on foreign currencies; and it can be easily extended to the case of options on commodities as well.

(ii) Note that the Black-Scholes-Merton equation does not involve the drift μ and therefore, the option price does not depend on the risk preferences of the investor.

(iii) Setting $q = 0$ in (3.86) leads to the celebrated Black-Scholes equation derived by Black and Scholes (1973).

(iv) Setting $f(S_t, t) = E_c(S_t, t)$ in (3.86) and by means of change of variables technique, the Black-Scholes-Merton model for the price of the European call option denoted by $E_c(S_t, t)$ is obtained as

$$E_c(S_t, t) = Se^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \tag{3.87}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.88}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.89}$$

(v) The Black-Scholes-Merton model for the price of the European put option denoted by $E_p(S_t, t)$ can be obtained directly using the put-call parity relationship for European options (3.77) as

$$E_p(S_t, t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - Se^{-q(T-t)}\mathcal{N}(-d_1) \tag{3.90}$$

with d_1 and d_2 as defined in (3.88) and (3.89), respectively.

(vi) More details on the derivation of the Black-Scholes model for the price of European call and put options on stocks that pay continuous dividend yield can be found in (Merton (1973)).

Chapter 4

Results

In this chapter, It was shown that the stock dynamics of power options followed a lognormal distribution. The generalized fundamental valuation equation for the price of power options with non-dividend and dividend yields, respectively was derived. By means of risk-free probability measure, the valuation formula for power call option in the Black-Scholes model framework was obtained. The Mellin transform method was used to obtain the integral representations for the price of the European power put option which pays both non-dividend and dividend yields, respectively. It was also shown that the expression for the European power put option reduced to the fundamental valuation formula by means of the convolution property of the Mellin transform method. The Mellin transform method was extended to obtain the integral representations for the price and the optimal exercise boundary (free boundary) of the American power put option with non-dividend and dividend yields, respectively. It was shown that the integral equation of the American power options matched with the existing characterizations of the integral equations of Kim (1990) and Carr et al. (1992) for $n = 1$. The integral representation for the price of the American power put option with non-dividend and dividend yields, respectively was used to derive the optimal exercise boundary and the analytical valuation formula for the perpetual American power put option. A closed-form solution for the price of the American power put option with dividend yield was obtained. The Mellin transform method in higher dimensions was used to obtain the integral representation for the price of put options on a basket of multi-dividend paying stocks. Other related methods for options valuation were considered. Some numerical experiments and discussion of results were also presented.

4.1 Power Options Valuation

Power options can be classified as European or American. European power option can be exercised only at the expiry date while American power option can be exercised before or at the expiry date. The first result on power option showed that the stock dynamics followed a lognormal distribution.

Theorem 4.1.1

Let S_t^n denote the underlying asset price for power option, σ the volatility, r the risk-free interest rate, n the power of the option, q the dividend yield and W_t the Brownian motion. If the underlying asset price S_t^n follows a random process in

$$dS_t^n = \left(n(r-q) + \frac{n(n-1)\sigma^2}{2} \right) S_t^n dt + n\sigma S_t^n dW_t \tag{4.1}$$

then the explicit formula for the evolution of the underlying asset price is given by

$$S_T^n = S_0^n \exp\left(n\left(r - q - \frac{\sigma^2}{2}\right)T + n\sigma W_T\right) \tag{4.2}$$

Proof: Let

$$u(S_t^n, t) = \ln S_t^n \tag{4.3}$$

Differentiating (4.3) yields

$$\frac{\partial u(S_t^n, t)}{\partial S_t^n} = \frac{1}{S_t^n} \tag{4.4}$$

$$\frac{\partial^2 u(S_t^n, t)}{\partial (S_t^n)^2} = \frac{-1}{(S_t^n)^2} \tag{4.5}$$

$$\frac{\partial u(S_t^n, t)}{\partial t} = 0 \tag{4.6}$$

Recall from the Itô's lemma (3.72) for plain vanilla option and using (4.1) for any derivative $u(S_t^n, t)$ leads to

$$\begin{aligned} du(S_t^n, t) = &\left(\frac{\partial u(S_t^n, t)}{\partial t} + g(S_t^n, t)\frac{\partial u(S_t^n, t)}{\partial S_t^n} + \frac{h^2(S_t^n, t)}{2}\frac{\partial^2 u(S_t^n, t)}{\partial (S_t^n)^2}\right)dt \\ &+ h(S_t^n, t)\frac{\partial u(S_t^n, t)}{\partial S_t^n}dW_t \end{aligned} \tag{4.7}$$

From (4.1),

$$g(S_t^n, t) = \left(n(r - q) + \frac{1}{2}n(n-1)\sigma^2\right)S_t^n, \;\; h(S_t^n, t) = n\sigma S_t^n \tag{4.8}$$

Substituting (4.3), (4.4), (4.5), (4.6) and (4.8) into (4.7) and rearranging the terms yields

$$\begin{aligned} d(\ln S_t^n) = &\left(\left(n(r - q) + \frac{1}{2}n(n-1)\sigma^2\right)S_t^n\left(\frac{1}{S_t^n}\right)\right)dt \\ &+ \left(\frac{1}{2}n^2\sigma^2(S_t^n)^2\left(\frac{-1}{(S_t^n)^2}\right)\right)dt + n\sigma S_t^n\left(\frac{1}{S_t^n}\right)dW_t \end{aligned} \tag{4.9}$$

Therefore,

$$d(\ln S_t^n) = \left(n(r-q) - \frac{1}{2}n\sigma^2\right) dt + n\sigma dW_t \tag{4.10}$$

Thus, $\ln S_t^n$ is a Brownian motion with drift parameter $\left(n(r-q) - \frac{1}{2}n\sigma^2\right)$ and variance parameter $(n\sigma)^2$. To derive an explicit formula for the evolution of the underlying asset price, Integrating (4.10) from 0 to T to obtain

$$\int_0^T d(\ln S_t^n) = \int_0^T \left(n(r-q) - \frac{1}{2}n\sigma^2\right) dt + \int_0^T n\sigma dW_t \tag{4.11}$$

$$\ln S_T^n - \ln S_0^n = \left(n(r-q) - \frac{1}{2}n\sigma^2\right) T + n\sigma W_T \tag{4.12}$$

$$\ln\left(\frac{S_T^n}{S_0^n}\right) = n\left(r - q - \frac{1}{2}\sigma^2\right) T + n\sigma W_T \tag{4.13}$$

Taking the exponential of both sides of (4.13) leads to a relation

$$\left(\frac{S_T^n}{S_0^n}\right) = \exp\left[n\left(r - q - \frac{1}{2}\sigma^2\right) T + n\sigma W_T\right] \tag{4.14}$$

Therefore,

$$S_T^n = S_0^n \exp\left[n\left(r - q - \frac{1}{2}\sigma^2\right) T + n\sigma W_T\right] \tag{4.15}$$

Equation (4.15) is the required explicit formula for the evolution of the underlying asset price.

Remark 4.1.1

(i) Equation (4.15) can also be written as

$$S_T^n = S_0^n \exp\left[n\left(r - q - \frac{1}{2}\sigma^2\right) T + n\sigma Z\sqrt{T}\right] \tag{4.16}$$

where $Z \sim N(0,1)$[1].

[1]This equation (4.16) showed that the stock dynamic follows a lognormal distribution.

(ii) Setting $n = 1$, (4.16) becomes

$$S_T = S_0 \exp\left[\left(r - q - \frac{1}{2}\sigma^2\right)T + \sigma Z\sqrt{T}\right] \tag{4.17}$$

Equation (4.17) shows that plain vanilla option follows a lognormal distribution.

(iii) For the case of non-dividend yield, (4.16) and (4.17) become, respectively

$$S_T^n = S_0^n \exp\left[n\left(r - \frac{1}{2}\sigma^2\right)T + n\sigma Z\sqrt{T}\right] \tag{4.18}$$

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma Z\sqrt{T}\right] \tag{4.19}$$

The generalized fundamental valuation equation for the price of power option was given by the following result.

Theorem 4.1.2

Let the underlying asset price S_t^n follows a lognormal distribution

$$dS_t^n = \left(n(r-q) + \frac{n(n-1)\sigma^2}{2}\right)S_t^n dt + n\sigma S_t^n dW_t$$

Using the Itô's lemma given by (4.7), then the Black-Scholes-Merton-like partial differential equation for any derivative $v(S_t^n, t)$ written on S_t^n for power option is obtained as

$$\frac{\partial v(S_t^n,t)}{\partial t} + n\left((r-q) + \frac{(n-1)\sigma^2}{2}\right)S_t^n \frac{\partial v(S_t^n,t)}{\partial S_t^n}$$
$$+\frac{1}{2}\sigma^2 n^2 (S_t^n)^2 \frac{\partial^2 v(S_t^n,t)}{\partial (S_t^n)^2} = rv(S_t^n,t) \tag{4.20}$$

Proof: Let us write the value of the power option as $v(S_t^n, t, \sigma, q, K, \mu, T, r)$, where $S_t^n, t, \sigma, q, K, \mu, T$ and r are underlying asset price, current time, volatility, dividend yield, strike price, drift parameter, time to expiry and risk-free interest rate, respectively. As the price of the underlying asset falls by the amount of the dividend yield, the asset price dynamics based on the geometric Brownian motion becomes:

$$\frac{dS_t^n}{S_t^n} = \left(n(r-q) + \frac{n(n-1)\sigma^2}{2}\right)dt + n\sigma dW_t \tag{4.21}$$

Using the Itô lemma given by (4.7) with

$$g(S_t^n, t) = \left(n(r-q) + \frac{n(n-1)\sigma^2}{2}\right)S_t^n, h(S_t^n, t) = n\sigma S_t^n \tag{4.22}$$

and setting $u(S_t^n, t) = v(S_t^n, t)$ yields

$$\begin{aligned} dv(S_t^n, t) &= \frac{\partial v(S_t^n, t)}{\partial t}dt + \left(n(r-q) + \frac{n(n-1)\sigma^2}{2}\right)S_t^n\frac{\partial v(S_t^n, t)}{\partial S_t^n}dt \\ &+ \frac{(n\sigma S_t^n)^2}{2}\frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2}dt + n\sigma S_t^n\frac{\partial v(S_t^n, t)}{\partial S_t^n}dW_t \end{aligned} \tag{4.23}$$

Using the assumption of Baz and Chacko (2004) as follows: Assume that the dynamics of marginal utility in the economy at time t are determined by

$$\frac{d\varepsilon_t}{\varepsilon_t} = f(\varepsilon_t, S_t^n)dt + g(\varepsilon_t, S_t^n)dW_t$$

where $f(\varepsilon_t, S_t^n) = -r$ and $g(\varepsilon_t, S_t^n) = \frac{(r-\mu)}{\sigma}$. Hence the dynamics of the pricing kernel is obtained as

$$\frac{d\varepsilon_t}{\varepsilon_t} = -rdt + \frac{(r-\mu)}{\sigma}dW_t \tag{4.24}$$

The stochastic process for $v(S_t^n, t)\varepsilon_t$ is given by

$$d(v(S_t^n, t)\varepsilon_t) = \varepsilon_t dv(S_t^n, t) + v(S_t^n, t)d\varepsilon_t + d\langle v(S_t^n, t), \varepsilon_t\rangle \tag{4.25}$$

Substituting (4.23) and (4.24) into (4.25) leads to

$$\begin{aligned} d(v(S_t^n, t)\varepsilon_t) &= \varepsilon_t \left(\frac{\partial v(S_t^n, t)}{\partial t} dt + \left(n(r-q) + \frac{n(n-1)\sigma^2}{2} \right) S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} dt \right) \\ &+ \varepsilon_t \left(\frac{(n\sigma S_t^n)^2}{2} \frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2} dt + n\sigma S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} dW_t \right) \\ &+ v(S_t^n, t) \left(-rdt + \frac{(r-\mu)}{\sigma} \right) \varepsilon_t dW_t \end{aligned}$$

$$\begin{aligned} d(v(S_t^n, t)\varepsilon_t) &= \varepsilon_t \left(\frac{\partial v(S_t^n, t)}{\partial t} + \left(n(r-q) + \frac{n(n-1)\sigma^2}{2} \right) S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} \right) dt \\ &+ \varepsilon_t \left(\frac{(n\sigma S_t^n)^2}{2} \frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2} - rv(S_t^n, t) \right) dt \\ &+ \varepsilon_t \left(n\sigma S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} + v(S_t^n, t) \frac{(r-\mu)}{\sigma} \right) dW_t \end{aligned} \tag{4.26}$$

Using the fact that $v(S_t^n, t)\varepsilon_t$ is martingale, then the drift coefficient is zero. Therefore

$$\begin{aligned} \frac{\partial v(S_t^n, t)}{\partial t} &+ \left(n(r-q) + \frac{n(n-1)\sigma^2}{2} \right) S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} \\ &+ \frac{(n\sigma S_t^n)^2}{2} \frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2} - rv(S_t^n, t) = 0 \end{aligned} \tag{4.27}$$

Equation (4.27) is called the generalized fundamental valuation equation for the price of power option with dividend yield.

Remark 4.1.2

(i) Alternative method of obtaining the fundamental valuation equation (4.27) using Girsanov's theorem was shown in the following result (Baz and Chacko (2004)).

Theorem 4.1.3

When an economy with a pricing kernel defined by (4.24) is transformed to a risk-neutral economy, any stochastic process X_t (whether X_t is the price of a traded security or not) whose dynamics are characterized by

$$\frac{dX_t}{X_t} = h_1(X_t)dt + h_2(X_t)dW_t$$

in the original economy becomes transformed to the process

$$\frac{dX_t}{X_t} = (h_1(X_t) - g(\varepsilon_t, S_t^*)h_2(X_t))dt + h_2(X_t)dW_t^* \tag{4.28}$$

where W_t^* is simply a Brownian motion in the risk-neutral economy. Applying the Girsanov's theorem gives the stochastic process for the price of power option of the form

$$\begin{aligned}\frac{dv(S_t^n,t)}{v(S_t^n,t)} &= \frac{1}{v(S_t^n,t)}\left(\frac{\partial v(S_t^n,t)}{\partial t}\right)dt \\ &+ \frac{1}{v(S_t^n,t)}\left(\left(n(\mu-q)+\frac{1}{2}n(n-1)\sigma^2\right)S_t^n\frac{\partial v(S_t^n,t)}{\partial S_t^n}\right)dt \\ &+ \frac{1}{v(S_t^n,t)}\left(\frac{n^2\sigma^2}{2}(S_t^n)^2\frac{\partial^2 v(S_t^n,t)}{\partial (S_t^n)^2} + n(r-\mu)S_t^n\frac{\partial v(S_t^n,t)}{\partial S_t^n}\right)dt \\ &+ \frac{1}{v(S_t^n,t)}\left(n\sigma S_t^n\frac{\partial v(S_t^n,t)}{\partial S_t^n}\right)dW_t\end{aligned}$$

$$\begin{aligned}&= \frac{1}{v(S_t^n,t)}\left(\frac{\partial v(S_t^n,t)}{\partial t} + \left(n(r-q)+\frac{1}{2}n(n-1)\sigma^2\right)S_t^n\frac{\partial v(S_t^n,t)}{\partial S_t^n}\right)dt \\ &+ \frac{1}{v(S_t^n,t)}\left(\left(\frac{n^2\sigma^2}{2}(S_t^n)^2\frac{\partial^2 v(S_t^n,t)}{\partial (S_t^n)^2}\right)dt + \left(n\sigma S_t^n\frac{\partial v(S_t^n,t)}{\partial S_t^n}\right)dW_t\right)\end{aligned} \tag{4.29}$$

In a risk neutral world, the expected return of any traded security must equal to the risk-free interest rate. That is

$$r = E_t^* \left[\frac{dv(S_t^n, t)}{v(S_t^n, t)} \right] \tag{4.30}$$

The expected instantaneous return of $\frac{dv(S_t^n,t)}{v(S_t^n,t)}$ is simply the drift term of the stochastic process. So,

$$r = \frac{1}{v(S_t^n, t)} \left(\frac{\partial v(S_t^n, t)}{\partial t} + n \left((r - q) + \frac{1}{2}(n-1)\sigma^2 \right) S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} \right)$$
$$+ \frac{1}{v(S_t^n, t)} \left(\frac{n^2\sigma^2}{2} (S_t^n)^2 \frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2} \right)$$

Hence,

$$\frac{\partial v(S_t^n, t)}{\partial t} + n \left((r - q) + \frac{1}{2}(n-1)\sigma^2 \right) S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n}$$
$$+ \frac{n^2\sigma^2}{2} (S_t^n)^2 \frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2} - rv(S_t^n, t) = 0$$

is the required fundamental valuation equation.

(ii) For the case of non-dividend yield where $q = 0$, (4.27) becomes

$$\begin{aligned} \frac{\partial v(S_t^n, t)}{\partial t} &+ n \left(r + \frac{1}{2}(n-1)\sigma^2 \right) S_t^n \frac{\partial v(S_t^n, t)}{\partial S_t^n} \\ &+ \frac{n^2\sigma^2}{2} (S_t^n)^2 \frac{\partial^2 v(S_t^n, t)}{\partial (S_t^n)^2} - rv(S_t^n, t) = 0 \end{aligned} \tag{4.31}$$

4.1.1 Valuation of Power Options in the Black-Scholes-Like Model

A new approach to derive the Black-Scholes-like model for the valuation of power call option via the risk-free probability measure was presented in the following result.

Theorem 4.1.4

By means of the risk-free probability measure Q, the Black-Scholes-like valuation formula for the price of power call option is given by

$$\begin{aligned} V_c^n(S_t^n, t) = & S_t^n e^{(n-1)\left(r+\frac{n\sigma^2}{2}\right)(T-t)} \mathcal{N}(d_{1,n}(S_t^n, K, (T-t))) \\ & - K e^{-r(T-t)} \mathcal{N}(d_{2,n}(S_t^n, K, (T-t))) \end{aligned} \tag{4.32}$$

with

$$d_{1,n}(S_t^n, K, (T-t)) = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

and

$$d_{2,n}(S_t^n, K, (T-t)) = d_{1,n}(S_t^n, K, (T-t)) - n\sigma\sqrt{(T-t)}$$

where $\mathcal{N}(.)$ is the normal cumulative distribution function of random variable.

Proof: The value of the power call option under the risk-free probability measure Q is given by

$$V_c^n(S_t^n, t) = \mathbf{E}^Q\left[e^{-r(T-t)} P_c^n(S_T^n, T)\right] \tag{4.33}$$

where n is positive and $\mathbf{E}$ is the expectation. Substituting the payoff at time to expiry T of a power option with exercise price K on an underlying asset S_T^n given by (3.79) into (4.33) yields

$$V_c^n(S_t^n, t) = \mathbf{E}^Q\left[e^{-r(T-t)}(S_T^n - K)^+\right]$$

The explicit formula for the evolution of the underlying asset price in (4.16) for the case $t \neq 0$ can be written as

$$S_T^n = S_t^n \exp\left[n\left(r - q - \frac{1}{2}\sigma^2\right)(T-t) + n\sigma Z\sqrt{T-t}\right]$$

The expected value of the stock price at time to expiry T under the risk-free probability measure Q is obtained as

$$\mathbf{E}^Q\left[e^{-r(T-t)}S_T^n\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r(T-t)} e^{\frac{-1}{2}z^2} S_T^n dZ$$

Using the last two relations, (4.33) becomes

$$V_c^n(S_t^n, t) = \int_{-\infty}^{\infty} \frac{e^{-r(T-t)}e^{-\frac{z^2}{2}}\left(S_t^n e^{\left(nZ\sigma\sqrt{(T-t)}+n\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)} - K\right)^+}{\sqrt{2\pi}} dZ \tag{4.34}$$

Since

$$Z \geq \frac{-\ln\left(\frac{S_t^n}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} = -d_{2,n}(S_t^n, K, (T-t)) = -d_{2,n}$$

this implies that

$$S_t^n \exp\left(nz\sigma\sqrt{(T-t)} + n\left(r - \frac{\sigma^2}{2}\right)(T-t)\right) \geq K$$

By changing the lower bound of integration, (4.34) yields

$$V_c^n(S_t^n, t) = \int_{-d_{2,n}}^{\infty} \frac{e^{-r(T-t)}e^{-\frac{z^2}{2}}\left(S_t^n e^{\left(nZ\sigma\sqrt{(T-t)}+n\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)} - K\right)}{\sqrt{2\pi}} dZ \tag{4.35}$$

Equation (4.35) can be expressed in the form

$$V_c^n(S_t^n, t) = A_1 + A_2 \tag{4.36}$$

where the first integral is

$$A_1 = e^{-r(T-t)} \int_{-d_{2,n}}^{\infty} \frac{e^{-\frac{Z^2}{2}} \left(S_t^n e^{\left(nZ\sigma\sqrt{(T-t)}+n\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)} \right)}{\sqrt{2\pi}} dZ$$

and the second integral is

$$A_2 = -e^{-r(T-t)} \int_{-d_{2,n}}^{\infty} \frac{1}{\sqrt{2\pi}} K e^{-\frac{Z^2}{2}} dZ$$

To find more classic representations of A_1 and A_2. Observe that the second integral

$$\begin{aligned} A_2 &= -e^{-r(T-t)} \int_{-d_{2,n}}^{\infty} \frac{1}{\sqrt{2\pi}} K e^{-\frac{Z^2}{2}} dZ \\ &= -e^{-r(T-t)} \int_{-\infty}^{d_{2,n}} \frac{1}{\sqrt{2\pi}} K e^{-\frac{u^2}{2}} du \end{aligned}$$

with the transformation $Z = -u$.

Thus,

$$A_2 = -Ke^{-r(T-t)} \mathcal{N}(d_{2,n}) \tag{4.37}$$

Simplifying A_1 further yields,

$$A_1 = S_t^n e^{(n-1)\left(r+\frac{1}{2}n\sigma^2\right)(T-t)} \int_{-d_{2,n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z-n\sigma\sqrt{(T-t)})^2} dZ$$

Substituting $Z = v + n\sigma\sqrt{(T-t)}$ into the last equation above, therefore

$$A_1 = S_t^n e^{(n-1)\left(r+\frac{1}{2}n\sigma^2\right)(T-t)} \int_{-d_{2,n}-n\sigma\sqrt{(T-t)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv$$

Setting $v = -u$, the second integral becomes

$$A_1 = S_t^n e^{(n-1)\left(r+\frac{1}{2}n\sigma^2\right)(T-t)} \int_{-\infty}^{d_{2,n}+n\sigma\sqrt{(T-t)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

Therefore,

$$A_1 = S_t^n e^{(n-1)\left(r+\frac{1}{2}n\sigma^2\right)(T-t)}\mathcal{N}(d_{1,n}) \tag{4.38}$$

where $d_{1,n} = d_{2,n} + n\sigma\sqrt{(T-t)}$. Thus, using (4.36), (4.37) and (4.38) with the fact that $d_{1,n} = d_{1,n}(S_t^n, K, (T-t)), d_{2,n} = d_{2,n}(S_t^n, K, (T-t))$, the valuation formula for the price of power call option in the Black-Scholes framework with constant volatility, σ and risk-free interest rate, r is obtained as

$$\begin{aligned} V_c^n(S_t^n, t) = S_t^n e^{(n-1)\left(r+\frac{n\sigma^2}{2}\right)(T-t)}\mathcal{N}(d_{1,n}(S_t^n, K, (T-t))) \\ - Ke^{-r(T-t)}\mathcal{N}(d_{2,n}(S_t^n, K, (T-t))) \end{aligned}$$

This completes the proof.

Remark 4.1.3

(i) By means of the put-call parity given by

$$V_c^n(S_t^n, t) + Ke^{-r(T-t)} = V_p^n(S_t^n, t) + S_t^n e^{(n-1)\left(r+\frac{n\sigma^2}{2}\right)(T-t)}$$

The price of power put option is obtained as

$$\begin{aligned} V_p^n(S_t^n, t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}(S_t^n, K, (T-t))) \\ - S_t^n e^{(n-1)\left(r+\frac{n\sigma^2}{2}\right)(T-t)}\mathcal{N}(-d_{1,n}(S_t^n, K, (T-t))) \end{aligned} \tag{4.39}$$

(ii) Equations (4.32) and (4.39) are for the cases of non-dividend paying stock.

(iii) For the case of dividend paying stock, (4.32) and (4.39) become the valuation formula for the price of power call and put options in the Black-Scholes-Merton-like framework respectively.

$$\begin{aligned} V_c^n(S_t^n, t) = S_t^n e^{\left((n-1)r - nq + \frac{n(n-1)\sigma^2}{2}\right)(T-t)}\mathcal{N}(d_{1,n}(S_t^n, K, (T-t))) \\ - Ke^{-r(T-t)}\mathcal{N}(d_{2,n}(S_t^n, K, (T-t))) \end{aligned}$$

and

$$V_p^n(S_t^n,t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}(S_t^n,K,(T-t)))$$
$$- S_t^n e^{\left((n-1)r-nq+\frac{n(n-1)\sigma^2}{2}\right)(T-t)}\mathcal{N}(-d_{1,n}(S_t^n,K,(T-t)))$$

with

$$d_{1,n}(S_t^n,K,(T-t)) = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r-q+\left(n-\frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

and

$$d_{2,n}(S_t^n,K,(T-t)) = d_{1,n}(S_t^n,K,(T-t)) - n\sigma\sqrt{(T-t)}$$

(iv) For $n = 1$, (4.32) and (4.39) become the fundamental valuation formula for plain vanilla call and put options with non-dividend yields, respectively.

4.1.2 Closed-Form Solutions for the Payoffs of Power Call and Put Options

The closed-form solutions for the payoffs of power call and put options was given by the following result.

Theorem 4.1.5

By means of the Mellin transforms, the closed-form solutions for the payoffs of power call and put options are obtained as

$$\mathcal{M}(P_c^n(S_T^n,T)) = \frac{K^{1-\omega}}{\omega(\omega-1)} \qquad (4.40)$$

and

$$\mathcal{M}(P_p^n(S_T^n,T)) = \frac{K^{1+\omega}}{\omega(\omega+1)} \qquad (4.41)$$

respectively.

Proof: Consider the payoff of the power call option given by (3.79) as

$$P_c^n(S_T^n, T) = (S_T^n - K)^+$$

Using the definition of the Mellin transform (3.1), the closed-form solution for the payoff of the power call option is obtained as follows:

$$\begin{aligned}
\mathcal{M}(P_c^n(S_T^n, T), -\omega) &= \int_0^\infty P_c^n(S_T^n, T)(S_T^n)^{-\omega-1} dS_T^n \\
&= \int_0^\infty (S_T^n - K)^+ (S_T^n)^{-\omega-1} dS_T^n \\
&= \int_K^\infty (S_T^n - K)(S_T^n)^{-\omega-1} dS_T^n \qquad (4.42) \\
&= \int_K^\infty S_T^n (S_T^n)^{-\omega-1} dS_T^n - \int_K^\infty K(S_T^n)^{-\omega-1} dS_T^n \\
&= \frac{K^{1-\omega}}{\omega(\omega-1)}
\end{aligned}$$

Equation (4.40) is established. Next, consider the payoff of the power put option given by (3.80) as

$$P_p^n(S_T^n, T) = (K - S_T^n)^+$$

Once again apply (3.1) to get the closed-form solution for the payoff of the power put option as:

$$\begin{aligned}
\mathcal{M}(P_p^n(S_T^n,T),\omega) &= \int_0^\infty P_p^n(S_T^n,T)(S_T^n)^{\omega-1}dS_T^n \\
&= \int_0^\infty (K-S_T^n)^+(S_T^n)^{\omega-1}dS_T^n \\
&= \int_0^K (K-S_T^n)(S_T^n)^{\omega-1}dS_T^n \\
&= \int_0^K K(S_T^n)^{\omega-1}dS_T^n - \int_0^K (S_T^n)^{\omega}dS_T^n \\
&= \frac{K^{1+\omega}}{\omega(\omega+1)}
\end{aligned} \tag{4.43}$$

This completes the proof.

Remark 4.1.4

Equations (4.42) and (4.43) hold for the case where the strike price K is used as transform variable.

4.1.3 Numerical Examples

Example 1

Consider a power option with Six months to expiration, underlying asset price of \$10, power of 2, strike price of \$100, risk-free interest rate of 8%, continuous dividend yield of 6% and expected volatility of the stock of 30%. Find the

(i) value of the power call option

(ii) value of the power put option

Solution:

$$S_t = \$10, K = \$100, n = 2, r = 0.08, q = 0.06, \sigma = 0.3, t = 0, T = 0.5$$

Using the analytic formula for the price of power call option given by

$$V_c^n(S_t^n, t) = S_t^n e^{\left((n-1)r - nq + \frac{n(n-1)\sigma^2}{2}\right)(T-t)} \mathcal{N}(d_{1,n}(S_t^n, K, (T-t)))$$
$$- Ke^{-r(T-t)}\mathcal{N}(d_{2,n}(S_t^n, K, (T-t)))$$

with

$$d_{1,n}(S_t^n, K, (T-t)) = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

and

$$d_{2,n}(S_t^n, K, (T-t)) = d_{1,n}(S_t^n, K, (T-t)) - n\sigma\sqrt{(T-t)}$$

Therefore,

$$d_{1,2}(S_t^2, K, (T-t)) = \frac{\ln\left(\frac{S_t^2}{K}\right) + 2\left(r - q + \left(\frac{3}{2}\right)\sigma^2\right)(T-t)}{2\sigma\sqrt{(T-t)}}$$

$$d_{1,2}(S_t^2, K, (T-t)) = \frac{\ln\left(\frac{10^2}{100}\right) + 2\left(0.08 - 0.06 + \left(\frac{3}{2}\right)0.3^2\right)(0.5)}{2(0.3)\sqrt{0.5}} = 0.3653,$$

$$d_{2,2}(S_t^2, K, (T-t)) = d_{1,2}(S_t^2, K, (T-t)) - 2\sigma\sqrt{(T-t)}$$
$$= 0.3653 - 2(3)\sqrt{0.5} = -0.0589$$

$$\mathcal{N}(d_{1,2}(S_t^2, K, (T-t))) = 0.6426, \mathcal{N}(d_{2,2}(S_t^2, K, (T-t))) = 0.4765$$

The value of the power call option is obtained as

$$V_c^2(S_t^2, t) = S_t^2 e^{\left(r - 2q + \sigma^2\right)(T-t)} \mathcal{N}(d_{1,2}(S_t^2, K, (T-t)))$$
$$- Ke^{-r(T-t)}\mathcal{N}(d_{2,2}(S_t^2, K, (T-t)))$$

$$V_c^2(S_t^2, t) = 10^2(0.6426)e^{\left(0.08-0.12+0.3^2\right)(0.5)} - 100(0.4765)e^{-0.08(0.5)}$$
$$= 20.1051$$

Next, to get the value of the power put option given by

$$V_p^n(S_t^n, t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}(S_t^n, K, (T-t)))$$
$$- S_t^n e^{\left((n-1)r-nq+\frac{n(n-1)\sigma^2}{2}\right)(T-t)}\mathcal{N}(-d_{1,n}(S_t^n, K, (T-t)))$$

For $n = 2$, (4.39) yields

$$V_p^2(S_t^2, t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,2}(S_t^2, K, (T-t)))$$
$$- S_t^2 e^{\left(r-2q+\sigma^2\right)(T-t)}\mathcal{N}(-d_{1,2}(S_t^2, K, (T-t)))$$

where

$$\mathcal{N}(-d_{1,2}(S_t^2, K, (T-t))) = 0.3574, \mathcal{N}(-d_{2,2}(S_t^2, K, (T-t))) = 0.5235$$

Therefore, the value of the power put option is obtained as

$$V_p^2(S_t^2, t) = 100(0.5235)e^{-0.08(0.5)} - 10^2(0.3574)e^{\left(0.08-2(0.06)+0.3^2\right)(0.5)} = 13.6525$$

Example 2

Consider the valuation of the power call and put options with the following parameters; $S_t = \$10, K = \$100, \sigma = \{0.10, 0, 15, 0.20, 0.25, 0.30\}$, $r = 0.08, q = 0.06, T = 0.5, t = 0,$ $n = \{1.90, 1.95, 2.00, 2.05, 2.10\}$

Calculate the call and put values of the power options.

Solution:

The results generated using the above parameters for power call and put options are shown in the Tables 4.1 and 4.2, respectively.

Table 4.1: The price of power call option.

n/σ	0.10	0.15	0.20	0.25	0.30
1.90	0.3102	1.4522	3.2047	5.3446	7.7621
1.95	1.9320	4.2990	6.9724	9.8596	12.9351
2.00	6.7862	9.8585	13.0957	16.5067	20.1051
2.05	15.8587	18.6128	21.8980	25.5429	29.4939
2.10	28.4341	30.4628	33.4555	37.1126	41.2849

Table 4.2: The price of power put option.

n/σ	0.10	0.15	0.20	0.25	0.30
1.90	18.27382	18.9972	20.1600	21.5351	23.0079
1.95	10.2890	12.1467	14.1021	16.9575	17.9810
2.00	4.3539	6.8086	9.1746	11.4533	13.6525
2.05	1.3089	3.3161	5.5476	7.8230	10.0774
2.10	0.2745	1.4031	3.1247	5.1286	7.2508

Remark 4.1.5

(i) From Table 4.1, it is observed that the higher the volatility, the higher the values of the power call option.

(ii) From Table 4.2, it is observed that the higher the volatility, the higher the values of the power put option.

4.2 The Mellin Transform Method for the Valuation of European Power Put Option with Non-Dividend Yield

The Mellin transform method for the valuation of European power put option which pay no dividend yield and its extension for the derivation of the Black-Scholes-like model by means of the convolution property was presented in this section. Despite the great interest for the valuation of option via transform methods, the Mellin transform method has received petite attention. This may relatively be because of the partial differential equation for pricing is formulated in terms of log-prices. Although the introduction of the Mellin transform method to options valuation is relatively new. The integral representation for the price of the European power put option with non-dividend yield via the Mellin transform method was given by the following result.

Theorem 4.2.1

Let S_t^n be the price of the underlying asset, K be the strike price, r be the risk-free interest rate and T be the time to expiry. Assume S_t^n yields no dividend, then the integral representation for the price of the European power put option $P_E^n(S_t^n, t)$ is given by

$$\begin{aligned} P_E^n(S_t^n, t) &= \mathcal{M}^{-1}(\tilde{P}_E^n(\omega, t)) \\ &= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \frac{K^{1+\omega}}{\omega(\omega+1)} \frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}}{(S_t^n)^\omega} d\omega \end{aligned}$$

Proof: Setting $v(S_t^n, t) = P_E^n(S_t^n, t)$ and $q = 0$ in (4.27) yields the partial differential equation for the price of European power put options of the form

$$\begin{aligned} &\frac{\partial P_E^n(S_t^n, t)}{\partial t} + n\left(r + \frac{(n-1)\sigma^2}{2}\right) S_t^n \frac{\partial P_E^n(S_t^n, t)}{\partial S_t^n} \\ &\qquad + \frac{(n\sigma S_t^n)^2}{2} \frac{\partial^2 P_E^n(S_t^n, t)}{\partial (S_t^n)^2} - rP_E^n(S_t^n, t) = 0 \end{aligned} \tag{4.44}$$

with the boundary conditions

$$\lim_{S_t^n \to \infty} P_E^n(S_t^n, t) = 0 \qquad \text{on } [0, T) \tag{4.45}$$

$$P_E^n(S_T^n, T) = (K - S_T^n)^+ \qquad \text{on } [0, \infty) \tag{4.46}$$

$$\lim_{S_t^n \to 0} P_E^n(S_t^n, t) = Ke^{-r(T-t)} \qquad \text{on } [0, T) \tag{4.47}$$

where $P_E^n(S_t^n, t)$ denote the price of the European power put option.

Let $\tilde{P}_E^n(\omega, t)$ be the Mellin transform of the European power put option which is defined by the relation (see section 3.1)

$$\mathcal{M}(P_E^n(S_t^n, t), \omega) = \tilde{P}_E^n(S_t^n, t) = \int_0^\infty P_E^n(S_t^n, t)(S_t^n)^{\omega-1} dS_t^n \tag{4.48}$$

where ω is a complex variable with $0 < \Re(\omega) < \infty$. Conversely the inversion formula for the Mellin transform in (4.48) is defined as

$$P_E^n(S_t^n, t) = \mathcal{M}(\tilde{P}_E^n(\omega, t)) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \tilde{P}_E^n(\omega, t)(S_t^n)^{-\omega} d\omega \tag{4.49}$$

Taking the Mellin transform of (4.44) to obtain

$$\mathcal{M}\left(\frac{\partial P_E^n(S_t^n,t)}{\partial t}+n\left(r+\frac{(n-1)\sigma^2}{2}\right)S_t^n\frac{\partial P_E^n(S_t^n,t)}{\partial S_t^n},\omega\right)$$

$$+\mathcal{M}\left(\frac{(n\sigma S_t^n)^2}{2}\frac{\partial^2 P_E^n(S_t^n,t)}{\partial (S_t^n)^2}-rP_E^n(S_t^n,t),\omega\right)=\mathcal{M}(0,\omega) \tag{4.50}$$

where

$$\mathcal{M}\left(\frac{\partial P_E^n(S_t^n,t)}{\partial t},\omega\right)=\int_0^\infty \frac{\partial P_E^n(S_t^n,t)}{\partial t}(S_t^n)^{\omega-1}dS_t^n=\frac{\partial \tilde{P}_E^n(\omega,t)}{\partial t} \tag{4.51}$$

$$\mathcal{M}\left(n\left(r+\frac{(n-1)\sigma^2}{2}\right)S_t^n\frac{\partial P_E^n(S_t^n,t)}{\partial S_t^n},\omega\right)$$

$$=\int_0^\infty \left(n\left(r+\frac{(n-1)\sigma^2}{2}\right)S_t^n\frac{\partial P_E^n(S_t^n,t)}{\partial S_t^n}\right)(S_t^n)^{\omega-1}dS_t^n$$

$$=-n\omega\left(r+\frac{(n-1)\sigma^2}{2}\right)\tilde{P}_E^n(\omega,t) \tag{4.52}$$

$$\mathcal{M}\left(\frac{(n\sigma S_t^n)^2}{2}\frac{\partial^2 P_E^n(S_t^n,t)}{\partial (S_t^n)^2},\omega\right)=\int_0^\infty\left(\frac{(n\sigma S_t^n)^2}{2}\frac{\partial^2 P_E^n(S_t^n,t)}{\partial (S_t^n)^2}\right)(S_t^n)^{\omega-1}dS_t^n$$

$$=\frac{(n\sigma)^2}{2}(\omega^2+\omega)\tilde{P}_E^n(\omega,t) \tag{4.53}$$

$$\mathcal{M}(rP_E^n(S_t^n,t),\omega)=\int_0^\infty rP_E^n(S_t^n,t)(S_t^n)^{\omega-1}dS_t^n=r\tilde{P}_E^n(\omega,t) \tag{4.54}$$

$$\mathcal{M}(0,\omega)=0 \tag{4.55}$$

Substituting (4.51), (4.52), (4.53), (4.54) and (4.55) into (4.50) yields

$$\frac{\partial \tilde{P}_E^n(\omega,t)}{\partial t}-n\omega\left(r+\frac{(n-1)\sigma^2}{2}\right)\tilde{P}_E^n(\omega,t)$$

$$+\frac{(n\sigma)^2}{2}(\omega^2+\omega)\tilde{P}_E^n(\omega,t)-r\tilde{P}_E^n(\omega,t)=0$$

$$\frac{\partial \tilde{P}_E^n(\omega,t)}{\partial t} = \left(n\omega\left(r+\frac{(n-1)\sigma^2}{2}\right)-\frac{(n\sigma)^2}{2}(\omega^2+\omega)+r\right)\tilde{P}_E^n(\omega,t)$$

$$\frac{\partial \tilde{P}_E^n(\omega,t)}{\partial t} = -\frac{(n\sigma)^2}{2}\left(\omega^2+\omega\left(1-\frac{(n-1)}{n}-\frac{2r}{n\sigma^2}\right)-\frac{2r}{n^2\sigma^2}\right)\tilde{P}_E^n(\omega,t) \tag{4.56}$$

Setting

$$\alpha_1 = \left(1-\frac{(n-1)}{n}-\frac{2r}{n\sigma^2}\right) \text{ and } \alpha_2 = \frac{2r}{n^2\sigma^2}$$

Then (4.56) becomes

$$\frac{\partial \tilde{P}_E^n(\omega,t)}{\partial t} = -\frac{(n\sigma)^2}{2}\left(\omega^2+\alpha_1\omega-\alpha_2\right)\tilde{P}_E^n(\omega,t) \tag{4.57}$$

Solving (4.57) and integrating from 0 to t using variables separable method yields

$$\int_0^t \frac{\partial \tilde{P}_E^n(\omega,\tau)}{\tilde{P}_E^n(\omega,\tau)} = -\int_0^t \frac{(n\sigma)^2}{2}\left(\omega^2+\alpha_1\omega-\alpha_2\right)\partial\tau$$

$$\ln\left(\frac{\tilde{P}_E^n(\omega,t)}{\tilde{P}_E^n(\omega,0)}\right) = \exp\left(-\frac{(n\sigma)^2}{2}\left(\omega^2+\alpha_1\omega-\alpha_2\right)t\right)$$

$$\tilde{P}_E^n(\omega,t) = \tilde{P}_E^n(\omega,0)\exp\left(-\frac{(n\sigma)^2}{2}\left(\omega^2+\alpha_1\omega-\alpha_2\right)t\right)$$

Let $\tilde{P}_E^n(\omega,0) = c(\omega)$, where $c(\omega)$ is a constant that depends on the terminal condition given by (4.46) which is of the form

$$P_E^n(S_T^n,T) = (K-S_T^n)^+ \quad \text{on } [0,\infty)$$

Therefore,

$$\tilde{P}_E^n(\omega,t) = c(\omega)\exp\left(-\frac{(n\sigma)^2}{2}\left(\omega^2+\alpha_1\omega-\alpha_2\right)t\right) \tag{4.58}$$

The constant $c(\omega)$ can be expressed as follows;

$$c(\omega) = \tilde{\phi}(\omega, t) \exp\left(\frac{(n\sigma)^2}{2}\left(\omega^2 + \alpha_1\omega - \alpha_2\right) T\right) \tag{4.59}$$

where

$$\tilde{\phi}(\omega, t) = \mathcal{M}(P_E^n(S_T^n, T), \omega) = \int_0^\infty (K - S_T^n)^+ (S_T^n)^{\omega-1} dS_T^n = \frac{K^{1+\omega}}{\omega(\omega+1)} \tag{4.60}$$

Equation (4.60) is independent of n. Substituting (4.60) into (4.59) gives

$$c(\omega) = \frac{K^{1+\omega}}{\omega(\omega+1)} \exp\left(\frac{(n\sigma)^2}{2}\left(\omega^2 + \alpha_1\omega - \alpha_2\right) T\right) \tag{4.61}$$

Using (4.58) and (4.61), therefore

$$\tilde{P}_E^n(\omega, t) = \frac{K^{1+\omega}}{\omega(\omega+1)} \exp\left(\frac{(n\sigma)^2}{2}\left(\omega^2 + \alpha_1\omega - \alpha_2\right)(T - t)\right) \tag{4.62}$$

Using the inversion formula of the Mellin transform defined by (4.49), then (4.62) becomes

$$\begin{aligned} P_E^n(S_t^n, t) &= \mathcal{M}^{-1}(\tilde{P}_E^n(\omega, t)) \\ &= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \frac{K^{1+\omega}}{\omega(\omega+1)} \frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}}{(S_t^n)^\omega} d\omega \end{aligned} \tag{4.63}$$

Equation (4.63) is the integral representation for the price of the European power put option with non-dividend yield, where $(S_t^n, t) \in \{(0,\infty) \times [0, T)\}, c \in (0, \infty)$ a constant and $\{\omega \in \mathbb{C} | 0 < \Re(\omega) < \infty\}$. This completes the proof.

4.2.1 The Black-Scholes-Like Formula for the Valuation of the European Power Put Option with Non-Dividend Yield

The Black-Scholes-like formula for the valuation of the European power put option which pays no dividend yield using the convolution property of the Mellin transform was presented in the following result.

Theorem 4.2.2

Let S_t^n be the price of the underlying asset, K the strike price, r the risk-free interest rate and T the time to expiry. Using the convolution property of the Mellin transform, the price of European power put option on a non-dividend yield is given by

$$P_E^n(S_t^n, t) = \int_0^\infty \phi(v)\xi_0\left(\frac{S_t^n}{v}\right)\frac{1}{v}dv. \tag{4.64}$$

then the Black-Scholes-like formula for the valuation of the European power put option on non-dividend paying stock is obtained as

$$P_E^n(S_t^n, t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) - S_t^n e^{(r(n-1)+\frac{1}{2}n(n-1)\sigma^2)(T-t))}\mathcal{N}(-d_{1,n}) \tag{4.65}$$

where

$$\mathcal{N}(-d_{1,n}) = 1 - \mathcal{N}(d_{1,n}), \mathcal{N}(-d_{2,n}) = 1 - \mathcal{N}(d_{2,n}),$$
$$d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}},$$

and

$$d_{2,n} = d_{1,n} - n\sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - \frac{1}{2}\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

Proof: Using the convolution property of the Mellin transform (see subsection 3.1.2) and follow the procedures of Panini and Srivastav (2004) and Frontczak and Schöbel (2008). The price of the European power put option which pays no dividend yield using the convolution property of the Mellin transform is given by (4.64) as

$$P_E^n(S_t^n, t) = \int_0^\infty \phi(v)\xi_0\left(\frac{S_t^n}{v}\right)\frac{1}{v}dv$$

where the values of $\phi(v)$ and $\xi_0\left(\frac{S_t^n}{v}\right)$ are to be determined. Let

$$\xi_0(S_t^n) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}}{(S_t^n)^{\omega}}d\omega \tag{4.66}$$

Setting

$$\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t) = \rho_1((\omega+\rho_2)^2-(\rho_2)^2-\alpha_2)$$

where $\rho_1 = \frac{1}{2}n^2\sigma^2$ and $\rho_2 = \frac{\alpha_1}{2}$, then (4.66) becomes

$$\xi_0(S_t^n) = (2\pi i)^{-1}e^{-\rho_1((\rho_2)^2+\alpha_2)}\int_{c-i\infty}^{c+i\infty}e^{\rho_1(\omega+\rho_2)^2}(S_t^n)^{-\omega}d\omega \tag{4.67}$$

Setting $G = \rho_1((\rho_2)^2+\alpha_2)$ and using the transform given by Erdéyi et al. (1954).

$$e^{\phi\omega^2} = \frac{1}{2\sqrt{\pi}}\int_0^{\infty}\frac{1}{\sqrt{\phi}}\exp\left(\frac{-(\ln S_t^n)^2}{4\phi}\right)(S_t^n)^{\omega-1}dS_t^n,\ \ \Re(\phi)\geq 0 \tag{4.68}$$

Equation (4.67) leads to

$$\xi_0(S_t^n) = \frac{e^{-G}(S_t^n)^{\rho_2}}{n\sigma\sqrt{2\pi(T-t)}}\exp\left(\frac{-1}{2}\left(\frac{\ln S_t^n}{n\sigma\sqrt{T-t}}\right)^2\right) \tag{4.69}$$

Similarly,

$$\xi_0\left(\frac{S_t^n}{v}\right) = \frac{e^{-G}\left(\frac{S_t^n}{v}\right)^{\rho_2}}{n\sigma\sqrt{2\pi(T-t)}}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right) \tag{4.70}$$

Using the terminal condition given by (4.46), then

$$\phi(v) = (K-v)^+ = \max(K-v,0) \tag{4.71}$$

Substituting (4.70) and (4.71) into (4.64) yields

$$
\begin{aligned}
P_E^n(S_t^n, t) &= \int_0^\infty (K - v)^+ \frac{e^{-G}\left(\frac{S_t^n}{v}\right)^{\rho_2}}{n\sigma\sqrt{2\pi(T-t)}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv \\
&= \frac{e^{-G}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K (K-v)\left(\frac{S_t^n}{v}\right)^{\rho_2} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv \\
&= \frac{e^{-G}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K K\left(\frac{S_t^n}{v}\right)^{\rho_2} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv \\
&\quad - \frac{e^{-G}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K v\left(\frac{S_t^n}{v}\right)^{\rho_2} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv \\
&= \frac{K(S_t^n)^{\rho_2}e^{-G}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K \frac{1}{v^{\rho_2+1}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)dv \\
&\quad - \frac{(S_t^n)^{\rho_2}e^{-G}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K \frac{1}{v^{\rho_2}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)dv
\end{aligned}
\tag{4.72}
$$

Setting

$$
\begin{cases}
\Omega = \frac{e^{-G}}{n\sigma\sqrt{2\pi(T-t)}} \\
\Omega_1 = \int_0^K \frac{1}{v^{\rho_2+1}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right) dv \\
\Omega_2 = \int_0^K \frac{1}{v^{\rho_2}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right) dv
\end{cases}
\tag{4.73}
$$

Equation (4.72) becomes

$$
P_E^n(S_t^n, t) = \Omega(K(S_t^n)^{\rho_2}\Omega_1 - (S_t^n)^{\rho_2}\Omega_2) \tag{4.74}
$$

Using the transformations

$$\lambda_1 = \frac{\ln\left(\frac{S_t^n}{v}\right) - \rho_2 n^2\sigma^2(T-1)}{n\sigma\sqrt{(T-t)}}$$

and

$$\lambda_2 = \frac{\ln\left(\frac{S_t^n}{v}\right) - (\rho_2 - 1)n^2\sigma^2(T-1)}{n\sigma\sqrt{(T-t)}}$$

to evaluate Ω_1 and Ω_2, respectively. Thus

$$\begin{aligned}\Omega_1 &= \frac{n\sigma\sqrt{2\pi(T-t)}}{e^{-G}}\frac{e^{-r(T-t)}}{(S_t^n)^{\rho_2}}\frac{1}{\sqrt{2\pi}}\int_{d_{2,n}}^{\infty} e^{\frac{-(\lambda_1)^2}{2}}d\lambda_1 \\ &= \frac{1}{\Omega(S_t^n)^{\rho_2}}e^{-r(T-t)}\mathcal{N}(-d_{2,n})\end{aligned} \tag{4.75}$$

and

$$\begin{aligned}\Omega_2 &= \frac{n\sigma\sqrt{2\pi(T-t)}}{e^{-G}}\frac{e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t)}}{(S_t^n)^{\rho_2}}\frac{1}{\sqrt{2\pi}}\int_{d_{1,n}}^{\infty} e^{\frac{-(\lambda_2)^2}{2}}d\lambda_2 \\ &= \frac{1}{\Omega(S_t^n)^{\rho_2-1}}e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t)}\mathcal{N}(-d_{1,n})\end{aligned} \tag{4.76}$$

Substituting (4.75) and (4.76) into (4.74) yields

$$P_E^n(S_t^n, t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) - S_t^n e^{(r(n-1)+\frac{1}{2}n(n-1)\sigma^2)(T-t))}\mathcal{N}(-d_{1,n})$$

with

$$d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

and

$$d_{2,n} = d_{1,n} - n\sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - \frac{1}{2}\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

Hence (4.65) is established.

Remark 4.2.1

(i) Setting $V_p^n(S_t^n, t) = P_E^n(S_t^n, t)$, the above result showed that the expression (4.63) reduced to the Black-Scholes-like valuation formula (4.39) for the price of the European power put option with non-dividend yield.

(ii) For $n = 1$, (4.65) becomes Black-Scholes model for the price of the plain European put option with non-dividend yield given by

$$P_E(S_t, t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S_t\mathcal{N}(-d_1)$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{(T-t)}$$

where $\mathcal{N}(.)$ is the normal distribution function.

4.3 The Mellin Transform Method for the Valuation of European Power Put Option with Dividend Yield

The integral representation for the price of the European power put option with dividend yield was given by the following result.

Theorem 4.3.1

Let S_t^n be the price of the underlying asset, K be the strike price, r be the risk-free interest rate, q be the dividend yield and T be the time to expiry.

Assume S_t^n yields dividend, then the integral representation for the price of the European power put option $E_p^n(S_t^n, t)$ is given by

$$\begin{aligned} E_p^n(S_t^n, t) &= \mathcal{M}^{-1}(\tilde{E}_p^n(\omega, t)) \\ &= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \frac{K^{1+\omega}}{\omega(\omega+1)} \frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}}{(S_t^n)^\omega} d\omega \end{aligned}$$

Proof: Consider the Black-Scholes-Merton-like partial differential equation for the price of the European power put option with dividend yield given by

$$\begin{aligned} \frac{\partial E_p^n(S_t^n, t)}{\partial t} &+ n\left(r - q + \frac{(n-1)\sigma^2}{2}\right) S_t^n \frac{\partial E_p^n(S_t^n, t)}{\partial S_t^n} \\ &+ \frac{(n\sigma S_t^n)^2}{2} \frac{\partial^2 E_p^n(S_t^n, t)}{\partial (S_t^n)^2} - rE_p^n(S_t^n, t) = 0 \end{aligned} \tag{4.77}$$

with the boundary conditions (4.45), (4.46) and (4.47). Taking the Mellin transform of (4.77) to obtain

$$\begin{aligned} &\mathcal{M}\left(\left(\frac{\partial E_p^n(S_t^n, t)}{\partial t} + n\left(\frac{1}{2}\sigma^2(n-1) + (r-q)\right) S_t^n \frac{\partial E_p^n(S_t^n, t)}{\partial S_t^n}\right), \omega\right) \\ &+\mathcal{M}\left(\left(\frac{1}{2}(\sigma n S_t^n)^2 \frac{\partial^2 E_p^n(S_t^n, t)}{\partial (S_t^n)^2} - rE_p^n(S_t^n, t)\right), \omega\right) = 0 \end{aligned} \tag{4.78}$$

Using (3.25), linearity, independence of time derivative and following the procedures for the case of non-dividend yield, (4.78) becomes

$$\begin{aligned} \frac{\partial \tilde{E}_p^n(\omega, t)}{\partial t} &- \left(\frac{1}{2}\sigma^2 n(n-1) + n(r-q)\right) \omega \tilde{E}_p^n(\omega, t) \\ &+ \frac{1}{2}n^2\sigma^2(\omega^2 + \omega)\tilde{E}_p^n(\omega, t) - r\tilde{E}_p^n(\omega, t) = 0 \end{aligned}$$

Rearranging terms, yields

$$\frac{\partial \tilde{E}_p^n(\omega, t)}{\partial t} = -\frac{1}{2}n^2\sigma^2\left(\omega^2 + \omega\left(1 - \frac{2(r-q)}{n\sigma^2} - \frac{(n-1)}{n}\right) - \frac{2r}{n^2\sigma^2}\right)\tilde{E}_p^n(\omega, t) \tag{4.79}$$

Setting

$$\alpha_1^* = 1 - \frac{2(r-q)}{n\sigma^2} - \frac{(n-1)}{n} \text{ and } \alpha_2 = \frac{2r}{n^2\sigma^2},$$

then (4.79) becomes

$$\frac{\partial \tilde{E}_p^n(\omega,t)}{\partial t} = -\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)\tilde{E}_p^n(\omega,t) \tag{4.80}$$

Separating the variables in (4.80) and integrating from 0 to t. The general solution of (4.80) is obtained as

$$\tilde{E}_p^n(\omega,t) = \tilde{E}_p^n(\omega,0)e^{-\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)t} \tag{4.81}$$

where $\tilde{E}_p^n(\omega,0) = m(\omega)$, a constant that depends on the final time condition given by (4.46).

Therefore,

$$\tilde{E}_p^n(\omega,t) = m(\omega)e^{-\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)t} \tag{4.82}$$

But

$$m(\omega) = \mathcal{M}(E_p^n(S_T^n,T),\omega)e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)T}$$

Substituting (4.60) into the last expression leads to a relation

$$m(\omega) = \frac{K^{1+\omega}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)T} \tag{4.83}$$

Substituting (4.83) into (4.82) yields

$$\tilde{E}_p^n(\omega,t) = \frac{K^{1+\omega}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)} \tag{4.84}$$

Applying the inverse Mellin transform (4.49), then (4.84) becomes

$$\begin{aligned} E_p^n(S_t^n,t) &= \mathcal{M}^{-1}(\tilde{E}_p^n(\omega,t)) \\ &= (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \frac{K^{1+\omega}}{\omega(\omega+1)} \frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}}{(S_t^n)^\omega}d\omega \end{aligned} \tag{4.85}$$

Equation (4.85) is the integral representation for the price of the European power put option with dividend yield using the Mellin transform method, where $(S_t^n, t) \in \{(0,\infty) \times [0,T)\}, c \in (0,\infty)$ a constant and $\{\omega \in \mathbb{C} | \Re(\omega) \in (0,\infty)\}$. This completes the proof.

4.3.1 Equivalence of the Black-Scholes-Merton-Like Valuation Formula

The following result showed that the expression (4.85) for the price of the European power put option with dividend yield reduced to the Black-Scholes-Merton-like valuation formula.

Theorem 4.3.2

Let S_t^n be the price of the underlying asset, K the strike price, r the risk-free interest rate, q the dividend yield and T the time to expiry. Using the convolution property of the Mellin transform, the price of European power put options on a dividend yield is given by

$$E_p^n(S_t^n, t) = \int_0^\infty \phi(v)\xi_0\left(\frac{S_t^n}{v}\right)\frac{1}{v}dv. \tag{4.86}$$

then, the Black-Scholes-Merton-like formula for the valuation of the European power put option on a dividend paying stock is given by

$$\begin{aligned} E_p^n(S_t^n, t) &= Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) \\ &\quad - S_t^n e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t))}\mathcal{N}(-d_{1,n}) \end{aligned} \tag{4.87}$$

where

$$d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

and

$$d_{2,n} = d_{1,n} - n\sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

Proof: The price of the European power put option which pays dividend yield using the convolution property of the Mellin transform is given by

$$E_p^n(S_t^n, t) = \int_0^\infty \phi(v)\xi_0\left(\frac{S_t^n}{v}\right)\frac{1}{v}dv.$$

where the values of $\phi(v)$ and $\xi_0\left(\frac{S_t^n}{v}\right)$ are to be determined. Let

$$\xi_0(S_t^n) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}}{(S_t^n)^\omega}d\omega \tag{4.88}$$

Setting

$$\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(T-t) = \rho_1((\omega + \rho_2^*)^2 - (\rho_2^*)^2 - \alpha_2)$$

where $\rho_1 = \frac{1}{2}n^2\sigma^2$ and $\rho_2^* = \frac{\alpha_1^*}{2}$, then (4.88) becomes

$$\xi_0(S_t^n) = (2\pi i)^{-1}e^{-\rho_1((\rho_2^*)^2+\alpha_2)}\int_{c-i\infty}^{c+i\infty} e^{\rho_1(\omega+\rho_2^*)^2}(S_t^n)^{-\omega}d\omega \tag{4.89}$$

Setting $G^* = \rho_1((\rho_2^*)^2 + \alpha_2)$ and using the transform given by Erdéyi et al. (1954).

$$e^{\phi\omega^2} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{\phi}}\exp\left(\frac{-(\ln S_t^n)^2}{4\phi}\right)(S_t^n)^{\omega-1}dS_t^n, \;\; \Re(\phi) \geq 0 \tag{4.90}$$

Equation (4.89) becomes

$$\xi_0(S_t^n) = \frac{e^{-G^*}(S_t^n)^{\rho_2^*}}{n\sigma\sqrt{2\pi(T-t)}}\exp\left(\frac{-1}{2}\left(\frac{\ln S_t^n}{n\sigma\sqrt{T-t}}\right)^2\right) \tag{4.91}$$

Similarly,

$$\xi_0\left(\frac{S_t^n}{v}\right) = \frac{e^{-G^*}\left(\frac{S_t^n}{v}\right)^{\rho_2^*}}{n\sigma\sqrt{2\pi(T-t)}}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right) \tag{4.92}$$

Using the terminal condition given by (4.46), then

$$\phi(v) = (K-v)^+ = \max(K-v,0) \tag{4.93}$$

Substituting (4.92) and (4.93) into (4.86) yields

$$E_p^n(S_t^n,t) = \int_0^\infty (K-v)^+\frac{e^{-G^*}\left(\frac{S_t^n}{v}\right)^{\rho_2^*}}{n\sigma\sqrt{2\pi(T-t)}}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv$$

$$E_p^n(S_t^n,t) = \frac{e^{-G^*}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K (K-v)\left(\frac{S_t^n}{v}\right)^{\rho_2^*}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv$$

$$= \frac{e^{-G^*}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K K\left(\frac{S_t^n}{v}\right)^{\rho_2^*}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv$$

$$-\frac{e^{-G^*}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K v\left(\frac{S_t^n}{v}\right)^{\rho_2^*}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)\frac{1}{v}dv$$

$$= \frac{K(S_t^n)^{\rho_2^*}e^{-G^*}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K \frac{1}{v^{\rho_2^*+1}}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)dv$$

$$-\frac{(S_t^n)^{\rho_2^*}e^{-G^*}}{n\sigma\sqrt{2\pi(T-t)}}\int_0^K \frac{1}{v^{\rho_2^*}}\exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right)dv \tag{4.94}$$

Setting

$$\begin{cases} \Omega^* = \frac{e^{-G^*}}{n\sigma\sqrt{2\pi(T-t)}} \\ \Omega_1^* = \int_0^K \frac{1}{v^{\rho_2^*+1}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right) dv \\ \Omega_2^* = \int_0^K \frac{1}{v^{\rho_2^*}} \exp\left(\frac{-1}{2}\left(\frac{\ln\left(\frac{S_t^n}{v}\right)}{n\sigma\sqrt{T-t}}\right)^2\right) dv \end{cases} \tag{4.95}$$

Equation (4.94) yields

$$E_p^n(S_t^n, t) = \Omega^*(K(S_t^n)^{\rho_2^*}\Omega_1^* - (S_t^n)^{\rho_2^*}\Omega_2^*) \tag{4.96}$$

Using the transformations

$$\lambda_1^* = \frac{\ln\left(\frac{S_t^n}{v}\right) - \rho_2^* n^2\sigma^2(T-1)}{n\sigma\sqrt{(T-t)}}$$

and

$$\lambda_2^* = \frac{\ln\left(\frac{S_t^n}{v}\right) - (\rho_2^*-1)n^2\sigma^2(T-1)}{n\sigma\sqrt{(T-t)}}$$

to evaluate Ω_1^* and Ω_2^*, respectively. Thus

$$\begin{aligned} \Omega_1^* &= \frac{n\sigma\sqrt{2\pi(T-t)}}{e^{-G^*}} \frac{e^{-r(T-t)}}{(S_t^n)^{\rho_2^*}} \frac{1}{\sqrt{2\pi}} \int_{d_{2,n}}^{\infty} e^{\frac{-(\lambda_1^*)^2}{2}} d\lambda_1^* \\ &= \frac{1}{\Omega^*(S_t^n)^{\rho_2^*}} e^{-r(T-t)} \mathcal{N}(-d_{2,n}) \end{aligned} \tag{4.97}$$

and

$$\Omega_2^* = \frac{n\sigma\sqrt{2\pi(T-t)}}{e^{-G^*}} \frac{e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t)}}{(S_t^n)^{\rho_2^*}} \frac{1}{\sqrt{2\pi}} \int_{d_{1,n}}^{\infty} e^{\frac{-(\lambda_2^*)^2}{2}} d\lambda_2^* \tag{4.98}$$

$$= \frac{1}{\Omega^*(S_t^n)^{\rho_2^*-1}} e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t)} \mathcal{N}(-d_{1,n})$$

Substituting (4.97) and (4.98) into (4.96) leads to a relation

$$E_p^n(S_t^n, t) = Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) - S_t^n e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t))}\mathcal{N}(-d_{1,n})$$

with

$$d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

and

$$d_{2,n} = d_{1,n} - n\sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}$$

Hence (4.87) is established.

Remark 4.3.1

(i) The above result showed that the expression (4.85) reduced to the Black-Scholes-Merton-like valuation formula for the price of the European power put option with dividend yield.

(ii) For $n = 1$, (4.85) becomes the Black-Scholes-Merton model for the price of the plain European put option on dividend paying stocks given by

$$E_p(S_t, t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S_t e^{-q(T-t)}\mathcal{N}(-d_1)$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \text{ and } d_2 = d_1 - \sigma\sqrt{(T-t)}$$

4.4 The Mellin Transform Method for the Valuation of the American Power Put Option with Non-Dividend Yield

Analytical approximations and numerical techniques have been proposed for the valuation of plain American put option but there is no known closed-form solution for the price of American power put option. The integral representation for the price of the American power put option and the integral equation to determine the free boundary of the option via the Mellin transform method for the case of non-dividend yield was given by the following result.

Theorem 4.4.1

Let S_t^n be the price of the underlying asset, K be the strike price, r be the risk-free interest rate and T be the time to expiry. Assume S_t^n yields no dividend, then the integral representation for the price of the American power put option $P_A^n(S_t^n, t)$ is given by

$$P_A^n(S_t^n, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega} d\omega$$

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega}\int_t^T \frac{(\hat{S}_t^n(y))^{\omega}}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)} dy d\omega$$

Proof: Consider the non-homogeneous Black-Scholes partial differential equation for the price of American power put option with non-dividend yield given by

$$\frac{\partial P_A^n(S_t^n, t)}{\partial t} + n\left(\frac{1}{2}\sigma^2(n-1)+r\right) S_t^n \frac{\partial P_A^n(S_t^n, t)}{\partial S_t^n}$$
$$+\frac{1}{2}(\sigma n S_t^n)^2 \frac{\partial^2 P_A^n(S_t^n, t)}{\partial (S_t^n)^2} - rP_A^n(S_t^n, t) = f(S_t^n, t) \tag{4.99}$$

where the early exercise function $f(S_t^n, t)$ defined on $(0,\infty)\times(0,T)$ is given by

$$f(S_t^n, t) = \begin{cases} -rK, & \text{if } 0 < S_t^n \leq \hat{S}_t^n \\ 0, & \text{if } S_t^n > \hat{S}_t^n. \end{cases} \tag{4.100}$$

The final time condition is given by

$P_A^n(S_T^n, T) = \phi(S_T^n) = \max(K - S_T^n, 0) = (K - S_T^n)^+$ on $[0, \infty)$.

The other boundary conditions are given by

$$\lim_{S_t^n \to \infty} P_A^n(S_t^n, t) = 0 \quad \text{on} \quad [0, T) \tag{4.101}$$

$$\lim_{S_t^n \to 0} P_A^n(S_t^n, t) = K \quad \text{on} \quad [0, T) \tag{4.102}$$

The free boundary $\hat{S}_t^n$ is determined by the value-matching condition and super-contact condition given by

$$P_A^n(\hat{S}_t^n, t) = K - \hat{S}_t^n \tag{4.103}$$

and

$$\left.\frac{\partial P_A^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_t^n} = -1 \tag{4.104}$$

respectively. Equations (4.103) and (4.104) ensure that the price of the power option is continuous across the free boundary and the slope of the price is continuous across the free boundary respectively. The two conditions are jointly referred to as the smooth pasting conditions. Applying the Mellin transform to (4.99) yields

$$\frac{\partial \tilde{P}_A^n(\omega, t)}{\partial t} + \frac{n^2\sigma^2}{2}\left(\omega^2 + \omega\left(1 - \frac{n-1}{n} - \frac{2r}{n\sigma^2}\right) - \frac{2r}{n^2\sigma^2}\right)\tilde{P}_A^n(\omega, t) = \tilde{f}(\omega, t) \tag{4.105}$$

Setting $\alpha_1 = \left(1 - \frac{n-1}{n} - \frac{2r}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$. Then (4.105) becomes

$$\frac{\partial \tilde{P}_A^n(\omega, t)}{\partial t} + \frac{n^2\sigma^2}{2}(\omega^2 + \omega\alpha_1 - \alpha_2)\tilde{P}_A^n(\omega, t) = \tilde{f}(\omega, t) \tag{4.106}$$

The Mellin transform of the early exercise function in (4.106) is obtained as

$$\begin{aligned}\tilde{f}(\omega,t) &= \int_0^\infty f(S_t^n,t)(S_t^n)^{\omega-1}dS_t^n \\ &= \int_0^{\hat{S}_t^n} -rK(S_t^n)^{\omega-1}dS_t^n \\ &= \frac{-rK(\hat{S}_t^n)^\omega}{\omega}\end{aligned} \tag{4.107}$$

Solving further and from the theory of differential equation, the particular solution of (4.106) is obtained as

$$\tilde{P}_A^n(\omega,t)_{(p.sol)} = \int_t^T \frac{rK(\hat{S}_t^n)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dy \tag{4.108}$$

Similarly, the complementary solution of the left hand side of (4.106) is obtained as

$$\tilde{P}_A^n(\omega,t)_{comp.sol} = c(\omega)e^{-\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)t} \tag{4.109}$$

where $c(\omega)$ is the integration constant given by

$$c(\omega) = \tilde{\phi}(\omega,t)e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)T} \tag{4.110}$$

$\tilde{\phi}(\omega,t)$ is the Mellin transform of the final time condition and is given by

$$\begin{aligned}\tilde{\phi}(\omega,t) &= \int_0^\infty (K-S_T^n)^+(S_T^n)^{\omega-1}dS_T^n \\ &= \int_0^K (K-S_T^n)(S_T^n)^{\omega-1}dS_T^n \\ &= \frac{K^{\omega+1}}{\omega(\omega+1)}\end{aligned} \tag{4.111}$$

Using (4.110) and (4.111) in (4.109) yields

$$\tilde{P}_A^n(\omega,t)_{comp.sol} = \frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)} \tag{4.112}$$

Hence the general solution of (4.106) is given by

$$\begin{aligned}\tilde{P}_A^n(\omega,t) &= \tilde{P}_A^n(\omega,t)_{comp.sol} + \tilde{P}_A^n(\omega,t)_{(p.sol)}\\ &= \frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}\\ &+\int_t^T \frac{rK(\hat{S}_y^n)^\omega}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dy\end{aligned} \tag{4.113}$$

The Mellin inversion of (4.113) is obtained as

$$P_A^n(S_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega$$

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\hat{S}_y^n)^\omega}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega \tag{4.114}$$

where

$(S_t^n,t) \in \{(0,\infty)\times[0,T)\}, c \in (0,\infty)$ and $\{\omega \in \mathbb{C}|0 < \Re(\omega) < \infty\}$. This completes the proof.

Remark 4.4.1

(i) Equations (4.103) and (4.104) jointly ensure that the premature exercise of the American power put option on the endogenously determined early exercise boundary, $\hat{S}_t^n$, will be optimal and self-financing.

(ii) Equation (4.114) expresses the value of an American power put option as the sum of the value of a European power put option and the early exercise premium.

(iii) The first term in (4.114) is the integral representation for the price of the European power put option which pays no dividend yield[2]. The

[2]This stems from the minimum guaranteed payoff of the American power put option with non-dividend yield.

second term in (4.114) is called the early exercise premium for the American power put option with non-dividend yield[3]. Therefore (4.114) becomes

$$P_A^n(S_t^n,t) = P_E^n(S_t^n,t) + e_p^n(S_t^n,t) \tag{4.115}$$

where

$$P_E^n(S_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega$$

$$e_p^n(S_t^n,t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\hat{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega$$

(iv) Setting $S_t^n = \hat{S}_t^n$ in (4.115) and using the value-matching condition given by (4.103), the integral representation for the free boundary of the American power put option with non-dividend yield is obtained as

$$\hat{S}_t^n = K - P_E^n(\hat{S}_t^n,t)$$
$$-\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\hat{S}_t^n)^{-\omega}\int_t^T\frac{(\hat{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega \tag{4.116}$$

where

$$P_E^n(\hat{S}_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(T-t)}(\hat{S}_t^n)^{-\omega}d\omega$$

(v) The American power put option $P_A^n(S_t^n,t)$ which pays no dividend yield satisfies the decomposition

$$P_A^n(S_t^n,t) = P_E^n(S_t^n,t)$$

[3]This is the value attributable to the right of exercising the option early.

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\hat{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega$$

where $\alpha_1 = \left(1-\frac{n-1}{n}-\frac{2r}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$, $(S_t^n, t) \in \{(0,\infty)\times[0,T)\}$, $c \in (0,\infty)$ and $\{\omega \in \mathbb{C}|0 < \Re(\omega) < \infty\}$.

4.5 The Mellin Transform Method for the Valuation of the American Power Put Option with Dividend Yield

The integral representation for the price of the American power put option which pays dividend yield using the Mellin transform method was given by the following result.

Theorem 4.5.1

Let S_t^n be the price of the underlying asset, K be the strike price, r be the risk-free interest rate, q be the dividend yield and T be the time to maturity. Assume S_t^n yields dividend, then the integral representation for the price of the American power put option $A_p^n(S_t^n, t)$ is given by

$$A_p^n(S_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega$$

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\bar{S}_t^n(y))^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$

$$-\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\bar{S}_t^n(y))^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \qquad (4.117)$$

Proof: Consider the non-homogeneous Black-Scholes-Merton-like partial differential equation for the price of American power put option with dividend yield given by

$$\frac{\partial A_p^n(S_t^n,t)}{\partial t} + n\left(\frac{1}{2}\sigma^2(n-1)+(r-q)\right) S_t^n \frac{\partial A_p^n(S_t^n,t)}{\partial S_t^n}$$

$$+\frac{1}{2}(\sigma n S_t^n)^2 \frac{\partial^2 A_p^n(S_t^n,t)}{\partial (S_t^n)^2} - rA_p^n(S_t^n,t) = f^*(S_t^n,t) \qquad (4.118)$$

where

$$f^*(S_t^n,t) = \begin{cases} -rK + qS_t^n, & \text{if } \ 0 < S_t^n \leq \bar{S}_t^n \\ 0, & \text{if } \ S_t^n > \bar{S}_t^n \end{cases} \qquad (4.119)$$

on $(0,\infty)\times[0,T)$ and $\bar{S}_t^n$ the free boundary of the American power put option with dividend yield. The final time condition is given by

$A_p^n(S_T^n,T) = \phi(S_T^n) = \max(K - S_T^n, 0) = (K - S_T^n)^+$ on $[0,\infty)$.

The other conditions are given by

$$\begin{aligned} \lim_{S_t^n\to\infty} A_p^n(S_t^n,t) &= 0 \quad \text{on} \quad [0,T) \\ \lim_{S_t^n\to 0} A_p^n(S_t^n,t) &= K \quad \text{on} \quad [0,T) \end{aligned}$$

with the value-matching condition and super-contact condition given by

$$A_p^n(\bar{S}_t^n,t) = K - \bar{S}_t^n \qquad (4.120)$$

and

$$\left.\frac{\partial A_p^n(S_t^n,t)}{\partial \bar{S}_t^n}\right|_{S_t^n=\bar{S}_t^n} = -1, \qquad (4.121)$$

The Mellin transform of (4.118) gives

$$\frac{\partial \tilde{A}_p^n(\omega,t)}{\partial t} + \frac{n^2\sigma^2}{2}\left(\omega^2 + \omega\left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right) - \frac{2r}{n^2\sigma^2}\right)\tilde{A}_p^n(\omega,t)$$
$$= \tilde{f}^*(\omega,t) \qquad (4.122)$$

Putting $\alpha_1^* = \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$, (4.122) yields

$$\frac{\partial \tilde{A}_p^n(\omega,t)}{\partial t} + \frac{n^2\sigma^2}{2}(\omega^2 + \omega\alpha_1^* - \alpha_2)\tilde{A}_p^n(\omega,t) = \tilde{f}^*(\omega,t) \tag{4.123}$$

where

$$\begin{aligned}\tilde{f}^*(\omega,t) &= \int_0^\infty f^*(S_t^n,t)(S_t^n)^{\omega-1}dS_t^n \\ &= \int_0^{\bar{S}_t^n}(-rK+q)(S_t^n)^{\omega-1}dS_t^n \\ &= \frac{-rK(\bar{S}_t^n)^\omega}{\omega} + \frac{q(\bar{S}_t^n)^{\omega+1}}{\omega+1}\end{aligned} \tag{4.124}$$

Following the same procedures for the case of non-dividend yield, the general solution of (4.123) is obtained as

$$\begin{aligned}\tilde{A}_p^n(\omega,t) = &\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)} \\ &+ \int_t^T \frac{rK(\bar{S}_y^n)^\omega}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dy \\ &- \int_t^T \frac{q(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dy\end{aligned} \tag{4.125}$$

The Mellin inversion of (4.125) leads to

$$\begin{aligned}A_p^n(S_t^n,t) = &\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\bar{S}_y^n)^\omega}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega\end{aligned} \tag{4.126}$$

Equation (4.126) is the integral representation for the price of American power put option with dividend yield, where $(S_t^n,t) \in \{(0,\infty)\times[0,T)\}, c \in$

$(0, \infty)$ and $\{\omega \in \mathbb{C} | 0 < \Re(\omega) < \infty\}$.

Remark 4.5.1

(i) Equations (4.120) and (4.121) jointly ensure that the premature exercise of the American power put option on the endogenously determined early exercise boundary, $\bar{S}_t^n$, will be optimal and self-financing.

(ii) Equation (4.126) expresses the value of an American power put option as the sum of the value of a European power put option and the early exercise premium. The early exercise premium can be viewed as the value of a contingent claim that allows interest earned on the strike price to be exchanged for dividends paid by the asset whenever the asset price is above the optimal exercise boundary (free boundary).

(iii) The first term in (4.126) is the integral representation for the price of the European power put option with dividend yield[4]. The last two terms denote the early exercise premium for the American power put option with dividend yield[5]. Therefore (4.126) becomes

$$A_p^n(S_t^n, t) = E_p^n(S_t^n, t) + \mathbf{e}_p^n(S_t^n, t) \tag{4.127}$$

where

$$E_p^n(S_t^n, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)} (S_t^n)^{-\omega} d\omega$$

[4]This stems from the minimum guaranteed payoff of the American power put option with dividend yield.

[5]This is the value attributable to the right of exercising the option early.

$$\mathbf{e}_p^n(S_t^n,t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$
$$-\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$

(iv) Setting $S_t^n = \bar{S}_t^n$ in (4.127) and using the value-matching condition given by (4.120), the integral representation for the free boundary of the American power put option with dividend yield is obtained as

$$\begin{aligned}\bar{S}_t^n &= K - E_p^n(\bar{S}_t^n,t)\\ &-\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega\\ &+\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega\end{aligned} \tag{4.128}$$

where

$$E_p^n(\bar{S}_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(\bar{S}_t^n)^{-\omega}d\omega$$

(v) The American power put option $A_p^n(S_t^n,t)$ which pays dividend yield satisfies the decomposition

$$A_p^n(S_t^n,t) = E_p^n(S_t^n,t)$$
$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$
$$-\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$

where $\alpha_1^* = \left(1-\frac{n-1}{n}-\frac{2(r-q)}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$, $(S_t^n,t) \in \{(0,\infty)\times[0,T)\}, c\in(0,\infty)$ and $\{\omega\in\mathbb{C}|0<\Re(\omega)<\infty\}$.

In the following results, some special cases of integral representation for price of American power put option with non-dividend yield (4.114) and integral representation for price of American power put option with dividend yield (4.126) was considered.

Theorem 4.5.2

If $\tau \to T - t$ and $n = 1$, then

(i) the integral representation for the price of American power put option which pays no dividend yield (4.114) reduces to the integral equation derived by Kim (1990) for the price of the plain American put option given by

$$P_A(S_\tau, \tau) = P_E(S_\tau, \tau) + \int_0^\tau rKe^{-r\eta}\mathcal{N}(-d_\eta)d\eta \tag{4.129}$$

where

$$d_\eta = \frac{\ln\left(\frac{S_\tau}{\hat{S}_{(\tau-\eta)}}\right) + \left(r - \frac{\sigma^2}{2}\right)\eta}{\sigma\sqrt{\eta}} \tag{4.130}$$

(ii) the free boundary for the American power put option which pays no dividend yield (4.116) reduces to the integral equation derived by Kim (1990) for the price of the plain American put option given by

$$\hat{S}_\tau = K - P_E(\hat{S}_\tau, \tau) - \int_0^\tau rKe^{-r\eta}\mathcal{N}(-\hat{d}_\eta)d\eta \tag{4.131}$$

where

$$\hat{d}_\eta = \frac{\ln\left(\frac{\hat{S}_\tau}{\hat{S}_{(\tau-\eta)}}\right) + \left(r - \frac{\sigma^2}{2}\right)\eta}{\sigma\sqrt{\eta}} \tag{4.132}$$

Proof: Setting $n = 1$ and $\tau = T - t$ in (4.114) yields

$$P_A(S_\tau, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)}(S_\tau)^{-\omega} d\omega$$

$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_\tau)^{-\omega}\int_0^\tau \frac{(\hat{S}_y)^\omega}{\omega} e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(\tau-y)} dy d\omega \qquad (4.133)$$

where $\alpha_1 = \left(1 - \frac{2r}{\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{\sigma^2}$. Equation (4.133) can be written as

$$P_A(S_\tau, \tau) = P_E(S_\tau, \tau) + e_p(S_\tau, \tau) \qquad (4.134)$$

where $P_E(S_\tau, \tau)$ and $e_p(S_\tau, \tau)$ denote the price of the European put option with no dividend yield and early exercise premium for the American put option with no dividend yield respectively. Let

$$e_p(S_\tau, \tau) = \int_0^\tau \Omega(S_\tau, \hat{S}_y, \tau, y) dy \qquad (4.135)$$

where

$$\Omega(S_\tau, \hat{S}_y, \tau, y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\omega, y)\tilde{\xi}(\omega, y) S_\tau^{-\omega} d\omega dy \qquad (4.136)$$

The early exercise function is given by

$$f(S_\tau, y) = \begin{cases} -rK, & \text{if } S_\tau \in (0, \hat{S}_y] \\ 0, & \text{if } S_\tau > \hat{S}_y \end{cases} \qquad (4.137)$$

and

$$\tilde{\xi}(\omega, y) = e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(\tau-y)} \qquad (4.138)$$

Using the convolution property of the Mellin transform, (4.136) becomes

$$\Omega(S_\tau, \hat{S}_y, \tau, y) = \int_0^\infty f(v, y)\xi\left(\frac{S_\tau}{v}, y\right)\frac{1}{v} dv \qquad (4.139)$$

Using (4.137) and substituting

$$\xi(S_\tau, y) = e^{-\frac{\sigma^2}{2}(\tau-y)\left(\frac{\alpha_2+1}{2}\right)^2} \frac{S_\tau^{\frac{1-\alpha_2}{2}}}{\sigma\sqrt{2\pi(\tau-y)}} e^{-\frac{1}{2}\left(\frac{\ln S_\tau}{\sigma\sqrt{\tau-y}}\right)^2} \tag{4.140}$$

into (4.139) yields

$$\Omega(S_\tau, \hat{S}_y, \tau, y) = rK\int_0^{\hat{S}_y} \frac{e^{-\frac{\sigma^2}{2}(\tau-y)\left(\frac{\alpha_2+1}{2}\right)^2}}{v^{\left(1+\frac{1-\alpha_2}{2}\right)}} \frac{S_\tau^{\frac{1-\alpha_2}{2}}}{\sigma\sqrt{2\pi(\tau-y)}} e^{-\frac{1}{2}\left(\frac{\ln(S_\tau v^{-1})}{\sigma\sqrt{\tau-y}}\right)^2} dv \tag{4.141}$$

Using the transformation given by

$$\lambda = \frac{1}{\sigma\sqrt{\tau-y}}\left(\ln\left(\frac{S_\tau}{v}\right) - \sigma^2(\tau-y)\frac{1-\alpha_2}{2}\right) \tag{4.142}$$

Equation (4.141) becomes

$$\begin{aligned}\Omega(S_\tau, \hat{S}_y, \tau, y) &= rKe^{-r(\tau-y)}\frac{1}{\sqrt{2\pi}}\int_{d_y}^{\infty} e^{-\frac{\lambda^2}{2}} d\lambda \\ &= rKe^{-r(\tau-y)}\mathcal{N}(-d_y)\end{aligned} \tag{4.143}$$

Substituting (4.143) into (4.135) to obtain the early exercise premium for the American put option with non-dividend yield as

$$e_p(S_\tau, \tau) = rK\int_0^{\tau} e^{-r(\tau-y)}\mathcal{N}(-d_y)dy \tag{4.144}$$

where

$$d_y = \frac{\ln\left(\frac{S_\tau}{\hat{S}_y}\right) + \left(r - \frac{\sigma^2}{2}\right)(\tau-y)}{\sigma\sqrt{\tau-y}} \tag{4.145}$$

Setting $\eta = \tau - y$, then (4.144) becomes

$$e_p(S_\tau, \tau) = \int_0^{\tau} rKe^{-r\eta}\mathcal{N}(-d_\eta)d\eta \tag{4.146}$$

where

$$d_\eta = \frac{\ln\left(\frac{S_\tau}{\hat{S}_{(\tau-\eta)}}\right) + \left(r - \frac{\sigma^2}{2}\right)\eta}{\sigma\sqrt{\eta}}$$

Substituting (4.146) into (4.134) yields the integral equation (4.129) obtained by Kim (1990) as

$$P_A(S_\tau, \tau) = P_E(S_\tau, \tau) + \int_0^\tau rKe^{-r\eta}\mathcal{N}(-d_\eta)d\eta$$

Hence (i) is established.

For the second reduction, setting $S_\tau = \hat{S}_\tau$ in the last integral equation above and using the value-matching condition given by

$$P_A(\hat{S}_\tau, \tau) = K - \hat{S}_\tau,$$

the free boundary $\hat{S}_\tau$ of the American put option which pays no dividend yield (4.131) derived by Kim (1990) is obtained as

$$\hat{S}_\tau = K - P_A(\hat{S}_\tau, \tau) - \int_0^\tau rKe^{-r\eta}\mathcal{N}(-\hat{d}_\eta)d\eta$$

where

$$\hat{d}_\eta = \frac{\ln\left(\frac{\hat{S}_\tau}{\hat{S}_{(\tau-\eta)}}\right) + \left(r - \frac{\sigma^2}{2}\right)\eta}{\sigma\sqrt{\eta}}$$

The following result showed that the free boundary/optimal exercise boundary satisfied the ex-expiration date.

Theorem 4.5.3

If $\tau = T - t$, then the optimal exercise boundary $\bar{S}_\tau$ of the American power put option with dividend yield for $n = 1$ satisfies

$$\lim_{\tau\to 0} \bar{S}_\tau = K \min\left(1, \frac{r}{q}\right) \tag{4.147}$$

Proof: Let $\tau = T - t$ and $n = 1$, (4.128) becomes

$$\bar{S}_\tau = K - E_p(\bar{S}_\tau, \tau)$$

$$-\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau)^{-\omega}\int_0^\tau \frac{(\bar{S}_y)^\omega}{\omega}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega$$

$$+\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau)^{-\omega}\int_0^\tau \frac{(\bar{S}_y)^{\omega+1}}{\omega+1}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega \tag{4.148}$$

where $\alpha_1^* = \left(1-\frac{2(r-q)}{\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{\sigma^2}$. Factorizing and rearranging, (4.148) becomes

$$\bar{S}_\tau = K\left(\frac{1+e^{-r\tau}(\mathcal{N}(d_2(\bar{S}_\tau,K,\tau))-1)-rI_\tau}{1+e^{-q\tau}(\mathcal{N}(d_1(\bar{S}_\tau,K,\tau))-1)-qJ_\tau}\right) \tag{4.149}$$

where

$$d_1(\bar{S}_\tau,K,\tau) = \frac{\ln\left(\frac{\bar{S}_\tau}{K}\right)+\left(r-q+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \tag{4.150}$$

$$d_2(\bar{S}_\tau,K,\tau) = \frac{\ln\left(\frac{\bar{S}_\tau}{K}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \tag{4.151}$$

$$A_\tau = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau)^{-\omega}\int_0^\tau \frac{(\bar{S}_y)^\omega}{\omega}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega \tag{4.152}$$

and

$$B_\tau = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau)^{-\omega}\int_0^\tau \frac{(\bar{S}_y)^{\omega+1}}{\omega+1}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega \tag{4.153}$$

Notice first that critical stock price is bounded from above, that is $\bar{S}_\tau \leq K, \forall\, \tau > 0$. Taking the limits of (4.150) and (4.151) as $\tau \to 0$ yields

$$\lim_{\tau\to 0} d_1(\bar{S}_\tau,K,\tau) = \begin{cases} 0, & \text{for } \bar{S}(0) = K \\ -\infty, & \text{for } \bar{S}(0) < K \end{cases} \tag{4.154}$$

and

$$\lim_{\tau\to 0} d_2(\bar{S}_\tau,K,\tau) = \begin{cases} 0, & \text{for } \bar{S}(0) = K \\ -\infty, & \text{for } \bar{S}(0) < K \end{cases} \tag{4.155}$$

respectively. If

$$\lim_{\tau\to 0} \bar{S}_\tau = K \tag{4.156}$$

Therefore,

$$\lim_{\tau\to 0}\mathcal{N}(d_1(\bar{S}_\tau,K,\tau))=\lim_{\tau\to 0}\mathcal{N}(d_2(\bar{S}_\tau,K,\tau))=\frac{1}{2} \tag{4.157}$$

Using (4.157), the limit of (4.149) is obtained as

$$\lim_{\tau\to 0}\bar{S}_\tau = K\lim_{\tau\to 0}\left(\frac{1+e^{-r\tau}(\mathcal{N}(d_2(\bar{S}_\tau,K,\tau))-1)-rA_\tau}{1+e^{-q\tau}(\mathcal{N}(d_1(\bar{S}_\tau,K,\tau))-1)-qB_\tau}\right)$$

$$\lim_{\tau\to 0}\bar{S}_\tau = K\left(\frac{\frac{1}{2}-\lim_{\tau\to 0}(rA_\tau)}{\frac{1}{2}-\lim_{\tau\to 0}(qB_\tau)}\right) \tag{4.158}$$

Since

$$\lim_{\tau\to 0}A_\tau = 0$$

and

$$\lim_{\tau\to 0}B_\tau = 0$$

Equation (4.158) becomes

$$\lim_{\tau\to 0}\frac{\bar{S}_\tau}{K}=1 \tag{4.159}$$

If

$$\lim_{\tau\to 0}\bar{S}_\tau < K$$

then

$$\lim_{\tau\to 0}\frac{\bar{S}_\tau}{K}=\left(\frac{r}{q}\right)\lim_{\tau\to 0}\left(\frac{A_\tau}{B_\tau}\right) \tag{4.160}$$

The first integral A_τ can also be written as

$$A_\tau=\int_0^\tau\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau)^{-\omega}\frac{(\bar{S}_y)^\omega}{\omega}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}d\omega dy \tag{4.161}$$

Applying the residue theorem of complex number given by

$$\frac{1}{2\pi i}\int_{\delta\omega}f(\omega)d\omega=\sum_{j=0}^{k}Res(f,\omega_j),\omega\in\mathbb{C} \tag{4.162}$$

Then the inner integral in (4.161) becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau)^{-\omega}\frac{(\bar{S}_y)^{\omega}}{\omega}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega = e^{-r(\tau-y)} \tag{4.163}$$

Substituting (4.163) into (4.161) and solving further yields

$$A_\tau = \frac{(1-e^{-r\tau})}{r} \tag{4.164}$$

Similarly,

$$B_\tau = \frac{(1-e^{-q\tau})}{q} \tag{4.165}$$

Substituting (4.164) and (4.165) into (4.160) yields

$$\lim_{\tau\to 0}\frac{\bar{S}_\tau}{K} = \left(\frac{r}{q}\right)\lim_{\tau\to 0}\left(\frac{\frac{1-e^{-r\tau}}{r}}{\frac{1-e^{-q\tau}}{q}}\right) = \lim_{\tau\to 0}\left(\frac{1-e^{-r\tau}}{1-e^{-q\tau}}\right) = 1 \tag{4.166}$$

For $q \leq r$, (4.159) is obtained. Using the L'Hospital rule, for $q > r$, (4.166) becomes

$$\lim_{\tau\to 0}\frac{\bar{S}_\tau}{K} = \frac{r}{q} \tag{4.167}$$

Combining (4.159) and (4.167), then

$$\lim_{\tau\to 0}\bar{S}_\tau = K\min\left(1,\frac{r}{q}\right)$$

Hence (4.147) is established.

Remark 4.5.2

(i) The above result confirms the formula of Kim and Yu (1996).

(ii) The ex-expiration date early exercise boundary for the American put option is given by (4.147).

The following result showed the behaviour of the optimal exercise boundary $\bar{S}_\tau$ of the American power put option with $n = 1$ near time to expiry.

Theorem 4.5.4

If the underlying asset price follows a lognormal diffusion process and the risk-free interest rate is a positive constant, then the optimal exercise boundary of the American power put option with $n = 1$ at maturity is given by

$$\lim_{\tau \to 0} \bar{S}_\tau = \begin{cases} \frac{rK}{q}, & \text{for} \quad q > r \\ K, & \text{for} \quad q \leq r \end{cases} \tag{4.168}$$

Proof: Let $\tau = T - t$, consider (4.149) which is of the form

$$\bar{S}_\tau = K \left(\frac{1 + e^{-r\tau}(\mathcal{N}(d_2(\bar{S}_\tau, K, \tau)) - 1) - rA\tau}{1 + e^{-q\tau}(\mathcal{N}(d_1(\bar{S}_\tau, K, \tau)) - 1) - qB_\tau} \right)$$

If $q > r$, the limit of the right hand side of (4.149) as $\tau \to 0$ can be evaluated using the L'Hospital's rule to get

$$\lim_{\tau \to 0} \bar{S}_\tau = \frac{rK}{q} \tag{4.169}$$

If $q \leq r$, the limit of the right hand side of (4.149) as $\tau \to 0$ is obtained directly as

$$\lim_{\tau \to 0} \bar{S}_\tau = K \tag{4.170}$$

Using (4.169) and (4.170), the optimal exercise boundary of the American power put option with $n = 1$ at time to expiry is obtained as

$$\lim_{\tau \to 0} \bar{S}_\tau = \begin{cases} \frac{rK}{q}, & \text{for} \quad q > r \\ K, & \text{for} \quad q \leq r \end{cases}$$

Hence (4.168) is established.

Remark 4.5.3

(i) From (4.169), it is observed that large dividend payouts reduce the incentives of early exercise.

(ii) From (4.170), it is observed that it is not possible for the underlying asset price at expiration to fall below K without crossing the exercise boundary at an earlier time.

The following result showed that the integral representation given by (4.126) reduced to the integral equation derived by Kim (1990) for the valuation of plain American put option.

Theorem 4.5.5

The integral representation for the price of the American power put option which pays dividend yield given by

$$\begin{aligned} A_p^n(S_t^n, t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \end{aligned}$$

can be reduced to integral representation derived by Kim (1990).

$$\begin{aligned} A_p(S_\tau, \tau) &= E_p(S_\tau, \tau) + \int_0^\tau rKe^{-r(\tau-\eta)}N(-d_2(S_\tau, \bar{S}_\eta, \tau-\eta))d\eta \\ &- \int_0^\tau qS_\tau e^{-q(\tau-\eta)}N(-d_1(S_\tau, \bar{S}_\eta, \tau-\eta))d\eta \qquad (4.171) \end{aligned}$$

where

$$\begin{aligned}
d_1(S_\tau, \bar{S}_\eta, \tau-\eta) &= \frac{\ln\left(\frac{S_\tau}{\bar{S}_{(\tau-\eta)}}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(\tau-\eta)}{\sigma\sqrt{\tau-\eta}} \\
d_2(S_\tau, \bar{S}_\eta, \tau-\eta) &= \frac{\ln\left(\frac{S_\tau}{\bar{S}_{(\tau-\eta)}}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(\tau-\eta)}{\sigma\sqrt{\tau-\eta}} \\
\tau &= T - t \\
\bar{S}_\tau &\leq S_\tau
\end{aligned}$$

Proof: Setting $\tau = T - t$, then (4.126) becomes

$$\begin{aligned}
A_p^n(S_\tau^n, \tau) = E_p^n(S_\tau^n, \tau) \\
+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_\tau^n)^{-\omega}\int_0^\tau \frac{(\bar{S}_y^n)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega \\
-\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_0^\tau \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega
\end{aligned} \tag{4.172}$$

where $\alpha_1^* = \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$, $(S_\tau^n, \tau) \in \{(0,\infty)\times[0,T]\}$, $c \in (0,\infty)$ and $\{\omega \in \mathbb{C} | 0 < \Re(\omega) < \infty\}$.

Using the procedures of Frontczak and Schöbel (2008), (4.172) can be written as

$$A_p^n(S_\tau^n, \tau) = E_p^n(S_\tau^n, \tau) - \int_0^\tau \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}^*(\omega, y)\tilde{\xi}(\omega, y)(S_\tau^n)^{-\omega}d\omega dy \tag{4.173}$$

with the Mellin transform of $f^*(S_\tau^n, y)$ and $\xi(S_\tau^n, y)$ given by

$$\tilde{f}^*(\omega, y) = \frac{-rK(\bar{S}_y^n)^\omega}{\omega} + \frac{q}{\omega+1}(\bar{S}_y^n)^{\omega+1} \tag{4.174}$$

$$\tilde{\xi}(\omega, y) = e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)} \tag{4.175}$$

respectively. Using the convolution theorem of the Mellin transform, yields

$$A_p^n(S_\tau^n,\tau) = E_p^n(S_\tau^n,\tau) - \int_0^\tau \int_0^\infty f^*(v,y)\xi\left(\frac{S_\tau^n}{v},y\right)\frac{1}{v}dvdy$$

The price of the American power put option which pays dividend yield can be expressed as

$$A_p^n(S_\tau^n,\tau) = E_p^n(S_\tau^n,\tau) - \int_0^\tau I(S_\tau^n,y)dy \tag{4.176}$$

The integral $I(S_\tau^n,y)$ is evaluated as follows

$$I(S_\tau^n,y) = \int_0^\infty f^*(v,y)\xi\left(\frac{S_\tau^n}{v},y\right)\frac{1}{v}dv$$

$$I(S_\tau^n,y) = -rKe^{-\rho_1((\rho_2^*)^2+\alpha_2)}\frac{(S_\tau^n)^{\rho_2^*}}{\sigma\sqrt{2\pi(\tau-y)}}\int_0^{\bar{S}_y^n}\frac{1}{v^{\rho_2+1}}e^{-\frac{1}{2}\left(\frac{\ln\frac{S_\tau^n}{v}}{\sigma\sqrt{\tau-y}}\right)^2}dv$$

$$+qe^{-\rho_1((\rho_2^*)^2+\alpha_2)}\frac{(S_\tau^n)^{\rho_2^*}}{\sigma\sqrt{2\pi(\tau-y)}}\int_0^{\bar{S}_y^n}\frac{1}{v^{\rho_2^*}}e^{-\frac{1}{2}\left(\frac{\ln\frac{S_\tau^n}{v}}{\sigma\sqrt{\tau-y}}\right)^2}dv \tag{4.177}$$

where $\rho_1 = \frac{n^2\sigma^2}{2}(\tau-y)$, $\rho_2^* = \frac{\alpha_1^*}{2} = \frac{1}{2}\left(1-\frac{n-1}{n}-\frac{2(r-q)}{n\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{n^2\sigma^2}$. Using the following variables transformation given by

$$\lambda_1 = \frac{1}{n\sigma\sqrt{\tau-y}}\left(\ln\left(\frac{S_\tau^n}{v}\right) - \rho_2 n^2\sigma^2(\tau-y)\right)$$

and

$$\lambda_2 = \frac{1}{n\sigma\sqrt{\tau-y}}\left(\ln\left(\frac{S_\tau^n}{v}\right) - (\rho_2-1)n^2\sigma^2(\tau-y)\right)$$

for the first and second integrals in (4.177) respectively, to obtain

$$I(S_\tau^n,y) = -rKe^{-r(\tau-y)}\mathcal{N}(-d_{2,n}(S_\tau^n,\bar{S}_y^n,\tau-y))$$

$$+qe^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(\tau-y)}\mathcal{N}(-d_{1,n}(S_\tau^n,\bar{S}_y^n,\tau-y)) \tag{4.178}$$

Substituting (4.178) into (4.176) yields

$$A_p^n(S_\tau^n,\tau) = E_p^n(S_\tau^n,\tau) + \int_0^\tau rKe^{-r(\tau-y)}\mathcal{N}(-d_{2,n}(S_\tau^n,\bar{S}_y^n,\tau-y))dy$$
$$-\int_0^\tau qe^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(\tau-y)}\mathcal{N}(-d_{1,n}(S_\tau^n,\bar{S}_y^n,\tau-y))dy \tag{4.179}$$

By changing the variable y to η, (4.179) becomes

$$A_p^n(S_\tau^n,\tau) = E_p^n(S_\tau^n,\tau) + \int_0^\tau rKe^{-r(\tau-\eta)}\mathcal{N}(-d_{2,n}(S_\tau^n,\bar{S}_\eta^n,\tau-\eta))d\eta$$
$$-\int_0^\tau qe^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(\tau-\eta)}\mathcal{N}(-d_{1,n}(S_\tau^n,\bar{S}_\eta^n,\tau-\eta))d\eta \tag{4.180}$$

Hence, by setting $n=1$, this proves (4.171).

The following result showed that the integral representation (4.126) and decomposition derived by Carr et al. (1992) for the price of American put option are equivalent.

Theorem 4.5.6

If $\tau = T-t$, $S_\tau \geq \bar{S}_\tau$ and $n=1$, then the integral representation for the price of American power put option which pays dividend yield

$$A_p^n(S_t^n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K^{\omega+1}}{\omega(\omega+1)}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega$$
$$+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\bar{S}_y^n)^\omega}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$
$$-\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T\frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega$$

reduces to the decomposition derived by Carr et al. (1992) for the price of the plain American put option given by

$$A_p(S_\tau,\tau) = (K-S_\tau)^+ + \frac{\sigma^2 S_\tau}{2}\int_0^\tau\frac{e^{-q(\tau-\eta)}}{\sigma\sqrt{\tau-\eta}}\acute{\mathcal{N}}(-d_1(S,K,\tau-\eta))d\eta$$

$$+\int_0^{\tau} rKe^{-r(\tau-\eta)}[\mathcal{N}(-d_2(S_\tau,\bar{S}(\eta),\tau-\eta))-\mathcal{N}(-d_2(S_\tau,K,\tau-\eta))]d\eta$$
$$-\int_0^{\tau} qS_\tau e^{-q(\tau-\eta)}[\mathcal{N}(-d_1(S_\tau,\bar{S}(\eta),\tau-\eta))-\mathcal{N}(-d_1(S_\tau,K,\tau-\eta))]d\eta \quad (4.181)$$

where

$$\left.\begin{aligned} d_1(x,z,t) &= \frac{\ln\left(\frac{x}{z}\right)+(r-q-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \\ d_2(x,z,t) &= d_1(x,z,t)-\sigma\sqrt{t} \end{aligned}\right\} \quad (4.182)$$

Proof: Setting $\tau = T-t$, $n=1$ in (4.126) leads to

$$\begin{aligned} A_p(S_\tau,\tau) &= E_p(S_\tau,\tau) \\ &+\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_\tau)^{-\omega}\int_0^{\tau}\frac{(\bar{S}_y)^{\omega}}{\omega}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega \\ &-\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_\tau)^{-\omega}\int_0^{\tau}\frac{(\bar{S}_y)^{\omega+1}}{\omega+1}e^{\frac{1}{2}\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(\tau-y)}dyd\omega \end{aligned} \quad (4.183)$$

where $\alpha_1^* = \left(1-\frac{2(r-q)}{\sigma^2}\right)$ and $\alpha_2 = \frac{2r}{\sigma^2}$, $(S_\tau,\tau)\in\{(0,\infty)\times[0,\tau)\}, c\in(0,\infty)$ and $\{\omega\in\mathbb{C}|0<Re(\omega)<\infty\}$. Following the procedures of Frontczak and Schöbel (2008), the price for the European put option can be expressed as

$$\begin{aligned} E_p(S_\tau,\tau) &= K.H(K-S_\tau)-K.H(K-S_\tau) \\ &+Ke^{-r\tau}\mathcal{N}(-d_2(S_\tau,K,\tau))-Se^{-q\tau}\mathcal{N}(-d_1(S_\tau,K,\tau)) \end{aligned} \quad (4.184)$$

where $H(y)$ is the Heaviside step function given by

$$H(y)=\begin{cases} 0, & \text{for } y<0 \\ \frac{1}{2}, & \text{for } y=0 \\ 1, & \text{for } y>0 \end{cases} \quad (4.185)$$

The reason for the factor $\frac{1}{2}$ at the point of discontinuity will become clearly below.

$$\lim_{\tau\to 0} d_1(S_\tau,K,\tau)=\lim_{\tau\to 0} d_2(S_\tau,K,\tau)=\begin{cases} -\infty, & \text{for } S_\tau<K \\ 0, & \text{for } S_\tau=K \\ \infty, & \text{for } S_\tau>K \end{cases} \quad (4.186)$$

Equation (4.184) leads to a relation

$$\begin{aligned}
E_p(S_\tau,\tau) &= K.H(K-S_\tau) - Se^{-q\tau}\mathcal{N}(-d_1(S_\tau,K,\tau)) \\
&+ [Ke^{-r\eta}\mathcal{N}(-d_2(S_\tau,K,\tau))]\Big|_0^\tau \\
&= K.H(K-S_\tau) - Se^{-q\tau}\mathcal{N}(-d_1(S_\tau,K,\tau)) \\
&- K\int_0^\tau re^{-r\eta}\mathcal{N}(-d_2(S_\eta,K,\eta))d\eta \\
&+ K\int_0^\tau \left(e^{-r\eta}\acute{\mathcal{N}}(-d_2(S_\eta,K,\eta))\frac{\partial(-d_1(S_\eta,K,\eta)-\sigma\sqrt{\eta})}{\partial\eta}\right)d\eta
\end{aligned}$$

Thus,

$$\begin{aligned}
E_p(S_\tau,\tau) &= K.H(K-S_\tau) - S_\tau e^{-q\tau}\mathcal{N}(-d_1(S_\tau,K,\tau)) \\
&- K\int_0^\tau re^{-r\eta}\mathcal{N}(-d_2(S_\eta,K,\eta))d\eta \\
&+ K\int_0^\tau \left(e^{-r\eta}\acute{\mathcal{N}}(-d_2(S_\eta,K,\eta))\frac{\partial(-d_1(S_\eta,K,\eta)}{\partial\eta}\right)d\eta \\
&+ K\int_0^\tau \left(e^{-r\eta}\acute{\mathcal{N}}(-d_2(S_\eta,K,\eta))\right)\frac{\sigma}{2\sqrt{\eta}}d\eta
\end{aligned} \tag{4.187}$$

where $\acute{\mathcal{N}}(y) = n(y)$ is the density function of a standard normal distributed random variable y and the following identities

$$\acute{\mathcal{N}}(-d_1(S_\eta,K,\eta)) = \acute{\mathcal{N}}(d_1(S_\eta,K,\eta)),$$

$$\acute{\mathcal{N}}(-d_2(S_\eta,K,\eta)) = \acute{\mathcal{N}}(d_2(S_\eta,K,\eta))$$

and

$$S_\eta e^{-q\eta}\acute{\mathcal{N}}(d_1(S_\eta,K,\eta)) = Ke^{-r\eta}\mathcal{N}(d_2(S_\eta,K,\eta)).$$

Therefore,

$$E_p(S_\tau,\tau) = (K - S_\tau)H(K - S_\tau) + S_\tau.H(X - S_\tau) - S_\tau e^{-q\tau}\mathcal{N}(-d_1(S_\tau, K, \tau))$$
$$- rK\int_0^\tau e^{-r\eta}\mathcal{N}(-d_2(S_\eta, K, \eta))d\eta$$
$$+ S_\tau\int_0^\tau \left(e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))\frac{\partial}{\partial\eta}(-d_1(S_\eta, K, \eta)\right)d\eta$$
$$+ S_\tau\int_0^\tau \left(e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))\right)\frac{\sigma}{2\sqrt{\eta}}d\eta$$

$$\begin{aligned} E_p(S_\tau,\tau) = (K - S_\tau)^+ &- rK\int_0^\tau e^{-r\eta}\mathcal{N}(-d_2(S_\eta, K, \eta))d\eta \\ &+ \frac{\sigma^2}{2}S_\tau\int_0^\tau \left(e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))\right)\frac{1}{\sigma\sqrt{\eta}}d\eta \\ &- S_\tau\left[e^{-q\tau}\acute{\mathcal{N}}(-d_1(S_\tau, K, \tau)) - H(K - S_\tau)\right] \\ &- \int_0^\tau \left(e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))\frac{\partial}{\partial\eta}(-d_1(S_\eta, K, \eta)\right)d\eta \end{aligned} \tag{4.188}$$

Solving (4.188) further yields

$$\begin{aligned} E_p(S_\tau,\tau) = (K - S_\tau)^+ &- rK\int_0^\tau e^{-r\eta}\mathcal{N}(-d_2(S_\eta, K, \eta))d\eta \\ &+ \frac{\sigma^2}{2}S_\tau\int_0^\tau \left(e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))\right)\frac{1}{\sigma\sqrt{\eta}}d\eta \\ &- S_\tau[e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))]|_0^\tau \\ &\int_0^\tau \left(e^{-q\eta}\acute{\mathcal{N}}(-d_1(S_\eta, K, \eta))\frac{\partial}{\partial\eta}(-d_1(S_\eta, K, \eta)\right)d\eta \end{aligned} \tag{4.189}$$

Changing the integration variable from η to $\tau - \eta$, (4.189) yields

$$\begin{aligned} E_p(S_\tau,\tau) = (K - S_\tau)^+ &- rK\int_0^\tau e^{-r(\tau-\eta)}\mathcal{N}(-d_2(S_\tau, K, \tau - \eta))d\eta \\ &+ \frac{\sigma^2}{2}S_\tau\int_0^\tau \left(e^{-q(\tau-\eta)}\acute{\mathcal{N}}(-d_1(S_\tau, K, \tau - \eta))\right)\frac{1}{\sigma\sqrt{\tau - \eta}}d\eta \\ &+ qS_\tau\int_0^\tau e^{-q(\tau-\eta)}\mathcal{N}(-d_1(S_\tau, K, \tau - \eta))d\eta \end{aligned} \tag{4.190}$$

Substituting (4.190) into (4.171) leads to

$$\begin{aligned}
A_p(S_\tau,\tau) &= (K-S_\tau)^+ - rK\int_0^\tau e^{-r(\tau-\eta)}\mathcal{N}(-d_2(S_\tau,K,\tau-\eta))d\eta \\
&+ \frac{\sigma^2}{2}S_\tau\int_0^\tau \left(e^{-q(\tau-\eta)}\acute{\mathcal{N}}(-d_1(S_\tau,K,\tau-\eta))\right)\frac{1}{\sigma\sqrt{\tau-\eta}}d\eta \\
&+ qS_\tau\int_0^\tau e^{-q(\tau-\eta)}\mathcal{N}(-d_1(S_\tau,K,\tau-\eta))d\eta \qquad (4.191)\\
&+ \int_0^\tau rKe^{-r(\tau-\eta)}N(-d_2(S_\tau,\bar{S}_\eta,\tau-\eta))d\eta \\
&- \int_0^\tau qS_\tau e^{-q(\tau-\eta)}N(-d_1(S_\tau,\bar{S}_\eta,\tau-\eta))d\eta
\end{aligned}$$

Rearranging terms, (4.191) becomes

$$\begin{aligned}
A_p(S_\tau,\tau) &= (K-S_\tau)^+ + \frac{\sigma^2 S_\tau}{2}\int_0^\tau \frac{e^{-q(\tau-\eta)}}{\sigma\sqrt{\tau-\eta}}\acute{\mathcal{N}}(-d_1(S,K,\tau-\eta))d\eta \\
&+ \int_0^\tau rKe^{-r(\tau-\eta)}[\mathcal{N}(-d_2(S_\tau,\bar{S}(\eta),\tau-\eta)) - \mathcal{N}(-d_2(S_\tau,K,\tau-\eta))]d\eta \\
&- \int_0^\tau qS_\tau e^{-q(\tau-\eta)}[\mathcal{N}(-d_1(S_\tau,\bar{S}(\eta),\tau-\eta)) - \mathcal{N}(-d_1(S_\tau,K,\tau-\eta))]d\eta
\end{aligned}$$

where $\tau = T-t, S_\tau > \bar{S}_\tau$, d_1 and d_2 are given by (4.182). Hence (4.126) reduces to (4.181). This completes the proof.

Remark 4.5.4

The integral representation given by (4.126) with $n=1$, (4.171) and (4.181) are equivalent.

The following result showed the behaviour of the free boundary of American power put option near maturity.

Theorem 4.5.7

If the underlying asset price follows a lognormal diffusion process and the risk-free interest rate is a positive constant, then the optimal exercise boundary (free boundary) of the American power put option with dividend yield at maturity is given by

$$\lim_{\tau \to 0} \frac{\bar{S}^n_\tau}{K} = \begin{cases} \frac{r}{q}, & \text{for} \quad q > r \\ 1, & \text{for} \quad q \leq r \end{cases} \tag{4.192}$$

Proof: Changing the time variable $\tau = T - t$ in (4.128) leads to

$$\begin{aligned} K - \bar{S}^n_\tau &= E^n_p(\bar{S}^n_\tau, \tau) \\ &+ \frac{rK}{2\pi i} \int_{c-i\infty}^{c+i\infty} (S^n_\tau)^{-\omega} \int_0^\tau \frac{(\bar{S}^n_y)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(\tau - y)} dy d\omega \\ &- \frac{q}{2\pi i} \int_{c-i\infty}^{c+i\infty} (S^n_t)^{-\omega} \int_0^\tau \frac{(\bar{S}^n_y)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(\tau - y)} dy d\omega \end{aligned} \tag{4.193}$$

where t is the current time, τ is the reversed time and T is the time to expiry. Let

$$A^n_\tau = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\bar{S}^n_\tau)^{-\omega} \int_0^\tau \frac{(\bar{S}^n_y)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(\tau - y)} dy d\omega \tag{4.194}$$

and

$$B^n_\tau = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\bar{S}^n_\tau)^{-\omega} \int_0^\tau \frac{(\bar{S}^n_y)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(\tau - y)} dy d\omega \tag{4.195}$$

Equation (4.193) becomes

$$K - \bar{S}^n_\tau = E^n_p(\bar{S}^n_\tau, \tau) + rKA^n_\tau - q\bar{S}^n_\tau B^n_\tau \tag{4.196}$$

where

$$\begin{aligned} E^n_p(\bar{S}^n_\tau, \tau) &= Ke^{-r\tau}\mathcal{N}(-d_{2,n}) \\ &\quad - \bar{S}^n_\tau e^{\left(r(n-1) - nq + \frac{1}{2}n(n-1)\sigma^2\right)\tau} \mathcal{N}(-d_{1,n}) \end{aligned} \tag{4.197}$$

with

$$\mathcal{N}(-d_{1,n}) = 1 - \mathcal{N}(d_{1,n}), \mathcal{N}(-d_{2,n}) = 1 - \mathcal{N}(d_{2,n}) \tag{4.198}$$

Substituting (4.197) and (4.198) into (4.196) yields

$$\begin{aligned} K - \bar{S}_\tau^n &= K e^{-r\tau}(1 - \mathcal{N}(d_{2,n})) \\ &- \bar{S}_\tau^n e^{\left(r(n-1) - nq + \frac{1}{2}n(n-1)\sigma^2\right)\tau}(1 - \mathcal{N}(d_{1,n})) \\ &+ rKA_\tau^n - q\bar{S}_\tau^n B_\tau^n \end{aligned}$$

Rearranging terms lead to a relation

$$\frac{\bar{S}_\tau^n}{K} = \frac{(1 - e^{-r\tau}(1 - \mathcal{N}(d_{2,n})) - rA_\tau^n)}{\left(1 - e^{\left(r(n-1) - nq + \frac{1}{2}n(n-1)\sigma^2\right)\tau}(1 - \mathcal{N}(d_{1,n})) - qB_\tau^n\right)} \tag{4.199}$$

For the first case, the implicit equation for $\bar{S}_\tau^n$ reads

$$\lim_{\tau \to 0}\left(\frac{\bar{S}_\tau^n}{K}\right) = \left(\frac{r}{q}\right)\lim_{\tau \to 0}\left(\frac{A_\tau^n}{B_\tau^n}\right)$$

The complex integrals for A_τ^n and B_τ^n are given by

$$A_\tau^n(c) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau^n)^{-\omega}\frac{(\bar{S}_y^n)^\omega}{\omega}e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(\tau - y)}d\omega$$

and

$$B_\tau^n(c) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_\tau^n)^{-\omega}\frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1}e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(\tau - y)}d\omega$$

respectively. By means of (4.162)

$$A_\tau^n(c) = e^{-r(\tau - y)}$$

and

$$B_\tau^n(c) = e^{(r(n-1) - nq + \frac{1}{2}n(n-1)\sigma^2)(\tau - y)}$$

Therefore (4.194) and (4.195) become respectively

$$\begin{aligned} A^n_\tau &= \int_0^\tau A^n_\tau(c)dy \\ &= \int_0^\tau e^{-r(\tau-y)}dy \\ &= \frac{1}{r}(1-e^{-r\tau}) \end{aligned}$$

and

$$\begin{aligned} B^n_\tau &= \int_0^\tau B^n_\tau(c)dy \\ &= \int_0^\tau e^{\left(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2\right)(\tau-y)}dy \\ &= \frac{e^{\left(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2\right)\tau}-1}{\left(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2\right)} \end{aligned}$$

Putting the results together leads to

$$\lim_{\tau\to 0}\left(\frac{\bar{S}^n_\tau}{K}\right) = \left(\frac{r}{q}\right)\lim_{\tau\to 0}\left(\frac{\frac{1}{r}(1-e^{-r\tau})}{\frac{e^{\left(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2\right)\tau}-1}{\left(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2\right)}}\right)$$

Using the L'Hospital rule;

$$\begin{aligned} \lim_{\tau\to 0}\left(\frac{\bar{S}^n_\tau}{K}\right) &= \left(\frac{r}{q}\right)\lim_{\tau\to 0}\left(\frac{e^{-r\tau}}{e^{\left(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2\right)\tau}}\right) \\ &= \frac{r}{q} \end{aligned}$$

Thus,

$$\lim_{\tau\to 0}\bar{S}^n_\tau = \frac{rK}{q} \quad \text{for} \quad q > r \tag{4.200}$$

For the second case, the limits of the constants $d_{1,n}$ and $d_{2,n}$ are obtained as follows;

$$\lim_{\tau\to 0} d_{1,n} = \lim_{\tau\to 0} d_{1,n}(\bar{S}^n_\tau, K, \tau) = \begin{cases} 0, & \text{for} \quad \bar{S}^n(0) = K \\ -\infty, & \text{for} \quad \bar{S}^n(0) < K \end{cases}$$

and

$$\lim_{\tau\to 0} d_{2,n} = \lim_{\tau\to 0} d_{2,n}(\bar{S}^n_\tau, K, \tau) = \begin{cases} 0, & \text{for} \quad \bar{S}^n(0) = K \\ -\infty, & \text{for} \quad \bar{S}^n(0) < K \end{cases}$$

Therefore,

$$\lim_{\tau\to 0} \mathcal{N}(d_{1,n}(\bar{S}^n_\tau, K, \tau)) = \lim_{\tau\to 0} \mathcal{N}(d_{2,n}(\bar{S}^n_\tau, K, \tau)) = \frac{1}{2}$$

Taking the limit of (4.199) as $\tau \to 0$ and by means of last relation yields

$$\lim_{\tau\to 0} \left(\frac{\bar{S}^n_\tau}{K}\right) = \frac{\frac{1}{2} - r \lim_{\tau\to 0} A^n_\tau}{\frac{1}{2} - q \lim_{\tau\to 0} B^n_\tau}$$

Since

$$\lim_{\tau\to 0} A^n_\tau = \lim_{\tau\to 0} B^n_\tau = 0$$

Hence,

$$\lim_{\tau\to 0} \left(\frac{\bar{S}^n_\tau}{K}\right) = 1$$

$$\lim_{\tau\to 0} \bar{S}^n_\tau = K \quad \text{for} \quad q \leq r \tag{4.201}$$

Using (4.200) and (4.201), the optimal exercise boundary of the American power put option at maturity is given by

$$\lim_{\tau\to 0} \bar{S}^n_\tau = \begin{cases} \frac{rK}{q}, & \text{for} \quad q > r \\ K, & \text{for} \quad q \leq r \end{cases}$$

Hence (4.192) is established.

Remark 4.5.5

(i) From (4.200), it is observed that when $q > r$ and $S^n_\tau < K$, the American power put can have a positive value at expiration given that it has not been exercised earlier.

(ii) From (4.201), it is observed that when $q \leq r$ and $S_\tau^n = K$, the American power put will have a zero payoff at expiration even if it has not been exercised earlier.

4.6 Perpetual American Power Put Option Valuation

Now, the applications of the integral representations in (4.114) and (4.126) to power options which have no expiry date are presented. The expression for the free boundary of the perpetual American power put option and its closed form solution for both non-dividend and dividend yields, using the Mellin transform method was given by the following result.

Theorem 4.6.1

Consider the perpetual American power put option with non-dividend yield. If $T \to \infty$ and $0 < \Re(\omega) < \omega_2$, then the free boundary of the perpetual American power put option is given by

$$\hat{S}_\infty^n = \hat{S}_\infty^n(t) = K\frac{\alpha_2}{(\omega_2 - \omega_1)} \tag{4.202}$$

and the price of the perpetual American power put option becomes

$$P_\infty^n(S_t^n, t) = \frac{\alpha_2 K}{\omega_2(\omega_2 - \omega_1)}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega_2} \quad \text{for} \quad \hat{S}_\infty^n < S_t^n \tag{4.203}$$

where

$$\alpha_2 = \frac{2r}{n^2\sigma^2} \tag{4.204}$$

Proof: The integral representation for the price of the American power put option which pays no dividend yield given by (4.114) can be expressed as

$$P_A^n(S_t^n, t) = P_E^n(S_t^n, t) + P_1^n(S_t^n, t) \tag{4.205}$$

where

$$\begin{aligned} P_E^n(S_t^n, t) &= Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) \\ &\quad - S_t^n e^{\left(r(n-1)+\frac{1}{2}n(n-1)\sigma^2\right)(T-t)}\mathcal{N}(-d_{1,n}) \end{aligned} \tag{4.206}$$

with

$$d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}}$$

$$d_{2,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - \frac{\sigma^2}{2}\right)(T-t)}{n\sigma\sqrt{T-t}}$$

and

$$P_1^n(S_t^n, t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\hat{S}_y^n)^{\omega}}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dyd\omega \tag{4.207}$$

For (4.205) to hold as $T \to \infty$, it is necessary that $\Re(\omega^2 + \alpha_1\omega - \alpha_2) < 0$, that is $0 < \Re(\omega) < \omega_2$, where ω_2 is one of the roots of $\omega^2 + \alpha_1\omega - \alpha_2 = 0$. Using the super-contact condition (4.104), the perpetual American power put option as $T \to \infty$ becomes

$$\left.\frac{\partial P_A^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = \left.\frac{\partial P_E^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} + \left.\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = -1 \tag{4.208}$$

where the free boundary $\hat{S}_t^n = \hat{S}_\infty^n$ is now independent of time. Now, Differentiating (4.206) with respect to S_t^n at $S_t^n = \hat{S}_\infty^n$ yields

$$\left.\frac{\partial P_E^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n=\hat{S}_\infty^n} = -e^{\left(r(n-1)+\frac{1}{2}n(n-1)\sigma^2\right)(T-t)}\mathcal{N}(-\hat{d}_{1,n}) \tag{4.209}$$

where

$$\hat{d}_{1,n} = \frac{\ln\left(\frac{\hat{S}^n_\infty}{K}\right) + n\left(r + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}} \tag{4.210}$$

As $T \to \infty, \hat{d}_{1,n} \to \infty$ and therefore

$$\left.\frac{\partial P^n_E(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \hat{S}^n_\infty} \to 0 \tag{4.211}$$

Also consider the $P^n_1(S^n_t, t)$ term,

$$\frac{\partial P^n_1(S^n_t, t)}{\partial S^n_t} = -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S^n_t)^{-1}\left(\int_t^T \left(\frac{S^n_t}{\hat{S}^n_y}\right)^{-\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dy\right)d\omega \tag{4.212}$$

Taking the limit of (4.212) as $T \to \infty$ yields

$$\frac{\partial P^n_1(S^n_t, t)}{\partial S^n_t} = -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S^n_t)^{-1}\left(\int_t^\infty \left(\frac{S^n_t}{\hat{S}^n_\infty}\right)^{-\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dy\right)d\omega \tag{4.213}$$

Therefore,

$$\begin{aligned}
\frac{\partial P^n_1(S^n_t, t)}{\partial S^n_t} &= -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S^n_t)^{-1}\left(\frac{S^n_t}{\hat{S}^n_\infty}\right)^{-\omega}\left(\left.\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}}{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)}\right|_t^\infty\right)d\omega \\
&= -\frac{rK}{2\pi i}\frac{2}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty} (S^n_t)^{-1}\left(\frac{S^n_t}{\hat{S}^n_\infty}\right)^{-\omega}\left(\left.\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}}{(\omega^2+\alpha_1\omega-\alpha_2)}\right|_t^\infty\right)d\omega \\
&= \frac{rK}{2\pi i}\frac{2}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty} (S^n_t)^{-1}\left(\frac{S^n_t}{\hat{S}^n_\infty}\right)^{-\omega}\frac{d\omega}{(\omega^2+\alpha_1\omega-\alpha_2)}
\end{aligned}$$

Thus,

$$\left.\frac{\partial P^n_1(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \hat{S}^n_\infty} = \frac{K}{2\pi i}\frac{2r}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{d\omega}{\hat{S}^n_\infty(\omega^2+\alpha_1\omega-\alpha_2)} \tag{4.214}$$

Since $\alpha_2 = \frac{2r}{n^2\sigma^2}$, (4.214) becomes

$$\left.\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = \frac{\alpha_2 K}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{d\omega}{\hat{S}_\infty^n(\omega^2 + \alpha_1\omega - \alpha_2)} \tag{4.215}$$

But $\omega^2 + \alpha_1\omega - \alpha_2 = (\omega - \omega_1)(\omega - \omega_2)$, where

$$\omega = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \tag{4.216}$$

$$\omega_1 = \frac{-\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \tag{4.217}$$

$$\omega_2 = \frac{-\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \tag{4.218}$$

The limiting cases ω_1 and ω_2 are the roots of $\omega^2 + \alpha_1\omega - \alpha_2$. Hence (4.215) becomes

$$\left.\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = \frac{\alpha_2 K}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{d\omega}{\hat{S}_\infty^n(\omega - \omega_1)(\omega - \omega_2)} \tag{4.219}$$

By applying the residue theorem in (4.162), then (4.219) leads to a relation

$$\left.\frac{\partial P_1^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = \alpha_2 \frac{K}{\hat{S}_\infty^n(\omega_1 - \omega_2)} \tag{4.220}$$

Substituting (4.211) and (4.220) into (4.208) gives

$$\left.\frac{\partial P_A^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \hat{S}_\infty^n} = 0 + \alpha_2 \frac{K}{\hat{S}_\infty^n(\omega_1 - \omega_2)} = -1$$

The free boundary of a perpetual American power put option is obtained as

$$\hat{S}_\infty^n = K\frac{\alpha_2}{(\omega_2 - \omega_1)} \tag{4.221}$$

Next, use (4.221) to derive an expression for the price of perpetual American power put option $P_\infty^n(S_t^n, t)$. Note that the price of a perpetual European power put option is zero, since it can never be exercised. Therefore, taking the limit as $T \to \infty$ in (4.205), the price of perpetual American power put option for $S_t^n > \hat{S}_\infty^n$ is given by

$$P_\infty(S_t^n, t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega}\frac{1}{\omega}\left(\int_t^\infty e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1\omega-\alpha_2)(y-t)}dy\right)d\omega \tag{4.222}$$

where $\Re(\omega^2+\alpha_1\omega-\alpha_2) < 0$. Integrating the inner integral (that is, the time variable) in (4.222) leads to

$$P_\infty(S_t^n, t) = -\frac{rK}{2\pi i}\frac{2}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega}\frac{d\omega}{\omega(\omega-\omega_1)(\omega-\omega_2)} \tag{4.223}$$

Once again applying the residue theorem (4.162) to get

$$P_\infty^n(S_t^n, t) = \frac{\alpha_2 K}{\omega_2(\omega_2-\omega_1)}\left(\frac{S_t^n}{\hat{S}_\infty^n}\right)^{-\omega_2} \text{ for } \hat{S}_\infty^n < S_t^n \tag{4.224}$$

Equation (4.224) is the price of a perpetual American power put option. This completes the proof.

Theorem 4.6.2

Consider the perpetual American power put option with dividend yield. If $T \to \infty$ and $0 < \Re(\omega) < \omega_2$, then the free boundary of the perpetual American power put option is given by

$$\bar{S}_\infty^n = \frac{\alpha_2(n^2\sigma^2(\omega_1-\omega_2)(\omega_2+1))}{(2q\omega_2-(n^2\sigma^2(\omega_1-\omega_2)(\omega_2+1)))(\omega_1-\omega_2)}K \tag{4.225}$$

and the price of perpetual American power put option equals

$$A_\infty^n(S_t^n, t) = \frac{1}{(\omega_2-\omega_1)}\left(\frac{S_t^n}{\bar{S}_\infty^n}\right)^{-\omega_2}\left(\frac{\alpha_2 K}{\omega_2}-\frac{2q}{n^2\sigma^2}\frac{\bar{S}_\infty^n}{(\omega_2+1)}\right) \text{ for } \bar{S}_\infty^n < S_t^n \tag{4.226}$$

where

$$\omega_1 = \frac{-\alpha_1^* - \sqrt{(\alpha_1^*)^2 + 4\alpha_2}}{2}$$

$$\omega_2 = \frac{-\alpha_1^* + \sqrt{(\alpha_1^*)^2 + 4\alpha_2}}{2}$$

and

$$\alpha_1^* = \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right), \alpha_2 = \frac{2r}{n^2\sigma^2}$$

Proof: The integral representation for the price of the American power put option which pays dividend yield given by (4.126) can be expressed as

$$A_p^n(S_t^n, t) = E_p^n(S_t^n, t) + Z_1^n(S_t^n, t) + Z_2^n(S_t^n, t) \tag{4.227}$$

where $E_p^n(S_t^n, t)$, $Z_1^n(S_t^n, t)$ and $Z_2^n(S_t^n, t)$ are given by

$$\begin{aligned} E_p^n(S_t^n, t) &= Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) \\ &\quad - S_t^n e^{\left(r(n-1) - nq + \frac{1}{2}n(n-1)\sigma^2\right)(T-t)}\mathcal{N}(-d_{1,n}) \end{aligned} \tag{4.228}$$

with

$$d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}}$$

$$d_{2,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{n\sigma\sqrt{T-t}}$$

$$Z_1^n(S_t^n, t) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega} \int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(y-t)} dy d\omega \tag{4.229}$$

and

$$Z_2^n(S_t^n, t) = -\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_t^n)^{-\omega} \int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2 + \alpha_1^*\omega - \alpha_2)(y-t)} dy d\omega \tag{4.230}$$

respectively. The roots of $\omega^2 + \alpha_1^*\omega - \alpha_2 = 0$ are

$$\omega_1 = \frac{-\alpha_1^* + \sqrt{(\alpha_1^*)^2 + 4\alpha_2}}{2}$$

and

$$\omega_2 = \frac{-\alpha_1^* - \sqrt{(\alpha_1^*)^2 + 4\alpha_2}}{2}$$

with

$$\alpha_1^* = \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right), \alpha_2 = \frac{2r}{n^2\sigma^2}$$

Thus,

$$\omega^2 + \alpha_1^*\omega - \alpha_2 = (\omega - \omega_1)(\omega - \omega_2)$$

Notice that for the valuation formula (4.227) to hold as $T \to \infty$, it is necessary that $\Re(\omega^2 + \alpha_1^*\omega - \alpha_2) < 0$, that is $0 < \Re(\omega) < \omega_2$. Using the super-contact condition given by (4.121) as $T \to \infty$, perpetual American power put which pays dividend yield becomes

$$\begin{aligned} \left.\frac{\partial A_p^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \bar{S}_\infty^n} &= \left.\frac{\partial E_p^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \bar{S}_\infty^n} + \left.\frac{\partial Z_1^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \bar{S}_\infty^n} \\ &+ \left.\frac{\partial Z_2^n(\bar{S}_\infty^n, t)}{\partial \bar{S}_\infty^n}\right|_{S_t^n = \bar{S}_\infty^n} = -1 \end{aligned} \tag{4.231}$$

where the free boundary $\bar{S}_\infty^n$ is now independent of time. Now, the derivative of the price of European power put option $E_p^n(S_t^n, t)$ which pays dividend yield with respect to S_t^n at $S_t^n = \bar{S}_\infty^n$ is determined as

$$\left.\frac{\partial E_p^n(S_t^n, t)}{\partial S_t^n}\right|_{S_t^n = \bar{S}_\infty^n} = -e^{\left(r(n-1) - nq + \frac{1}{2}n(n-1)\sigma^2\right)(T-t)}\mathcal{N}(-\bar{d}_{1,n})$$

where

$$\bar{d}_{1,n} = \frac{\ln\left(\frac{\bar{S}^n_\infty}{K}\right) + n\left(r - q + \left(n - \frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}}$$

As $T \to \infty, \bar{d}_{1,n} \to \infty$ and therefore

$$\left.\frac{\partial E^n_p(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \bar{S}^n_\infty} \to 0 \tag{4.232}$$

Now, differentiating (4.229) with respect to S^n_t and taking the limit as $T \to \infty$ to obtain

$$\begin{aligned}
\frac{\partial Z^n_1(S^n_t, t)}{\partial S^n_t} &= -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S^n_t)^{-1}\left(\int_t^\infty \left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha^*_1\omega-\alpha_2)(y-t)}dy\right)d\omega \\
&= -\frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S^n_t)^{-1}\left(\left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega}\left.\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha^*_1\omega-\alpha_2)(y-t)}}{\frac{n^2\sigma^2}{2}(\omega^2+\alpha^*_1\omega-\alpha_2)}\right|_t^\infty\right)d\omega
\end{aligned}$$

Therefore,

$$\left.\frac{\partial Z^n_1(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \bar{S}^n_\infty} = \frac{K}{2\pi i}\frac{2r}{n^2\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{d\omega}{\bar{S}^n_\infty(\omega^2+\alpha^*_1\omega-\alpha_2)} \tag{4.233}$$

Equation (4.233) can be expressed as

$$\left.\frac{\partial Z^n_1(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \bar{S}^n_\infty} = \frac{\alpha_2 K}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{d\omega}{\bar{S}^n_\infty(\omega-\omega_1)(\omega-\omega_2)}d\omega \tag{4.234}$$

where $\alpha_2 = \frac{2r}{n^2\sigma^2}$ and $\omega^2 + \alpha^*_1\omega - \alpha_2 = (\omega-\omega_1)(\omega-\omega_2)$. By the application of the residue theorem (4.162), (4.234) becomes

$$\left.\frac{\partial Z^n_1(S^n_t, t)}{\partial S^n_t}\right|_{S^n_t = \bar{S}^n_\infty} = \frac{\alpha_2 K}{\bar{S}^n_\infty(\omega_1-\omega_2)} \tag{4.235}$$

In the same manner, setting $T \to \infty$ and differentiating (4.230) with respect to S_t^n leads to

$$\begin{aligned}
\frac{\partial Z_2^n(S_t^n,t)}{\partial S_t^n} &= \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\int_t^\infty \frac{\omega}{\omega+1}\left(\frac{S_t^n}{\bar{S}_\infty^n}\right)^{-(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dy\right)d\omega \\
&= -\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{\omega+1}\left(\frac{S_t^n}{\bar{S}_\infty^n}\right)^{-(\omega+1)}\left(\left.\frac{e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}}{\frac{n^2\sigma^2}{2}(\omega^2+\alpha_1^*\omega-\alpha_2)}\right|_t^\infty\right)d\omega \\
&= -\frac{2}{n^2\sigma^2}\frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{\omega+1}\left(\frac{S_t^n}{\bar{S}_\infty^n}\right)^{-(\omega+1)}\frac{d\omega}{\omega^2+\alpha_1^*\omega-\alpha_2} \\
&= -\frac{2q}{n^2\sigma^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{\omega+1}\left(\frac{S_t^n}{\bar{S}_\infty^n}\right)^{-(\omega+1)}\frac{d\omega}{\omega^2+\alpha_1^*\omega-\alpha_2}
\end{aligned}$$

Therefore,

$$\left.\frac{\partial Z_2^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\bar{S}_\infty^n} = -\frac{2q}{n^2\sigma^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{(\omega+1)(\omega^2+\alpha_1^*\omega-\alpha_2)}d\omega \tag{4.236}$$

By the application of residue theorem (4.162), then (4.236) becomes

$$\left.\frac{\partial Z_2^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\bar{S}_\infty^n} = -\frac{2q}{n^2\sigma^2}\left(\frac{\omega_1}{(\omega_1+1)(\omega_1-\omega_2)} - \frac{1}{(\omega_1+1)(\omega_2+1)}\right) \tag{4.237}$$

Substituting (4.232), (4.235) and (4.237) into (4.231) and by means of the super contact condition, yields

$$\begin{aligned}
\left.\frac{\partial A_p^n(S_t^n,t)}{\partial S_t^n}\right|_{S_t^n=\bar{S}_\infty^n} &= \frac{\alpha_2 K}{\bar{S}_\infty^n(\omega_1-\omega_2)} \\
&\quad - \frac{2q}{n^2\sigma^2}\left(\frac{\omega_1}{(\omega_1+1)(\omega_1-\omega_2)} - \frac{1}{(\omega_1+1)(\omega_2+1)}\right) \\
&= -1
\end{aligned} \tag{4.238}$$

Simplifying (4.238) further leads to a relation

$$\frac{\alpha_2 K}{\bar{S}_\infty^n(\omega_1-\omega_2)} - \frac{2q}{n^2\sigma^2}\left(\frac{\omega_2}{(\omega_1-\omega_2)(\omega_2+1)}\right) = -1 \tag{4.239}$$

Therefore, the free boundary of perpetual American power put option which pays dividend yield is obtained as

$$\bar{S}^n_\infty = \frac{\alpha_2(n^2\sigma^2(\omega_1-\omega_2)(\omega_2+1))}{(2q\omega_2-(n^2\sigma^2(\omega_1-\omega_2)(\omega_2+1)))(\omega_1-\omega_2)}K$$

Hence (4.225) is established.

Once again using the fact that $\Re(\omega^2+\alpha_1^*\omega-\alpha_2)<0$, taking the limit $T\to\infty$ in (4.227) and integrating the time variable leads to the price for the perpetual American power put option with dividend yield for $S^n_t > \bar{S}^n_\infty$ given by

$$\begin{aligned} A^n_\infty(S^n_t,t) &= -\frac{\alpha_2 K}{2\pi i}\int_{c-i\infty}^{c+\infty}\left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega}\frac{d\omega}{\omega(\omega-\omega_1)(\omega-\omega_2)} \\ &+\frac{2q}{n^2\sigma^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\bar{S}^n_\infty\left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega}\frac{d\omega}{(\omega+1)(\omega-\omega_1)(\omega-\omega_2)} \end{aligned} \tag{4.240}$$

Using the residue theorem (4.162), then (4.240) becomes

$$\begin{aligned} A^n_\infty(S^n_t,t) &= \left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega_2}\frac{\alpha_2 K}{\omega_2(\omega_2-\omega_1)} \\ &-\frac{2q}{n^2\sigma^2}\left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega_2}\frac{\bar{S}^n_\infty}{(\omega_2+1)(\omega_2-\omega_1)} \end{aligned} \tag{4.241}$$

Hence, the price of perpetual American power put option is obtained as

$$A^n_\infty(S^n_t,t) = \frac{1}{(\omega_2-\omega_1)}\left(\frac{S^n_t}{\bar{S}^n_\infty}\right)^{-\omega_2}\left(\frac{\alpha_2 K}{\omega_2}-\frac{2q}{n^2\sigma^2}\frac{\bar{S}^n_\infty}{(\omega_1+1)}\right)$$

This completes the proof.

Remark 4.6.1

(i) Note that the price of a perpetual European power put option with non-dividend and dividend yields, respectively is zero, since it can never be exercised before expiration.

(ii) By setting $n = 1$ and $\hat{S}_\infty = S^*_\infty$ in (4.221) and (4.224), the free boundary and the price of the perpetual American put option with non-dividend yield derived by Panini and Srivastav (2005) are given by

$$S^*_\infty = \frac{k_1}{k_1 + 1}K, \quad \text{where } k_1 = \frac{2r}{\sigma^2} \tag{4.242}$$

and

$$P_\infty(S,t) = (K - S^*_\infty)\left(\frac{S}{S^*_\infty}\right)^{-\frac{2r}{\sigma^2}} \tag{4.243}$$

respectively.

(iii) Setting $n = 1$ and $\bar{S}_\infty = S^*_\infty$ in (4.225) and (4.226), the free boundary and the price of the American put option with dividend yield derived by Frontczak and Schöbel (2008) are obtained as

$$S^*_\infty = \frac{\omega_2}{\omega_2 + 1}K \tag{4.244}$$

with $\omega_2 = \frac{k_2-1}{2} + \frac{\sqrt{(k_2-1)^2+4k_1}}{2}, k_1 = \frac{2r}{\sigma^2}, k_2 = \frac{2(r-q)}{\sigma^2}$

and

$$A_\infty(S,t) = (K - S^*_\infty)\left(\frac{S}{S^*_\infty}\right)^{-\omega_2} \tag{4.245}$$

respectively.

4.7 Closed-Form Solution for the Price of the American Power Put Option

The numerical result for the valuation of American power put options on a stock with dividend yield is presented below:

From (4.126), the integral representation for the price of the American power put option which pays dividend yield is given by

$$\begin{aligned} A_p^n(S_t^n,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K^{\omega+1}}{\omega(\omega+1)} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(T-t)}(S_t^n)^{-\omega}d\omega \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \end{aligned} \tag{4.246}$$

where $E_p^n(S_t^n,t)$ is the integral representation for the price of the European power put option with dividend yield which reduces to the Black-Scholes-Merton-like valuation formula of the form:

$$\begin{aligned} E_p^n(S_t^n,t) &= Ke^{-r(T-t)}\mathcal{N}(-d_{2,n}) \\ &\quad - S_t^n e^{(r(n-1)-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t))}\mathcal{N}(-d_{1,n}) \end{aligned} \tag{4.247}$$

with

$$\left.\begin{aligned} &\mathcal{N}(-d_{1,n}) = 1-\mathcal{N}(d_{1,n}), \mathcal{N}(-d_{2,n}) = 1-\mathcal{N}(d_{2,n}), \\ &d_{1,n} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r-q+\left(n-\frac{1}{2}\right)\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}}, \\ &d_{2,n} = d_{1,n} - n\sigma\sqrt{(T-t)} = \frac{\ln\left(\frac{S_t^n}{K}\right) + n\left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{n\sigma\sqrt{(T-t)}} \end{aligned}\right\} \tag{4.248}$$

The free boundary $\bar{S}_t^n$ is determined as the solution of

$$\begin{aligned} K - \bar{S}_t^n &= E_p^n(\bar{S}_t^n,t) \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega}}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\bar{S}_t^n)^{-\omega}\int_t^T \frac{(\bar{S}_y^n)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)(y-t)}dyd\omega \end{aligned} \tag{4.249}$$

where

$$\left.\begin{aligned}\alpha_1^* &= \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right), \\ \alpha_2 &= \frac{2r}{n^2\sigma^2}\end{aligned}\right\} \tag{4.250}$$

To evaluate the integral (4.246), first transform the time variable t to $\tau = T - t$. Then (4.246) becomes

$$\begin{aligned} A_p^n(S_\tau^n, \tau) &= E_p^n(S_\tau^n, \tau) \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_\tau^n)^{-\omega} \int_0^\tau \frac{(\bar{S}_{\tau-\eta}^n)^\omega}{\omega} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)\eta} d\eta d\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty} (S_\tau^n)^{-\omega} \int_0^\tau \frac{(\bar{S}_{\tau-\eta}^n)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2(\omega^2+\alpha_1^*\omega-\alpha_2)\eta} d\eta d\omega \end{aligned} \tag{4.251}$$

where

$$\eta = \tau - y$$

Equation (4.251) is in the transformed coordinates. Rearranging terms and setting $R(\omega) = \omega^2 + \alpha_1^*\omega - \alpha_2 = \omega^2 + \left(1 - \frac{n-1}{n} - \frac{2(r-q)}{n\sigma^2}\right)\omega - \frac{2r}{n^2\sigma^2}$ in (4.251) yields

$$\begin{aligned} A_p^n(S_\tau^n, \tau) &= E_p^n(S_\tau^n, \tau) \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} \int_0^\tau \frac{1}{\omega}\left(\frac{S_\tau^n}{\bar{S}_{\tau-\eta}^n}\right)^{-\omega} e^{\frac{1}{2}n^2\sigma^2 R(\omega)\eta} d\eta d\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty} \int_0^\tau (S_\tau^n)^{-\omega} \frac{(\bar{S}_{\tau-\eta}^n)^{\omega+1}}{\omega+1} e^{\frac{1}{2}n^2\sigma^2 R(\omega)\eta} d\eta d\omega \end{aligned} \tag{4.252}$$

For $n = 1$, (4.252) becomes

$$\begin{aligned} A_p(S_\tau, \tau) &= E_p(S_\tau, \tau) \\ &+ \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty} \int_0^\tau \frac{1}{\omega}\left(\frac{S_\tau}{\bar{S}_{\tau-\eta}}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega \\ &- \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty} \int_0^\tau (S_\tau)^{-\omega} \frac{(\bar{S}_{\tau-\eta})^{\omega+1}}{\omega+1} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega \end{aligned} \tag{4.253}$$

where

$$R_0(\omega) = \omega^2 + \left(1 - \frac{2(r-q)}{\sigma^2}\right)\omega - \frac{2r}{\sigma^2}$$
$$= \omega^2 + (1-e_2)\omega - e_1 \tag{4.254}$$

$$e_1 = \frac{2r}{\sigma^2} \tag{4.255}$$

$$e_2 = \frac{2(r-q)}{\sigma^2} \tag{4.256}$$

Next, let

$$M_1 = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^{\tau}\frac{1}{\omega}\left(\frac{S_\tau}{\bar{S}_{\tau-\eta}}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega \tag{4.257}$$

$$M_2 = \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^{\tau}(S_\tau)^{-\omega}\frac{(\bar{S}_{\tau-\eta})^{\omega+1}}{\omega+1} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega \tag{4.258}$$

Therefore, (4.253) becomes

$$A_p(S_\tau,\tau) = E_p(S_\tau,\tau) + M_1 - M_2 \tag{4.259}$$

Consider (4.257) and setting

$$I_1(\eta) = \frac{rK}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\omega}\left(\frac{S_\tau}{\bar{S}_{\tau-\eta}}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\omega \tag{4.260}$$

Then (4.257) leads to a relation

$$M_1 = \int_0^{\tau} I_1(\eta) d\eta \tag{4.261}$$

To evaluate the integral in (4.260), let

$$\omega = c + ix \Rightarrow d\omega = i dx \tag{4.262}$$

Substituting (4.262) into (4.260) leads to

$$I_1(\eta) = \frac{rK}{2\pi} e^{-r\eta-\alpha c^2+\beta c} \int_{-\infty}^{\infty} \left(\frac{c - ix}{c^2 + x^2} \right) e^{-\alpha x^2 + i\beta x} dx \tag{4.263}$$

where

$$\left.\begin{aligned} \alpha &= \frac{\sigma^2 \eta}{2} \\ \beta &= \alpha \left(1 - \frac{2(r-q)}{\sigma^2} + 2c \right) - \ln \left(\frac{S_\tau}{\bar{S}_{\tau-\eta}} \right) \\ 1 &< c < \infty \end{aligned}\right\} \tag{4.264}$$

Following the procedures of Panini and Srivastav (2004) for the case of non-dividend yield and using the identity

$$e^{i\beta x} = \cos \beta x + i \sin \beta x \tag{4.265}$$

Therefore,

$$\begin{aligned} I_1(\eta) &= \frac{rK}{2\pi} e^{-r\eta-\alpha c^2+\beta c} \int_{-\infty}^{\infty} \left(\frac{(c - ix)(\cos \beta x + i \sin \beta x)}{c^2 + x^2} \right) e^{-\alpha x^2} dx \\ &= \frac{rK}{2\pi} e^{-r\eta-\alpha c^2+\beta c} \int_{-\infty}^{\infty} \left(\frac{(c \cos \beta x + x \sin \beta x)}{c^2 + x^2} \right) e^{-\alpha x^2} dx \end{aligned} \tag{4.266}$$

where the real part of the last integral is taken into consideration. For an efficient and better accuracy pricing of American power put option for the case of n=1, (4.266) is transformed to a form that permits the use of Gauss-Laguerre quadrature method as follows:

$$I_1(\eta) = \frac{rK}{\pi} e^{-r\eta-\alpha c^2+\beta c} \int_{0}^{\infty} \left(\frac{(c \cos \beta x + x \sin \beta x)}{c^2 + x^2} \right) e^{-\alpha x^2} dx$$

Setting $Q = e^{-r\eta - \alpha c^2 + \beta c}$, the last integral equation becomes

$$I_1(\eta) = Q\frac{rK}{\pi}\left(\int_0^\infty \left(\frac{c\cos\beta x}{c^2+x^2}\right)e^{-\alpha x^2}dx + \int_0^\infty \left(\frac{x\sin\beta x}{c^2+x^2}\right)e^{-\alpha x^2}dx\right) \tag{4.267}$$

Using the following standard integrals (Gradshteyn and Ryzhik (2007)):

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \tag{4.268}$$

and

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} \tag{4.269}$$

and by replacing $\frac{c}{c^2+x^2}$ with a cosine transform (Erdelyi et al. (1954)), (4.267) becomes

$$\begin{aligned} I_1(\eta) = Q\frac{rK}{\pi}\int_0^\infty\int_0^\infty e^{-\alpha x^2}e^{-cy}\cos\beta x\cos xy \;\; dxdy \\ + Q\frac{rK}{\pi}\int_0^\infty\int_0^\infty e^{-\alpha x^2}e^{-cy}\sin\beta x\sin xy \;\; dxdy \end{aligned} \tag{4.270}$$

Using the following product rules for sine and cosine functions given by

$$2\sin x\sin y = \cos(x-y) - \cos(x+y) \tag{4.271}$$

and

$$2\cos x\cos y = \cos(x-y) + \cos(x+y) \tag{4.272}$$

respectively to get

$$\begin{aligned} I_1(\eta) = Q\frac{rK}{2\pi}\int_0^\infty e^{-cy}\int_0^\infty (\cos(x(\beta-y)) + \cos(x(\beta+y)))e^{-\alpha x^2}dxdy \\ + Q\frac{rK}{2\pi}\int_0^\infty e^{-cy}\int_0^\infty (\cos(x(\beta-y)) - \cos(x(\beta+y)))e^{-\alpha x^2}dxdy \end{aligned} \tag{4.273}$$

Using the procedures of Erdelyi et al. (1954) and Gradshteyn and Ryzhik (2007), then

$$\int_0^\infty \cos(x(\beta + y))e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta+y)^2}{4\alpha}} \tag{4.274}$$

$$\int_0^\infty \cos(x(\beta - y))e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta-y)^2}{4\alpha}} \tag{4.275}$$

By means of (4.274) and (4.275), (4.273) yields

$$\begin{aligned} I_1(\eta) &= Q\frac{rK}{4\pi}\int_0^\infty e^{-cy}\left(\sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta-y)^2}{4\alpha}} + \sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta+y)^2}{4\alpha}}\right)dy \\ &+ Q\frac{rK}{4\pi}\int_0^\infty e^{-cy}\left(\sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta-y)^2}{4\alpha}} - \sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\beta+y)^2}{4\alpha}}\right)dy \\ &= Q\frac{rK}{2\sqrt{\alpha\pi}}\int_0^\infty e^{-\frac{(\beta-y)^2}{4\alpha}}e^{-cy}dy \end{aligned} \tag{4.276}$$

The integral in (4.276) can be evaluated accurately by means of a N-point Gauss-Laguerre quadrature method as follows:

$$\begin{aligned} \int_0^\infty e^{-\frac{(\beta-y)^2}{4\alpha}}e^{-cy}dy &= \frac{1}{c}\int_0^\infty e^{-y}\phi_0\left(\frac{y}{c}\right) \\ &\approx \frac{1}{c}\sum_{j=1}^{N}\omega_j\phi_0\left(\frac{y_j}{c}\right) \end{aligned} \tag{4.277}$$

Next, consider (4.258) given by

$$M_2 = \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\tau (S_\tau)^{-\omega}\frac{(\bar{S}_{\tau-\eta})^{\omega+1}}{\omega+1}e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta}d\eta d\omega$$

Let

$$I_2(\eta) = \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}(S_\tau)^{-\omega}\frac{(\bar{S}_{\tau-\eta})^{\omega+1}}{\omega+1}e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta}d\omega \tag{4.278}$$

Therefore,

$$M_2 = \int_0^\tau I_2(\eta)d\eta \tag{4.279}$$

Using (4.262) and following the above procedures, therefore

$$\begin{aligned} I_2(\eta) &= Q\frac{q}{\pi}\bar{S}_{\tau-\eta}\int_0^\infty \left(\frac{(c-1)\cos\beta x + x\sin\beta x}{(c-1)^2+x^2}\right)e^{-\alpha x^2}dx \\ &= Q\frac{q}{\pi}\bar{S}_{\tau-\eta}\int_0^\infty \left(\frac{(c-1)\cos\beta x}{(c-1)^2+x^2}\right)e^{-\alpha x^2}dx \\ &+ Q\frac{q}{\pi}\bar{S}_{\tau-\eta}\int_0^\infty \left(\frac{x\sin\beta x}{(c-1)^2+x^2}\right)e^{-\alpha x^2}dx \end{aligned} \tag{4.280}$$

Once again by means of the standard integrals given by (4.268) and (4.269) and replacing $\frac{c-1}{(c-1)^2+x^2}$ with a cosine transform (Erdelyi et al. (1954) and Gradshteyn and Ryzhik (2007)). Equation (4.280) becomes

$$\begin{aligned} I_2(\eta) &= Q\frac{q}{\pi}\bar{S}_{\tau-\eta}\int_0^\infty\int_0^\infty e^{-\alpha x^2}e^{-(c-1)y}\cos\beta x\cos xy \;\; dxdy \\ &+ Q\frac{q}{\pi}\bar{S}_{\tau-\eta}\int_0^\infty\int_0^\infty e^{-\alpha x^2}e^{-(c-1)y}\sin\beta x\sin xy \;\; dxdy \end{aligned} \tag{4.281}$$

Again, using the product rules for the sine and cosine functions given by (4.271) and (4.272) respectively to get

$$\begin{aligned} I_2(\eta) = Q\frac{q}{2\pi}\bar{S}_{\tau-\eta}\int_0^\infty e^{-(c-1)y}\int_0^\infty(\cos(x(\beta-y))+\cos(x(\beta+y)))e^{-\alpha x^2}dxdy \\ + Q\frac{q}{2\pi}\bar{S}_{\tau-\eta}\int_0^\infty e^{-(c-1)y}\int_0^\infty(\cos(x(\beta-y))-\cos(x(\beta+y)))e^{-\alpha x^2}dxdy \end{aligned} \tag{4.282}$$

Substituting (4.274) and (4.275) into (4.280) and solving further yields

$$I_2(\eta) = Q\frac{q}{2\sqrt{\alpha\pi}}\bar{S}_{\tau-\eta}\int_0^\infty e^{-\frac{(\beta-y)^2}{4\alpha}}e^{-(c-1)y}dy \tag{4.283}$$

Finally, for better accuracy the above integral in (4.283) can be approximated by means of the N-point Gauss-Laguerre quadrature method. Therefore,

$$\begin{aligned} \int_0^\infty e^{-\frac{(\beta-y)^2}{4\alpha}}e^{-(c-1)y}dy &= \frac{1}{c-1}\int_0^\infty e^{-y}\phi_0\left(\frac{y}{c-1}\right) \\ &\approx \frac{1}{c-1}\sum_{j=1}^{N}\omega_j\phi_0\left(\frac{y_j}{c-1}\right) \end{aligned} \tag{4.284}$$

where

$$\phi_0(y) = e^{-\frac{(\beta-y)^2}{4\alpha}}$$

ω_j and y_j are the weight and abscissa of the Gauss-Laguerre quadrature method. Substituting (4.277) and (4.284) into (4.276) and (4.283), respectively yields

$$I_1(\eta) = Q\frac{rK}{2\sqrt{\alpha\pi}}\frac{1}{c}\sum_{j=1}^{N}\omega_j\phi_0\left(\frac{y_j}{c}\right) \tag{4.285}$$

and

$$I_2(\eta) = Q\frac{q}{2\sqrt{\alpha\pi}}S_{\tau-\eta}\frac{1}{c-1}\sum_{j=1}^{N}\omega_j\phi_0\left(\frac{y_j}{c-1}\right) \tag{4.286}$$

Using (4.259), (4.261), (4.279), (4.285), (4.286) and the value of Q, then the following approximation for the price of the American power put option for the case of $n = 1$ is obtained as

$$\begin{aligned} A_p(S_\tau, \tau) &= E_p(S_\tau, \tau) \\ &+ \int_0^\tau e^{-r\eta-\alpha c^2+\beta c}\frac{rK}{2c\sqrt{\alpha\pi}}\sum_{j=1}^{N}\omega_j\phi_0\left(\frac{y_j}{c}\right)d\eta \\ &- \int_0^\tau e^{-r\eta-\alpha c^2+\beta c}\frac{q\bar{S}_{\tau-\eta}}{2(c-1)\sqrt{\alpha\pi}}\sum_{j=1}^{N}\omega_j\phi_0\left(\frac{y_j}{c-1}\right)d\eta \end{aligned} \tag{4.287}$$

Remark 4.7.1

(i) The integrals in (4.287) are evaluated by means of trapezoidal rule.

(ii) The weights $\omega_j, j = 1, 2, ..., N$ are determined by

$$\omega_j = \frac{y_j}{(N+1)^2 L_{N+1}(y_j)^2} \tag{4.288}$$

with $L_N(y)$, the N-th Laguerre polynomial defined by

$$L_N(y) = \frac{e^y}{N!}\frac{d^N}{dy^N}(e^{-x}y^N) \tag{4.289}$$

(iii) The calculation of the price of American power put option for the case of $n = 1$ assumes that $\bar{S}_\tau$ is known for all τ.

Setting $\tau = T - t$ and $n = 1$ in (4.249) yields

$$
\begin{aligned}
K - \bar{S}_\tau = \mathrm{E}_p(\bar{S}_\tau, \tau) \\
+ \frac{rK}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega}\left(\frac{\bar{S}_\tau}{\bar{S}_{\tau-\eta}}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega \\
- \frac{q}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} (\bar{S}_\tau)^{-\omega}\frac{(\bar{S}_{\tau-\eta})^{\omega+1}}{\omega+1} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega
\end{aligned}
\tag{4.290}
$$

where $\eta = \tau - y$ and $R_0(\omega) = \omega^2 + (1-e_2)\omega - e_1$. The recursive scheme for determining $\bar{S}_\tau$ using (4.290) is obtained as

$$
\begin{aligned}
\bar{S}_N(\tau) = K - E_p(\bar{S}_{N-1}(\tau), \tau) \\
- \frac{rK}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega}\left(\frac{\bar{S}_{N-1}(\tau)}{\bar{S}_{N-1}(\tau-\eta)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega \\
+ \frac{q}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} (\bar{S}_{N-1}(\tau))^{-\omega}\frac{(\bar{S}_{N-1}(\tau-\eta))^{\omega+1}}{\omega+1} e^{\frac{1}{2}\sigma^2 R_0(\omega)\eta} d\eta d\omega
\end{aligned}
\tag{4.291}
$$

where $N = 1, 2, ..$ and $\bar{S}_0(\tau) = K$ for every τ. As before, the outer integral in (4.291) is evaluated using trapezoidal rule and the inner integral is approximated using an N-point Gauss-Laguerre quadrature method, The stopping criterion for recursion is $\|\bar{S}_N - \bar{S}_{N-1}\|_2 \le \epsilon$.

(iv) The closed-form solution for the price of the American power put option with non-dividend yield for the case of $n = 1$ can be obtained by setting $q = 1$ in (4.287).

4.8 The Mellin Transform Method and Basket Put Options

A natural extension of the univariate Mellin transform exists for higher dimensions. The double Mellin transform was first introduced by Reed (1944). He proved conditions for which the Mellin transform and inverse exist. Basket options are becoming increasingly widespread in commodity and particularly energy markets. A basket option gives the holder the right, but not the obligation, to buy or sell a group of underlying assets. The payoff for a basket call option is given by

$$B_c = \left(\sum_{i=1}^{m} \alpha_i S_i - K \right)^+ \tag{4.292}$$

The payoff for a basket put option is given by

$$B_p = \left(K - \sum_{i=1}^{m} \alpha_i S_i \right)^+ \tag{4.293}$$

where α_i is the number of shares of asset i in the basket, S_i is the price of asset i in the basket and K is the strike price. Mellin transforms in higher dimensions will be used to derive expressions for put options on a basket of multi-dividend paying stocks. Assume that the underlying assets follow geometric Brownian motion with drift $\mu_1, \mu_2, ..., \mu_m$ and volatility $\sigma_1, \sigma_2, ..., \sigma_m$ respectively. So, for $i = 1, 2, .., m$,

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dW_i \tag{4.294}$$

where each W_i is a Brownian motion and dW_i are normally distributed random variables with mean zero, variance dt and $corr(dW_i, dW_j) = \rho_{ij}$, for

$\rho_{ij} \in [-1, 1]$ such that $\sum = \sigma\rho\sigma$. The risk-free drift $\mu_i = r - q_i - \frac{\delta_i^2}{2}$ ensures that no-arbitrage condition holds. For multivariate Brownian motion with drift, say $\mathbf{X}_t$, the characteristic function $\Phi(\mathbf{u}; t) = e^{-t\Psi(\mathbf{u})} = \mathbf{E}(e^{i\mathbf{u}'\mathbf{X}_t})$ is given by the exponent (Manuge (2013)):

$$\Psi(u) = \frac{1}{2}\mathbf{u}'\sum\mathbf{u} - i(\mu^*)'\mathbf{u}$$

The expression for the integral equation for the price of the European put $E_p(\mathbf{S}, t)$ on a basket of m-stocks $S_1, S_2, ..., S_m$ by means of the multidimensional Mellin transform was presented in the following result.

Theorem 4.8.1

Let $\mathbf{S} = (S_1, S_2, ..., S_m)'$ and $\omega^* = (\omega_1, \omega_2, ..., \omega_m)'$. The generalized Black-Scholes partial differential equation for the price of the European basket put option is given by

$$\frac{\partial E_p(\mathbf{S}, t)}{\partial t} + \sum_{i=1}^{m}(r - q_i)S_i\frac{\partial E_p(\mathbf{S}, t)}{\partial S_i}$$

$$+\frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}\sigma_i\sigma_j S_i S_j\frac{\partial^2 E_p(\mathbf{S}, t)}{\partial S_i \partial S_j} - rE_p(\mathbf{S}, t) = 0 \quad (4.295)$$

where $0 < S_1, S_2, ..., S_n < \infty,\ 0 \leq t \leq T$, with the boundary conditions

$$E_p(\mathbf{S}, T) = \phi(\mathbf{S}) = \left(K - \sum_{i=1}^{m} S_i\right)^+ \quad (4.296)$$

$$\lim_{\mathbf{S}\to 0} E_p(\mathbf{S}, t) = Ke^{-r(T-t)} \quad (4.297)$$

$$\lim_{\sum_{i=1}^{n} S_i\to\infty} E_p(\mathbf{S}, t) = 0 \quad (4.298)$$

Then, the expression for the integral equation for the price of the European put option on a basket of multi-dividend paying stocks is obtained as

$$E_p(\mathbf{S}, t) = \frac{1}{(2\pi i)^m} \int_{\gamma} \tilde{\phi}(\omega^*) e^{G(\omega^*)} \prod_{j=1}^{n} S_j^{-\omega_j} d\omega_j \tag{4.299}$$

where $\gamma = \times_{j=1}^{m} \gamma_j$ are strips in $\mathbb{C}^n$ defined by $\gamma_j = \{c_j + ib_j : c_j \in \mathbb{R}, b_j = \pm\infty\}$.

Proof: Let $\tilde{E}_p(\omega^*, t)$ denote the multi-dimensional Mellin transform of $E_p(\mathbf{S}, t)$ which is defined by the relation

$$\tilde{E}_p(\omega^*, t) = \int_{\mathbb{R}^{n+}} E_p(\mathbf{S}, t) \prod_{j=1}^{m} S_j^{\omega_j - 1} dS_j \tag{4.300}$$

The functions $E_p(\mathbf{S}, t)$ and $\tilde{E}_p(\omega^*, t)$ are called a Mellin transform pair. The multidimensional Mellin transform inversion of (4.300) is given by

$$E_p(\mathbf{S}, t) = \frac{1}{(2\pi i)^m} \int_{\gamma} \tilde{E}_p(\omega^*, t) \prod_{j=1}^{m} S_j^{-\omega_j} d\omega_j \tag{4.301}$$

Thus, to find the multidimensional Mellin transform of the generalized Black-Scholes equation, applying (4.300) to (4.295) to get

$$\frac{\partial \tilde{E}_p(\omega^*, t)}{\partial t} + G(\omega^*)\tilde{E}_p(\omega^*, t) = 0 \tag{4.302}$$

where

$$G(\omega^*) = \frac{1}{2} \sum_{i,j=1}^{m} \rho_{ij}\sigma_i\sigma_j\omega_i\omega_j - \sum_{i=1}^{m} \left((r - q_i) - \frac{\sigma_i^2}{2} \right) \omega_i - r \tag{4.303}$$

By means of the final time condition (4.296) and solving (4.302) further yields

$$\tilde{E}_p(\omega^*, t) = \tilde{\phi}(\omega^*) e^{G(\omega^*)(T-t)} \tag{4.304}$$

where $\tilde{\phi}(\omega^*)$ is the multidimensional Mellin transform of the final time condition obtained as

$$\tilde{\phi}(\omega^*) = \frac{B_m(\omega^*)K^{1+\sum_{j=1}^{n}\omega_j}}{\sum_{j=1}^{m}\omega_j\left(1+\sum_{j=1}^{m}\omega_j\right)} \tag{4.305}$$

with the multinomial beta function of n-variables

$$B_n(\omega^*) = \frac{\prod_{j=1}^{m}\Gamma(\omega_j)}{\Gamma\left(\sum_{j=1}^{m}\omega_j\right)} \tag{4.306}$$

Taking the multidimensional Mellin transform inversion of (4.304), then the expression for the integral equation for the price of the European put option on a basket of multi-dividend paying stocks is obtained as

$$E_p(\mathbf{S},t) = \frac{1}{(2\pi i)^m}\int_{\gamma}\tilde{\phi}(\omega^*)e^{G(\omega^*)(T-t)}\prod_{j=1}^{n}S_j^{-\omega_j}d\omega_j$$

Hence (4.299) is established.

Remark 4.8.1

(i) For m =1, (4.299) becomes the univariate Mellin-type formula for plain European put option given by

$$E_p(S,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}(\omega)e^{G(\omega)(T-t)}S^{-\omega}d\omega \tag{4.307}$$

with

$$\left.\begin{aligned} \tilde{\phi}(\omega) &= \frac{K^{\omega+1}}{\omega(\omega+1)} \\ G(\omega) &= \frac{1}{2}\sigma^2\omega^2 - \left((r-q)-\frac{\sigma^2}{2}\right)\omega - r \end{aligned}\right\} \tag{4.308}$$

(ii) For $m = 2$, (4.299) becomes the integral equation for the price of European put option on a basket of two-dividend paying stocks via the double Mellin transform of the form:

$$E_p(\mathbf{S}, t) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \tilde{\phi}(\omega^*) e^{G(\omega^*)(T-t)} \prod_{j=1}^{2} S_j^{-\omega_j} d\omega_j \tag{4.309}$$

with

$$\left.\begin{aligned}
&\tilde{\phi}(\omega^*) = \frac{B_2(\omega_1, \omega_2) K^{1+\sum_{j=1}^{2}\omega_j}}{\sum_{j=1}^{2}\omega_j (1 + \sum_{j=1}^{2}\omega_j)} \\
&G(\omega^*) = \tfrac{1}{2}\sum_{i,j=1}^{2} \rho_{ij}\sigma_i\sigma_j\omega_i\omega_j - \sum_{i=1}^{2}\left((r - q_i) - \tfrac{\sigma_i^2}{2}\right)\omega_i - r \\
&\mathbf{S} = (S_1, S_2)' \\
&\omega^* = (\omega_1, \omega_2)'
\end{aligned}\right\} \tag{4.310}$$

The payoff function for the European basket put option by means of multi-dimensional Mellin transform was given by the following result.

Theorem 4.8.2

Let the complex variable $\omega^* = (\omega_1, \omega_2, ..., \omega_m)'$ exist in an appropriate domain of convergence in $\mathbb{C}^n$, $\mathbf{S}$ be the current price of the underlying asset, $0 \leq t < T$ and $0 < K, T, \mathbf{S} < \infty$. For $\Re(\omega^*) > 0$, the multidimensional Mellin transform of the payoff function for the European basket put option is given by

$$\tilde{\phi}(\omega^*) = \frac{B_m(\omega^*) K^{1+\sum_{j=1}^{m}\omega_j}}{\sum_{j=1}^{m}\omega_j (1 + \sum_{j=1}^{m}\omega_j)} \tag{4.311}$$

Proof: Let the multidimensional Mellin transform of the European basket put payoff function be defined by

$$\tilde{\phi}(\omega^*) = \int_{\mathbb{R}^{n+}} \phi(\mathbf{S}) \prod_{j=1}^{m} S_j^{\omega_j - 1} dS_j \tag{4.312}$$

Substituting the final time condition of the European basket put option of the form

$$\phi(\mathbf{S}) = \left(K - \sum_{i=1}^{m} S_i\right)^+$$

into (4.312) yields

$$\tilde{\phi}(\omega^*) = \int_{\mathbb{R}^{n+}} \left(K - \sum_{i=1}^{n} S_i\right)^+ \prod_{j=1}^{m} S_j^{\omega_j - 1} dS_j \tag{4.313}$$

By simplifying (4.313) further, the multidimensional Mellin transform of the payoff function for the European basket put option is obtained as

$$\begin{aligned} \tilde{\phi}(\omega^*) &= \frac{\prod_{j=1}^{m} \Gamma(\omega_j) K^{1+\sum_{j=1}^{m} \omega_j}}{\Gamma\left(2 + \sum_{j=1}^{m} \omega_j\right)} \\ &= \frac{B_m(\omega^*) K^{1+\sum_{j=1}^{m} \omega_j}}{\sum_{j=1}^{m} \omega_j (1 + \sum_{j=1}^{m} \omega_j)} \end{aligned} \tag{4.314}$$

This completes the proof.

The integral representation for the price of the American put option on a basket of multi-dividend paying stocks was given by the following result.

Theorem 4.8.3

Let $\mathbf{S} = (S_1, S_2, ..., S_m)'$ and $\omega^* = (\omega_1, \omega_2, ..., \omega_m)'$. The generalized non-homogeneous Black-Scholes-Merton partial differential equation for the price of the American basket put option is given by

$$\frac{\partial A_p(\mathbf{S}, t)}{\partial t} + \sum_{i=1}^{m} (r - q_i) S_i \frac{\partial A_p(\mathbf{S}, t)}{\partial S_i}$$

$$+\frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 A_p(\mathbf{S},t)}{\partial S_i \partial S_j} - rA_p(\mathbf{S},t) = f(\mathbf{S},t) \tag{4.315}$$

where the early exercise function

$$f(\mathbf{S},t) = \begin{cases} -rK + \sum_{i=1}^{m} q_i S_i, & \text{if } 0 < \sum_{i=1}^{m} S_i \leq \bar{S} \\ 0, & \text{if } \bar{S} < \sum_{i=1}^{m} S_i < \infty \end{cases} \tag{4.316}$$

The boundary conditions imposed on (4.315) are

$$A_p(\mathbf{S},t) \to 0 \text{ as } \mathbf{S} \to \infty \tag{4.317}$$

$$A_p(\mathbf{S},T) = \phi(\mathbf{S}) = \left(K - \sum_{j=1}^{m} S_j\right)^{+} \tag{4.318}$$

The smooth pasting conditions along the boundary are

$$A(\mathbf{S},t)\Big|_{\sum_{i=1}^{m} S_i = \bar{S}} = K - \bar{S} \tag{4.319}$$

and

$$\frac{\partial A(\mathbf{S},t)}{\partial S_i}\Bigg|_{\sum_{i=1}^{m} S_i = \bar{S}} = -1 \tag{4.320}$$

The integral equation for the price of American basket put option with multi-dividend paying stocks is obtained as

$$A_p(\mathbf{S},t) = E_p(\mathbf{S},t) - \mathcal{M}^{-1}\left(\int_t^T \tilde{f}(\omega^*,y)e^{-(\Psi(\omega^* i)+r)(y-t)}dy\right) \tag{4.321}$$

Proof: As of the case of European basket put, the multi-dimensional Mellin transform of (4.315) yields

$$\frac{\partial \tilde{A}_p(\omega^*,t)}{\partial t} + G(\omega^*)\tilde{A}_p(\omega^*,t) = \tilde{f}(\omega^*,t) \tag{4.322}$$

where $G(\omega^*)$ is given by (4.303) which can be written as

$$\begin{aligned} G(\omega^*) &= \frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}\sigma_i\sigma_j\omega_i\omega_j - \sum_{i=1}^{m}\left((r-q_i)-\frac{\sigma_i^2}{2}\right)\omega_i - r \\ &= \frac{1}{2}(\omega^*)'\Sigma\omega^* + (\mu^*)'\omega^* - r \\ &= -(\Psi(\omega^* i) + r) \end{aligned} \tag{4.323}$$

Substituting (4.323) into (4.322) leads to

$$\frac{\partial \tilde{A}_p(\omega^*,t)}{\partial t} - (\Psi(\omega^* i)+r)\tilde{A}_p(\omega^*,t) = \tilde{f}(\omega^*,t) \tag{4.324}$$

where $\tilde{f}(\omega^*,t)$ is the multidimensional Mellin transform of the early exercise function

$$f(\mathbf{S},t) = f_a(\mathbf{S},t) + f_b(\mathbf{S},t) \tag{4.325}$$

with

$$f_a(\mathbf{S},t) = -rK$$

and

$$f_b(\mathbf{S},t) = \sum_{j=1}^{m} S_j q_j$$

Therefore,

$$\tilde{f}(\omega^*,t) = \tilde{f}_a(\omega^*,t) + \tilde{f}_b(\omega^*,t) \tag{4.326}$$

$$\begin{aligned} \tilde{f}_a(\omega^*,t) &= \int_{\mathbb{R}^{n+}} f_a(\mathbf{S},t)\prod_{j=1}^{m} S_j^{\omega_j-1}dS_j \\ &= -rK\int_0^{\bar{S}}\dots\int_0^{\bar{S}-\sum_{j=1}^{m-1}S_j} S_m^{\omega_n-1}\prod_{j=1}^{m} S_j^{\omega_j-1}dS_j \\ &= \frac{-rK\prod_{j=1}^{m}\Gamma(\omega_j)(\bar{S})^{\sum_{j=1}^{m}S_j}}{\sum_{j=1}^{m}S_j\Gamma(\sum_{j=1}^{m}S_j)} \\ &= \frac{-rKB_m(\omega^*)(\bar{S})^{\sum_{j=1}^{m}\omega_j}}{\sum_{j=1}^{m}\omega_j} \end{aligned} \tag{4.327}$$

where $\bar{S}$ is the boundary at time t. Similarly,

$$\tilde{f}_b(\omega^*, t) = \sum_{k=1}^{m} q_k \omega_k \frac{B_m(\omega^*)(\bar{S})^{\sum_{j=1}^{m}(1+\omega_j)}}{\sum_{j=1}^{m}\omega_j \left(\sum_{j=1}^{m}\omega_j + 1\right)} \tag{4.328}$$

Using (4.327) and (4.328), therefore (4.326) becomes

$$\tilde{f}(\omega^*, t) = \frac{-rKB_m(\omega^*)(\bar{S})^{\sum_{j=1}^{m}\omega_j}}{\sum_{j=1}^{m}\omega_j} + \sum_{k=1}^{m} q_k \omega_k \frac{B_m(\omega^*)(\bar{S})^{\sum_{j=1}^{m}(1+\omega_j)}}{\sum_{j=1}^{m}\omega_j \left(\sum_{j=1}^{m}\omega_j + 1\right)} \tag{4.329}$$

By means of (4.329), the final time condition (4.318) and Duhamel's principle[6] (John (1982)), the general solution of (4.324) is obtained as

$$\tilde{A}_p(\omega^*, t) = \tilde{E}_p(\omega^*, t) - \int_t^T \tilde{f}(\omega^*, y) e^{-(\Psi(\omega^* i)+r)(y-t)} dy \tag{4.330}$$

Taking the multidimensional Mellin transform of (4.330) leads to (4.321). This completes the proof.

Remark 4.8.2

(i) Note that, the first term in (4.321) is the price of the European basket put option.

(ii) By applying the value-matching condition (4.319) to (4.321), the value of $\bar{S}$ can be determined as a solution of the integral equation derived:

$$\begin{aligned} K - \bar{\mathbf{S}} &= \frac{1}{(2\pi i)^m} \int_{\gamma} \tilde{\phi}(\omega^*) e^{G(\omega^*)} \prod_{j=1}^{m} \bar{S}_j^{-\omega_j} d\omega_j \\ &\quad - \frac{1}{(2\pi i)^m} \int_{\gamma} \left(\int_t^T \tilde{f}(\omega^*, y) e^{-(\Psi(\omega^* i)+r)(y-t)} dy \right) \prod_{j=1}^{m} \bar{S}_j^{-\omega_j} d\omega_j \end{aligned} \tag{4.331}$$

[6]It reduced the problem of solving the American case to instead of solving the European case under different boundary conditions

with

$$\tilde{f}(\omega^*, y) = \frac{-rKB_m(\omega^*)(\bar{S}_y)^{\sum_{j=1}^m \omega_j}}{\sum_{j=1}^m \omega_j} + \sum_{k=1}^m q_k \omega_k \frac{B_m(\omega^*)(\bar{S}_y)^{\sum_{j=1}^m (1+\omega_j)}}{\sum_{j=1}^m \omega_j \left(\sum_{j=1}^m \omega_j + 1\right)} \tag{4.332}$$

(iii) Setting the free boundary to zero, (4.321) reduced to (4.299).

The closed-form solution for the price of the American basket put option was given by the following result.

Theorem 4.8.4

Let $\tau = T-t$, $\mathbf{S} = (S_1, S_2, ..., S_m)'$, $\omega^* = (\omega_1, \omega_2, ..., \omega_m)'$ and $0 < K, T, S_j, q_j < \infty$ for all $1 \leq j \leq m$. For Lipschitz payoff $\phi(\mathbf{S})$, the integral equation for the price of American basket put option with multi-dividend paying stocks given by

$$A_p(\mathbf{S}, t) = E_p(\mathbf{S}, t) - \mathcal{M}^{-1}\left(\int_t^T \tilde{f}(\omega^*, y) e^{-(\Psi(\omega^* i)+r)(y-t)} dy\right) \tag{4.333}$$

reduces to the approximation given by

$$\begin{aligned} A_p(\mathbf{S}, \tau) \simeq &\frac{(-1)^{\sum k} \Delta_b}{(2\pi i)^m} \mathcal{FFT}(\gamma \varsigma^E) e^{-r\tau - c' y_k} \\ &+ \frac{(-1)^{\sum k} \Delta_b \Delta_\tau}{(2\pi i)^m} \mathcal{FFT}\left(\sum_{l=0}^{M-1} \gamma \varsigma^{eep} e^{-r(\tau - t_l)}\right) e^{-c' y_k} \end{aligned} \tag{4.334}$$

Proof: From (4.333), write that

$$\begin{aligned}
A_p(\mathbf{S},t) &= \frac{1}{(2\pi i)^m}\int_\gamma \tilde{\phi}(\omega^*)e^{-(\Psi(\omega^* i)+r))(T-t)}\prod_{j=1}^m S_j^{-\omega_j}d\omega_j \\
&- \mathcal{M}^{-1}\left(\int_t^T \tilde{f}(\omega^*,y)e^{-(\Psi(\omega^* j)+r)(y-t)}dy\right) \\
&= \frac{1}{(2\pi i)^m}\int_\gamma \tilde{\phi}(\omega^*)e^{-(\Psi(\omega^* i)+r))(T-t)}\prod_{j=1}^m S_j^{-\omega_j}d\omega_j \\
&- \frac{1}{(2\pi i)^m}\int_\gamma\left(\int_t^T \tilde{f}(\omega^*,y)e^{-(\Psi(\omega^* i)+r)(y-t)}dy\right)\prod_{j=1}^m S_j^{-\omega_j}d\omega_j \\
&= \frac{1}{(2\pi i)^m}\int_\gamma e^{-r(T-t)}\tilde{\phi}(\omega^*)\Phi(\omega^* i, T-t)\prod_{j=1}^m S_j^{-\omega_j}d\omega_j \\
&- \frac{1}{(2\pi i)^m}\int_\gamma\left(\int_t^T \tilde{f}(\omega^*,y)\Phi(\omega^* i, y-t)\right)e^{-r(y-t)}\prod_{j=1}^m S_j^{-\omega_j}dyd\omega_j
\end{aligned} \tag{4.335}$$

Setting $\tau = T-t$, (4.335) yields

$$\begin{aligned}
A_p(\mathbf{S},\tau) = \frac{1}{(2\pi i)^m}\lim_{\mathbf{b}\to\infty}\int_{\mathbf{c}-i\mathbf{b}}^{\mathbf{c}+i\mathbf{b}} e^{-r\tau}\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)\mathbf{S}^{-\omega^*}d\omega^* \\
+ \frac{1}{(2\pi i)^m}\lim_{\mathbf{b}\to\infty}\int_{\mathbf{c}-i\mathbf{b}}^{\mathbf{c}+i\mathbf{b}}\left(\int_0^\tau \tilde{f}(\omega^*,y)\Phi(\omega^* i,\tau-y)\right)e^{-r(\tau-y)}\mathbf{S}^{-\omega^*}dyd\omega^*
\end{aligned} \tag{4.336}$$

where $\Phi(.)$ is the characteristic function of a multivariate Brownian motion with drift. By means of change of variables $\omega^* = \mathbf{c}+i\mathbf{b}, d\omega^* = id\mathbf{b}$. Then, (4.336) yields

$$\begin{aligned}A_p(\mathbf{S},\tau) &= \frac{1}{(2\pi)^m}\lim_{\mathbf{b}\to\infty}\int_{-\mathbf{b}}^{\mathbf{b}} e^{-r\tau}\tilde{\phi}(\mathbf{c}+i\mathbf{b})\Phi(\mathbf{c}i-\mathbf{b},\tau)\mathbf{S}^{-(\mathbf{c}+i\mathbf{b})}d\mathbf{b}\\&+\frac{1}{(2\pi)^m}\lim_{\mathbf{b}\to\infty}\int_{-\mathbf{b}}^{\mathbf{b}}\int_0^{\tau}\tilde{f}(\mathbf{c}+i\mathbf{b},y)\Phi(\mathbf{c}i-\mathbf{b},\tau-y)e^{-r(\tau-y)}\mathbf{S}^{-(\mathbf{c}+i\mathbf{b})}dyd\mathbf{b}\\&=\frac{1}{(2\pi)^m}\lim_{\mathbf{b}\to\infty}\int_{-\mathbf{b}}^{\mathbf{b}} e^{-r\tau}\tilde{\phi}(\mathbf{c}+i\mathbf{b})\Phi(\mathbf{c}i-\mathbf{b},\tau)e^{-(\mathbf{c}+i\mathbf{b})'\ln(\mathbf{S})}d\mathbf{b}\\&+\frac{1}{(2\pi)^m}\lim_{\mathbf{b}\to\infty}\int_{-\mathbf{b}}^{\mathbf{b}}\int_0^{\tau}\tilde{f}(\mathbf{c}+i\mathbf{b},y)\Phi(\mathbf{c}i-\mathbf{b},\tau-y)e^{-r(\tau-y)}e^{-(\mathbf{c}+i\mathbf{b})'\ln(\mathbf{S})}dyd\mathbf{b}\end{aligned} \tag{4.337}$$

Discretizing the integrals over $\mathbf{b}$ and y and by means of Trapezoidal rule, (4.337) becomes

$$A_p(\mathbf{S},\tau) \simeq \frac{\Delta_{\mathbf{b}}e^{-r\tau}}{(2\pi)^m}\sum_{j_1,\dots,j_m=0}^{N-1}\tilde{\phi}(\mathbf{c}+i\mathbf{b}_j)\Phi(\mathbf{c}i-\mathbf{b}_j,\tau)I$$

$$+\frac{\Delta_{\mathbf{b}}\Delta_{\tau}}{(2\pi)^m}\sum_{j_1,\dots,j_m=0}^{N-1}\sum_{l=0}^{M-1}\tilde{f}(\mathbf{c}+i\mathbf{b}_j)\Phi(\mathbf{c}i-\mathbf{b}_j,\tau-t_l)e^{-r(\tau-t_l)}I \tag{4.338}$$

where

$$I = e^{-\mathbf{c}'\ln(\mathbf{S})-i\mathbf{b}_j'\ln(\mathbf{S})} \tag{4.339}$$

$t_l = 0,\dots,M-1$ by step-size $h=\frac{L}{M-1}$, $\Delta_\tau = \frac{h}{2}$, $\mathbf{b}_j = (b_{j_1},\dots,b_{j_m})$, $b_{j_i} = \left(j_i - \frac{N}{2}\right)\Delta_i$ for $j_i = 0,\dots,N-1$ and $\Delta_{\mathbf{b}} = \prod_{i=1}^{n}\Delta_i$. Note that, the grid of each sum in j_i is bounded by N. Next, the use of the Fast Fourier Transform (FFT) will be considered as follows. Let the reciprocal lattice for the log initial prices be defined as

$$\ln(\mathbf{S}) = y_k = (y_{k_1},\dots,y_{k_n}) \tag{4.340}$$

where

$$y_{k_i} = \left(k_i - \frac{N}{2}\right)\lambda_i \tag{4.341}$$

Therefore, the multiple sum over the lattice is used for the approximation of the multiple integral.

$$\mathbf{B} = \{\mathbf{b}_j = (b_{j_1}, ..., b_{j_n})|\mathbf{j} = (j_1, ..., j_n) \in \{0, ..., N-1\}^m\} \tag{4.342}$$

For FFT to produce an acceptable error, the lattice spacing Δ_i and the number of points on the lattice must be carefully chosen. The reciprocal lattice $\mathbf{S}$ and the value of the strike price K for computation are log-prices

$$\mathbf{S} = \{\mathbf{y}_j = (y_{j_1}, ..., y_{j_n})|\mathbf{k} = (k_1, ..., k_n) \in \{0, ..., N-1\}^m\} \tag{4.343}$$

By choosing,

$$\Delta_i = \frac{2\pi}{N\lambda_i} \tag{4.344}$$

Equation (4.338) becomes

$$\begin{aligned} A_p(\mathbf{S}, t) \simeq{}& \frac{(-1)^{\sum k}\Delta_b e^{-r\tau}}{(2\pi)^m} \sum_{j_1,\dots,j_m}^{N-1} \varsigma^e e^{-\mathbf{c}'\mathbf{y}_k} e^{-\frac{-2\pi i\mathbf{j}'\mathbf{k}}{N}} \\ &+ \frac{(-1)^{\sum k}\Delta_b e^{-r\tau}}{(2\pi)^m} \sum_{j_1,\dots,j_m}^{N-1} \sum_{l=0}^{M-1} \varsigma^{eep} e^{-r(\tau-t_l)-\mathbf{c}'\mathbf{y}_k} e^{-\frac{-2\pi i\mathbf{j}'\mathbf{k}}{N}} \end{aligned} \tag{4.345}$$

where

$$\varsigma^e(\mathbf{j}) = (-1)^{\sum \mathbf{j}}\tilde{\phi}(\mathbf{c} + i\mathbf{b}_j)\Phi(\mathbf{c}i - \mathbf{b}_j, \tau) \tag{4.346}$$

and

$$\varsigma^{epp}(\mathbf{j}, t_l) = (-1)^{\sum \mathbf{j}}\tilde{f}(\mathbf{c} + i\mathbf{b}_j, \tau - t_l)\Phi(\mathbf{c}i - \mathbf{b}_j, t_l) \tag{4.347}$$

To compute the value of American basket put option, two FFT procedures must be computed with input arrays $\varsigma^{e}(\mathbf{j})$ and $\varsigma^{epp}(\mathbf{j}, t_l)$. Introducing the composite Simpson's rule allows the integrand to be approximated using quadratic polynomials rather than line segments. The price of American basket put option is obtained as

$$\begin{aligned} A_p(\mathbf{S}, t) \simeq & \frac{(-1)^{\sum k}\Delta_b}{(2\pi)^m}\mathcal{FFT}(\gamma\varsigma^{e})e^{-r\tau-\mathbf{c}' y_k} \\ & + \frac{(-1)^{\sum k}\Delta_b\Delta_\tau}{(2\pi)^m}\mathcal{FFT}\left(\sum_{l=0}^{M-1}\gamma\varsigma^{eep}e^{-r(\tau-t_l)}\right)e^{-\mathbf{c}'\mathbf{y}_k} \end{aligned} \tag{4.348}$$

where

$$\gamma = \frac{(3 + (-1)^{1+\sum \mathbf{j}} - \delta_{\sum \mathbf{j}})}{3} \tag{4.349}$$

with

$$\delta_{\sum \mathbf{j}} = \begin{cases} 1, & \text{if } \sum \mathbf{j} = 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence, (4.334) is established.

Remark 4.8.3

(i) Equation (4.348) is the valuation formula for the price of American basket put option.

(iii) Equation (4.334) computes an $N \times N$ matrix of option prices at varying initial asset prices.

(iv) The number of matrix corresponds to the number of underlying assets of the option.

(v) By means of the multidimensional Mellin transform method, the price of the European basket put option on a basket of multi-dividend yields can be approximated as

$$E_p(\mathbf{S},\tau) \simeq \frac{(-1)^{\sum k}\Delta_b}{(2\pi)^m}\mathcal{FFT}(\gamma\varsigma^e)e^{-r\tau-\mathbf{c}'y_k} \tag{4.350}$$

4.8.1 Greeks

In financial mathematics, option sensitivities also known as Greeks describe the relationship between the value of an option and changes in one of its underlying parameters. They are easily obtained for plain vanilla put option with dividend paying stocks. Setting $\tau = T - t$, the integral representation for the price of the European basket put option with multi-dividend paying stocks in (4.299) can be written as

$$E_p(\mathbf{S},\tau) = e^{-r\tau}\mathcal{M}^{-1}(\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)) \tag{4.351}$$

By inducing appropriate derivative operator on the complex integral in (4.351) and using the procedures of Manuge and Kim (2014) for the case of American basket put, the following Greeks for the European basket put option was obtained as follows:

(i) Delta, the rate of change between the option's price and the underlying asset price is given by

$$\Delta_1 = \frac{\partial E_p(\mathbf{S},\tau)}{\partial S_i} = -e^{-r\tau}\mathcal{M}^{-1}\left(\frac{\omega_i}{S_i}\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)\right)$$

Similarly, the cross partial derivative with respect to two independent assets is given by

$$\Delta_2 = \frac{\partial^2 E_p(\mathbf{S},\tau)}{\partial S_i \partial S_j} = -e^{-r\tau}\mathcal{M}^{-1}\left(\frac{\omega_i}{S_i}\frac{\omega_i}{S_j}\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)\right)$$

(ii) Gamma, the second derivative of the value function with respect to the underlying asset price is given by

$$\Gamma = \frac{\partial^2 E_p(\mathbf{S},\tau)}{\partial S_i^2} = e^{-r\tau}\mathcal{M}^{-1}(\omega_i(1-\omega_i)\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)S_i^{-2})$$

(iii) Theta, the rate of change between an option portfolio and time, or time sensitivity is given by

$$\Theta = -\frac{\partial E_p(\mathbf{S},\tau)}{\partial \tau} = e^{-r\tau}\mathcal{M}^{-1}(\tilde{\phi}(\omega^*)(\Psi(\omega^* i)+r)\Phi(\omega^* i,\tau))$$

(iv) Rho, the derivative of the option value with respect to the risk-free interest rate is given by

$$\rho = \frac{\partial E_p(\mathbf{S},\tau)}{\partial r} = -\tau e^{-r\tau}\mathcal{M}^{-1}\left(\sum_{j=1}^{m}(\omega_j-1)\tau\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)\right)$$

(v) Vega, the first derivative with respect to volatility is given by

$$\nu = \frac{\partial E_p(\mathbf{S},\tau)}{\partial \sigma_i} = \tau e^{-r\tau}\mathcal{M}^{-1}\left(\left(\frac{1}{2}\sum_{i,j=1}^{m}\rho_{i,j}\sigma_j\omega_i\omega_j\right)\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)\right)$$
$$+\tau e^{-r\tau}\mathcal{M}^{-1}\left(\left(\sum_{i=1}^{m}\sigma_i\omega_i(\omega_i-1)\right)\tilde{\phi}(\omega^*)\Phi(\omega^* i,\tau)\right), i \neq j$$

4.9 Other Related Methods for Options Valuation

4.9.1 Double Transform Method for the Valuation of Asian Option

A simple expression for the double transform by means of Fourier and Laplace transforms, (with respect to the logarithm of the strike and time to maturity, respectively) of the price of continuously monitored Asian options was obtained. The double transform is expressed in terms of Gamma functions only. The computation of the price requires a multivariate numerical inversion. The following result showed how double transform can be used for the valuation of Asian option.

Theorem 4.9.1

The double transform for the price of Asian option $c(k,h;a_f)$ for $\lambda > 2\gamma(\gamma+v)$ is obtained as

$$\mathcal{L}(\mathcal{F}(c(k,h;a_f);k \to \gamma);h \to \lambda)) = C(\gamma + ia_f, \lambda) \tag{4.352}$$

where

$$C(\gamma + ia_f, \lambda) = \frac{4\Gamma(i(\gamma + ia_f))\Gamma(\frac{\mu+\nu}{2} + 1)\Gamma(\frac{\mu-\nu}{2} - 1 - i(\gamma + ia_f))}{\sigma^2 \lambda 2^{(1+i(\gamma+ia_f))}\Gamma(\frac{\mu+\nu}{2} + 2 + i(\gamma + ia_f))\Gamma(\frac{\mu-\nu}{2})}$$

where $\Gamma(.)$ is the gamma function of complex argument and $\mu^2 = 2\lambda + \nu^2$.

Proof: To price Asian option, compute a double transform with respect to time to expiry and logarithm of the strike. Begin with the assumption that the risk-neutral process for the underlying asset is given by a stochastic differential equation.

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{4.353}$$

where W_t is a Brownian motion or wiener process, r is the risk-free interest rate, t is the time and σ is the volatility. Under this condition, in order to price continuously monitored Asian option, the probability density function of the random variable S will be needed, that is

$$A_t = \int_0^t \exp\left(\left(r - \frac{\sigma^2}{2}\right)s + \sigma W_s\right) ds \tag{4.354}$$

The payoff of a fixed strike Asian option is given by

$$P_A = \max\left(\frac{S_0 A_t}{t} - K, 0\right) \tag{4.355}$$

The case of floating strike Asian options is characterized by a payoff $\max\left(\frac{S_0 A_t}{t} - S_t, 0\right)$. The presence of a continuous dividend yield q can be taken into account in order to replace r by $r - q$ and the spot price by $S_0 e^{-qt}$. If the risk-free interest rate or volatility is not constant, then the pricing of the Asian option becomes more difficult. The price of the Asian option can be obtained by computing the discounted expected value:

$$e^{-rt} E_0 \max\left(\frac{S_0 A_t}{t} - K, 0\right) = e^{-rt} \frac{S_0}{t} E_0 \max(A_t - J, 0) \tag{4.356}$$

where E_0 is the expected value under the risk-neutral probability measure and $J = (\frac{K}{S_0})t$. In order to compute this expectation, let A_t be expressed as

$$A_t = \frac{4}{\sigma^2} D_h^{(v)} \tag{4.357}$$

where

$$D_h^{(v)} = \int_0^h e^{2(W_s+vs)} ds \tag{4.358}$$

$h = \frac{\sigma^2 t}{4}$ and $v = \frac{2r}{\sigma^2 - 1}$. Thus

$$\begin{aligned} E_0(A_t - J)^+ &= E_0 \max\left(\frac{4}{\sigma^2} D_h^{(v)} - J, 0\right) \\ &= \frac{4}{\sigma^2} E_0 \max\left(D_h^{(v)} - J_0, 0\right) \\ &= \frac{4}{\sigma^2} \int_{J_0}^{\infty} (x - J_0) f_D(x,h) dx \end{aligned} \tag{4.359}$$

where f_D is the density function of the random variable $D_h^{(v)}$ and $J = \frac{4J_0}{\sigma^2}$. After a final change of variable, $w = \ln x$, define a function of the form:

$$c(k,h) = \frac{4}{\sigma^2} \int_k^{\infty} (e^w - e^k) f_{\ln D}(w,h) dw \tag{4.360}$$

where $k = \ln J_0$. Using the fact that the density law of the logarithm of a random variable is related to the density of the same random variable by the relation:

$$f_{\ln D} = f_D(e^{\omega}, h)e^{\omega}, -\infty < \omega < \infty \tag{4.361}$$

Compute the analytical expression of the double transform $c(k,h)$ for Laplace and Fourier with respect to h and k respectively. Following Fu et al. (1999), multiplying (4.360) by an exponentially decaying function $e^{-a_f k}$, $c(k,h)$ becomes square integrable in k over the negative axis. Therefore, replacing the function $c(k,h)$ with $c(k,h;a_f)$, where $c(k,h;a_f) \equiv c(k,h)e^{-a_f k}$, $a_f > 0$. Therefore,

$$\mathcal{L}(\mathcal{F}(c(k,h;a_f); k \to \gamma); h \to \lambda)) = \int_0^{\infty} e^{-\lambda h} \int_{-\infty}^{\infty} e^{i\gamma k} c(k,h;a_f) dk dh \tag{4.362}$$

Solving (4.362) further, the double transform of $c(k, h; a_f)$ is obtained as

$$\mathcal{L}(\mathcal{F}(c(k, h; a_f); k \to \gamma); h \to \lambda) = C(\gamma + ia_f, \lambda)$$

This completes the proof.

The double numerical inversion for the price of Asian option was given by the following result.

Theorem 4.9.2

The double numerical inversion for the price of Asian option is given by

$$c(k, h) \approx \frac{e^{0.5(g_f+g_p)}}{4kh} \sum_{m=-\infty}^{\infty} (-1)^m \left(\sum_{s=-\infty}^{\infty} (-1)^s C\left(\frac{s\pi}{k} + \frac{ig_f}{2k}, a + \frac{is\pi}{h} \right) \right) \tag{4.363}$$

Proof: To obtain the function $c(k, h)$ by the double numerical inversion, begin with the price of the Asian option given by

$$e^{-rt} E_0 \max\left(\frac{S_0 A_t}{t} - K, 0 \right) = e^{-rt} \frac{S_0}{t} e^{a_f k} c(k, h; a_f) \bigg|_{k=\ln(\frac{K\sigma^2 t}{4}), h=\frac{\sigma^2 t}{4}} \tag{4.364}$$

The numerical Inversion of the double transform in (4.352) can be performed as follows:

Given the transform $C(\gamma, \lambda)$, the Fourier inverse can be computed with respect to γ numerically. Then invert the Laplace transform with respect to λ by using the numerical univariate inversion formula. Let $\mathcal{L}^{-1}(.)$ and $\mathcal{F}^{-1}(.)$ denote respectively the Laplace and Fourier inverses, then the price of Asian option denoted by $c(k, h)$ gives;

$$c(k, h) = e^{a_f k} \mathcal{L}^{-1}(\mathcal{F}^{-1}(C(\gamma + ia_f, \lambda); \gamma \to k); \lambda \to h) \tag{4.365}$$

Also $c(k,h)$ can be defined as

$$c(k,h) := e^{a_f k}\mathcal{L}^{-1}(\mathcal{F}^{-1}(C(\gamma + ia_f, \lambda))) \tag{4.366}$$

Using the the definition of the univariate Fourier inversion formula, (4.366) leads to a relation

$$c(k,h) = e^{a_f k}\mathcal{L}^{-1}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma k}C(\gamma + ia_f, \lambda)d\gamma\right) \tag{4.367}$$

Discretizing the inversion integral by a step size Δ_f, to get

$$c(k,h) = e^{a_f k}\mathcal{L}^{-1}\left(\frac{\Delta_f}{2\pi}\sum_{s=-\infty}^{\infty} e^{-i\Delta_f sk}C(\Delta_f s + ia_f, \lambda)\right) \tag{4.368}$$

Setting $\Delta_f = \frac{\pi}{k}$ and $a_f = \frac{g_f}{2k}$, then

$$c(k,h) = e^{0.5g_f}\mathcal{L}^{-1}\left(\frac{1}{2k}\sum_{s=-\infty}^{\infty}(-1)^s C\left(\frac{s\pi}{k} + \frac{ig_f}{2k}, \lambda\right)\right) \tag{4.369}$$

Taking the Laplace inversion of (4.369) yields

$$c(k,h) = \frac{e^{0.5g_f}}{2\pi i}\int_{a_p - i\infty}^{a_p + i\infty}\left(\frac{1}{2k}\sum_{s=-\infty}^{\infty}(-1)^s C\left(\frac{s\pi}{k} + \frac{ig_f}{2k}, \lambda\right)\right)e^{\lambda h}d\lambda \tag{4.370}$$

Setting

$$\lambda = a_p + iw \Rightarrow d\lambda = idw \tag{4.371}$$

where a_p is at the right of the largest singularity of the function $C(\gamma, \lambda)$. By means of (4.371), (4.370) becomes

$$c(k,h) = \frac{e^{0.5g_f + a_p h}}{4\pi k}\int_{-\infty}^{\infty} e^{iw}\left(\sum_{s=-\infty}^{\infty}(-1)^s C\left(\frac{s\pi}{k} + \frac{ig_f}{2k}, a_p + iw\right)\right)dw \tag{4.372}$$

Equation (4.372) can be approximated again using the trapezoidal rule with step size $\Delta_p = \frac{\pi}{h}$ and by setting $a_p = \frac{g_p}{2h}$, (4.363) is established.

Remark 4.9.1

(i) The parameters a_f and a_p control the discretization error and must be carefully chosen.

(ii) The numerical inversion of the double transform of (4.352) can be performed by resorting to the multivariate version of the Fourier Euler algorithm since it gives a much faster convergence for infinite sums (Abate and Whitt (1992), Choudhury et al. (1994)). Specifically, the Euler sum provides an estimate $E(m,n)$ of the series

$$\sum_{s=1}^{\infty}(-1)^s a_s$$

with

$$E(m,n) = \sum_{j=0}^{n-1} \binom{j}{n} 2^{-n} S_{m+j} \tag{4.373}$$

and

$$S_i = \sum_{j=0}^{n-1}(-1)^j a_j \tag{4.374}$$

The use of the Euler algorithm requires $(m+n)$ evaluation of the complex function a_j. In particular, Fourier and Laplace inversions require $(m_f + n_f)(m_p + n_p)$ evaluations of the double transform. The computational cost of the inversion is directly related to this product. In order to avoid numerical difficulties in the computation of the binomial coefficient in the Euler algorithm, let

$$n_f = m_f + 15 \tag{4.375}$$

$$n_p = m_p + 15 \tag{4.376}$$

where the choice of m_f and m_p has to be tuned according to the volatility level.

(iii) After some algebra, the delta of the Asian option becomes

$$\begin{aligned}\Delta(S_0, K, t, r, \sigma) &= e^{-rt}\frac{\partial}{\partial S_0}(E_0 \max(A_t - J, 0)) \\ &= \frac{e^{-rt}}{t}\left(c(k,h) - \frac{\partial c(k,h)}{\partial k}\right)\Bigg|_{k=\ln(\frac{K\sigma^2 t}{4S_0}), h=\frac{\sigma^2 t}{4}}\end{aligned} \tag{4.377}$$

Also the gamma of the Asian option is obtained as

$$\begin{aligned}\Gamma(S_0, K, t, r, \sigma) &= e^{-rt}\frac{\partial^2}{\partial S_0^2}(E_0 \max(A_t - J, 0)) \\ &= \frac{e^{-rt}}{S_0 t}\left(\frac{\partial c(k,h)}{\partial k} - \frac{\partial^2 c(k,h)}{\partial k^2}\right)\Bigg|_{k=\ln(\frac{K\sigma^2 t}{4S_0}), h=\frac{\sigma^2 t}{4}}\end{aligned} \tag{4.378}$$

4.9.2 Application of the Fourier Transform Method in the Valuation of European Call Option

The Fourier pricing techniques and Fourier inversion methods for density calculations add a versatile tool to the set of advanced techniques for pricing and management of financial derivatives. Stein and Stein (1991) and Heston (1993) started the ball rolling with their use of Fourier transforms in finance to analytically value European options on stocks with stochastic volatility.

The fast Fourier transform method is a numerical approach for pricing options which utilizes the characteristic function of the underlying instruments price process. This approach was introduced by Carr and Madan (1999). The Fast Fourier transform method assumes that the characteristic function of the log-price is given analytically. Consider the valuation of European call option. Let the risk-neutral density of $s = \log S_T$ be $f(s)$, where S_T is the underlying asset price at time to expiry/maturity T. The characteristics function of the density is given by

$$\varphi_T(v) := \int_{-\infty}^{\infty} e^{ivs} f(s) ds \tag{4.379}$$

The price of a European call option under the risk-neutral valuation with maturity T and strike price K denoted by $C_T(p)$ is given by

$$\begin{aligned} C_T(p) &= e^{-rt}\mathbf{E}[(S_T - K)^+] \\ &= e^{-rT}\mathbf{E}[(e^s - K)^+] \\ &= \int_{-\infty}^{\infty} e^{-rT}(e^s - K)^+ f(s) ds \\ &= \int_{-\infty}^{\infty} e^{-rT}(e^s - K) f(s) ds \end{aligned} \tag{4.380}$$

where p is the logarithm of the strike price K. That is

$$p \equiv \log_e K \Rightarrow K \equiv e^p \tag{4.381}$$

Substituting (4.381) into (4.380) yields

$$C_T(p) = \int_{p}^{\infty} e^{-rT}(e^s - e^p) f(s) ds \tag{4.382}$$

in which the expectation is taken with respect to some risk-neutral measure. Since,

$$\lim_{K\to\infty} C_T(K) = \lim_{K\to\infty} C_T(e^p) = S_0 \tag{4.383}$$

The integral representation given by (4.382) is not square integrable. Therefore, $C_T(e^p) \notin L^1$ as $C_T(e^p)$ does not tend to zero for $p \to -\infty$. Consider a modified version of the call price in (4.382) given by

$$c_T(p) \equiv e^{ap} C_T(p), a > 0 \tag{4.384}$$

Equation (4.384) is square integrable in p over the entire real line. Using (3.54) and (3.55), then

$$\mathcal{F}(c_T(v)) = \tilde{c}_T(v) = \int_{-\infty}^{\infty} e^{ivp} c_T(p) dp \tag{4.385}$$

and

$$c_T(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ivp} \tilde{c}_T(p) dp \tag{4.386}$$

Substituting (4.384) into (4.383) leads to a new call value in the Fourier transform domain as

$$\tilde{c}_T(v) = \int_{-\infty}^{\infty} e^{ivp} e^{ap} C_T(p) dp \tag{4.387}$$

Using (4.382) and (4.387) leads to a relation

$$\begin{aligned}\tilde{c}_T(v) &= \int_{-\infty}^{\infty} e^{ivp} e^{ap} \int_{p}^{\infty} e^{-rT}(e^s - e^p) f(s) ds dp \\ &= \int_{-\infty}^{\infty} e^{-rT} f(s) \int_{p}^{\infty} e^{ivp} e^{ap}(e^s - e^p) ds dp\end{aligned} \tag{4.388}$$

Solving (4.388) further yields

$$\begin{aligned}\tilde{c}_T(v) &= \int_{-\infty}^{\infty} e^{-rT} f(s) \int_p^{\infty} e^{ivp}(e^{s+ap} - e^{p+ap}) ds dp \\ &= \int_{-\infty}^{\infty} e^{-rT} f(s) \left(\frac{e^{(a+1+iv)s}}{a+iv} - \frac{e^{(a+1+iv)s}}{a+1+iv} \right) ds\end{aligned}$$

Since for $a > 0$,

$$\lim_{p\to-\infty} |e^{(iv+a)p}| = \lim_{p\to-\infty} |e^{(iv+1+a)p}| = \lim_{p\to-\infty} |e^{(1+a)p}| = 0$$

Therefore,

$$\tilde{c}_T(v) = \frac{e^{-rT}\varphi_T(v-(a+1)i)}{a^2+a-v^2+i(2a+1)v} \tag{4.389}$$

where φ_T is the characteristic function of the log S_T given by (4.379). Now, the desired option price in terms of $\tilde{c}_T(v)$ can be obtained using the Fourier inversion of the form:

$$\begin{aligned}C_T(p) &= \frac{e^{-ap}}{2\pi} \int_{-\infty}^{\infty} \Re(e^{-ivp}\tilde{c}_T(v))dv \\ &= \frac{e^{-ap}}{\pi} \int_0^{\infty} \Re(e^{-ivp}\tilde{c}_T(v))dv\end{aligned} \tag{4.390}$$

Substituting (4.389) into (4.390) yields

$$C_T(p) = \frac{e^{-ap}}{\pi} \int_0^{\infty} \Re\left(e^{-ivp} \frac{e^{-rT}\varphi_T(v-(a+1)i)}{a^2+a-v^2+i(2a+1)v} \right) dv \tag{4.391}$$

By recognizing that the call price is real (even in real part, odd in imaginary). Due to the condition a, (4.391) is well defined. After discretizing and using the Simpson's $\frac{1}{3}$ rule, (4.391) can be computed numerically by means of the fast Fourier transform as

$$C_T(p_u) \simeq \frac{e^{-ap_u}}{\pi} \sum_{j=1}^{N} e^{\left(\frac{-2\pi i(j-i)(u-1)}{N} + ibv_j\right)} \tilde{c}_T(v_j) \frac{\eta}{3}[3 + (-1)^j - \delta_{j-1}] \tag{4.392}$$

with $v_j = \eta(j-1), p_u = -b + \lambda(u-1), b = \frac{N\lambda}{2}, \lambda = \frac{2\pi}{\eta N}$ and δ_{j-1} is the Kronecker delta function defined as

$$\delta_{j-1} = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

where parameters η and N determine the fineness and size of the grid, thus defining the upper limit of integration.

Remark 4.9.2

(i) A sufficient condition for $c_T(p)$ to be square integrable is given by $\tilde{c}_T(0)$ being finite.This is equivalent to $\mathbf{E}^Q(S_T^{a+1}) < \infty$. Carr and Madan (1999) established that if the integrability parameter $a = 0$, the denominator of (4.389) vanishes when $p = 0$, including a singularity in the integrand. Since the fast Fourier transform evaluates the integrand at $p = 0$, the use of the factor e^{ap} is required.

(ii) The prices of vanilla puts can be obtained by means of put-call parity (3.77). However, one can easily obtain the price $P_T(K)$ of a vanilla put by Carr-Madan inversion by choosing negative value for a.

A sufficient condition for the call value $c_T(p)$ in the Fourier domain to be square integrable was presented in the following result.

Lemma 4.9.1

Let $a > 0$. The Fourier transform of $c_T(p)$ exists if $\mathbf{E}S_T^{a+1} < \infty$.

Proof: First note that $\mathbf{E}S_T^{a+1} < \infty \Rightarrow c_T(0) < \infty$, since

$$\begin{aligned}\tilde{c}_T(0) &= \frac{e^{-rT|\varphi_T(-(a+1)i)|}}{a^2+a} \\ &= \frac{e^{-rT}\mathbf{E}S_T^{a+1}}{a^2+a}\end{aligned} \tag{4.393}$$

where (4.393) follows from

$$\begin{aligned}\mathbf{E}S_T^{a+1} &= |\varphi_T(-(a+1)i)| \\ &= \left|\mathbf{E}e^{(-(a+1)i)i\log S_T}\right| \\ &= \left|\mathbf{E}e^{(a+1)\log S_T}\right|\end{aligned} \tag{4.394}$$

Also it follows from (4.385) that

$$\tilde{c}_T(0) = \int_{-\infty}^{\infty} c_T(p)dp \tag{4.395}$$

Combining this with $\tilde{c}_T(0) < \infty$ completes the proof.

Remark 4.9.3

(i) The dynamics of the stock price S_t in a risk-free Black-Scholes world follows geometric Brownian motion with a non-dividend yield is of the form

$$dS_t = rS_t dt + \sigma S_t dW_t, 0 < S_t < \infty$$

Utilizing the Itô's formula, S_T can be solved explicitly as:

$$S_T = e^{((r-0.5\sigma^2)T+\log S_0+\sigma W_T)}$$

from which S_T is lognormally distributed. Hence for the characteristic function $\varphi_T(u)$ of $\log S_T$ leads to a relation

$$\varphi_T(u) = e^{i((r-0.5\sigma^2)T+\log S_0)u-0.5\sigma^2 Tu^2}$$

(ii) For the Black-Scholes model, the integrand in (4.391) reduces to

$$BS_{int} = \frac{\exp(-0.5\sigma^2 T v^2 + 0.5a^2\sigma^2 T + as + aTr + 0.5\sigma^2 Ta + s)}{a^4 + 2a^3 + 2a^2v^2 + a^2 + 2av^2 + v^4 + v^2} g(a, p, r, s, \sigma, T, v) \tag{4.396}$$

where

$$\begin{aligned} g(a, p, r, s, \sigma, T, v) &= (a^2 + a - v^2)\cos((p - (\sigma^2 aT + s + rT + 0.5\sigma^2 T))v) \\ &\quad - v(2a + 1)\sin((p - (\sigma^2 aT + s + rT + 0.5\sigma^2 T))v) \end{aligned} \tag{4.397}$$

From (4.397), more fluctuating integrand can be obtained by increasing any of the parameters σ, T, a, s and r. The magnitudes of these fluctuations get larger which can be seen from the exponential term in (4.396). Pictures can be of help in understanding these observations. The most striking observations are visualized next. Unless stated otherwise, the following plots are generated based on the parameters $S = 100, K = 100, T = 1, \sigma = 0.4, r = 0.05, a = 3.5$. In fact, for practical ranges of the above parameters only (the interplay of T, $\frac{S}{K}$ and a) have noticeable influences on the integrand. The influence of T on the Black-Scholes integrand is shown in Figure 4.1. As anticipated, more fluctuations and larger functional values are obtained.

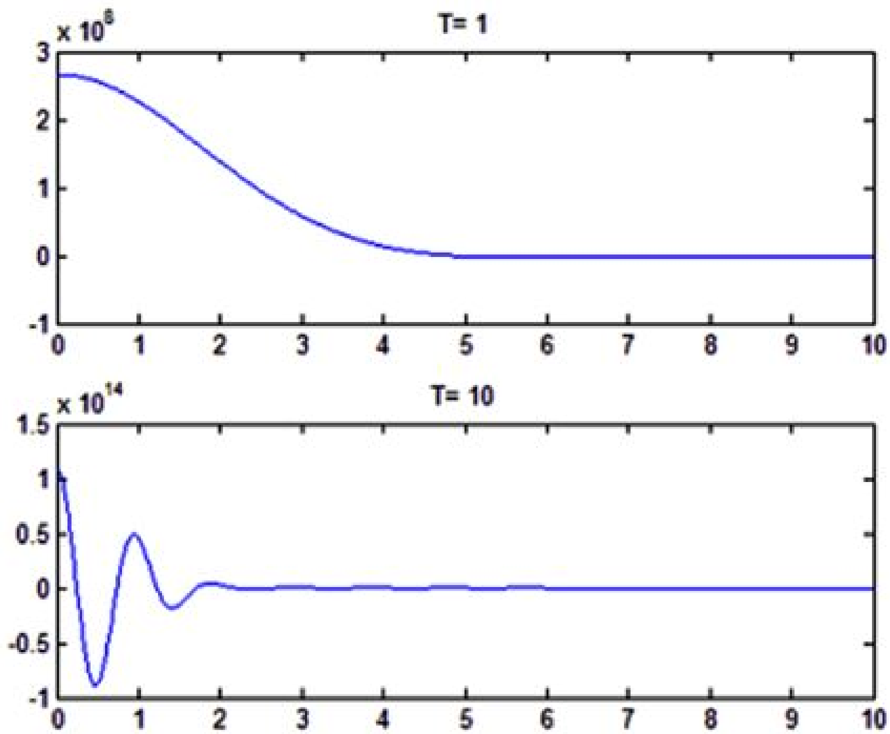

Figure 4.1: The influence of T on the Black-Scholes integrand. Lower: T=10, Upper: T=1.

(iii) The strike p appears solely in the sine and cosine terms in (4.397). Since $K \to 0 \Leftrightarrow p \to -\infty$. It is observed that both the cosine and sine terms will fluctuate rapidly as $K \to 0$. This will cause the integrand to be extremely oscillatory, while the absolute values do not grow in magnitude. Nonetheless, this is sufficient to pose a huge problem from a quadrature point of view. The same is true when $K \to \infty$. This latter case is of less practical interest however. In fact, it is the so-called moneyness $\frac{S}{K}$ that determines the oscillatory nature of the integrand. The influence of K on the Black-Scholes integrand is shown in Figure 4.2.

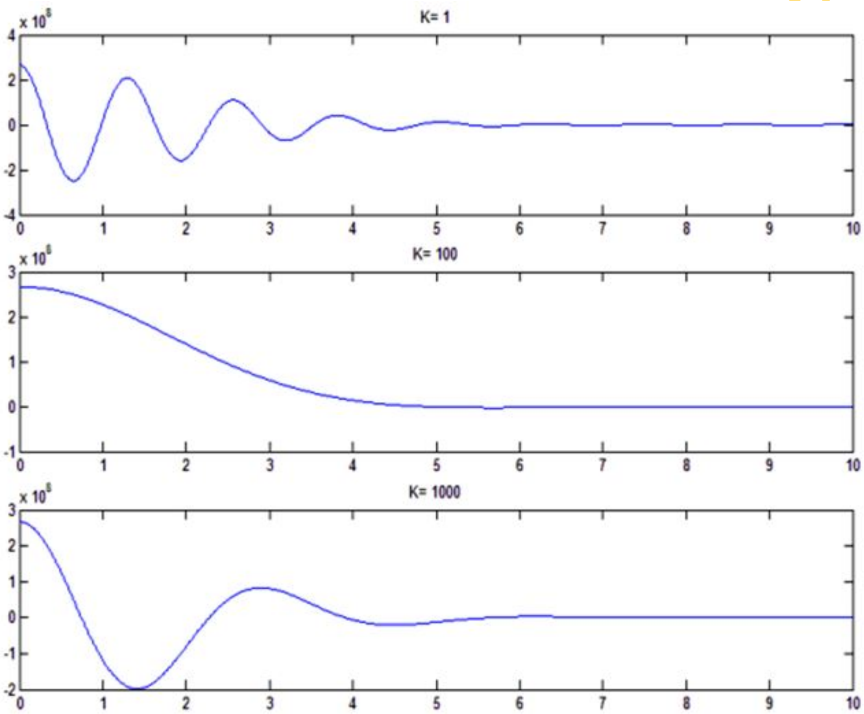

Figure 4.2: The influence of K on the Black-Scholes integrand. Lower: K=1000, Middle: $K = 100$, Upper: K=1.

(iv) At this point it is unavoidable to comment on the choice of the integrability parameter a. A small value of a is favourable since this reduces both the oscillations and the magnitudes hereof. However choosing a too small can turn the integrand into a sort of impulse function, which is not tractable at all from a numerical integration point of view. This follows from the fact that in the origin $v = 0$, the Black-Scholes integrand in (4.396) becomes

$$BS_{int} = \frac{\exp(0.5a^2\sigma^2 T + as + aTr + 0.5\sigma^2 Ta + s)}{a(a+1)} \tag{4.398}$$

Taking the limit of (4.398) as $a \to 0$ yields

$$\lim_{a\to 0}(BS_{int}) = \infty \tag{4.399}$$

Similarly, (4.398) tends to ∞ as $a \to \infty$

$$\lim_{a\to \infty}(BS_{int}) = \infty \tag{4.400}$$

On the other hand, for $v > 0$ and by letting $a \to 0$, the integrand (4.396) becomes

$$\lim_{a\to 0}(BS_{int}) = \frac{\exp(-0.5\sigma^2 T v^2 + s)(-v^2\cos((p-m_0)v) - v\sin((p-m_0))v)}{v^4 + v^2} \tag{4.401}$$

with

$$m_0 = \sigma^2 aT + s + Tr + 0.5\sigma^2 T$$

Equation (4.401) decreases very fast as a function of v because of the exponential term (ManWo Ng (2005)). The Black-Scholes integrand resembles more of the impulse function as shown in Figure 4.3 below.

For the integrand depicted, consider $S = 100, K = 100, T = 1,$

$\sigma = 0.4, r = 0.05.$

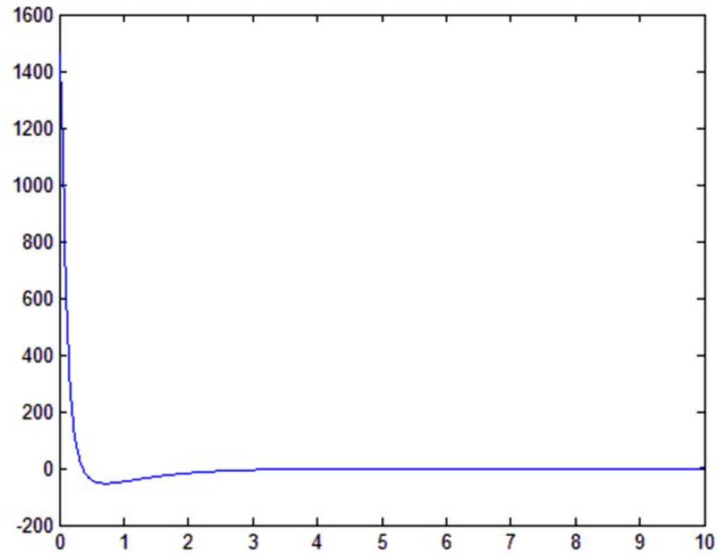

Figure 4.3: The Black-Scholes integrand resembles more of the impulse function as $a \to 0$.

(v) In order to determine a good value for a; it is proposed to (numerically) minimize the maximum of the integrand, that is to solve the following optimization problem:

$$\min_{a>0}\left(\frac{\exp(0.5a^2\sigma^2 + as + aTr + \sigma^2 Tr + s)}{a(a+1)}\right)$$

which intuitively would yield a nice integrand in the sense that both variations in function values as well as oscillations are reduced. Note that in this strategy the dependence of a on k have been discarded. The flavour of the function to be minimized is shown in the Figure 4.4, where $r = 0.05, T = 1, \sigma = 0.15, S = 100$. One possible way to solve the optimization problem is "setting the derivative to zero" (ManWo Ng (2005)).

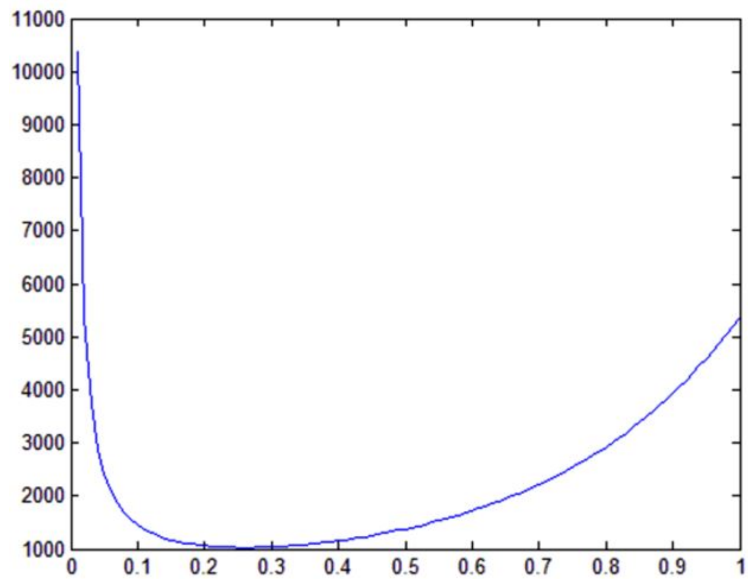

Figure 4.4: A typical function one has to face when the maximum of the Black-Scholes integrand is to be minimized.

The following result showed how the Black-Scholes integrand attained its maximum at $v = 0$.

Lemma 4.9.2

Let $v \geq 0$. The Black-Scholes integrand given by

$$BS_{int} = \Re\left(e^{-ivp}\frac{e^{-rT}\varphi_T(v-(a+1)i)}{a^2+a-v^2+i(2a+1)v}\right) \tag{4.402}$$

attains its maximum at $v = 0$, where $\varphi_T(v) = e^{i((r-0.5\sigma^2)T+\log S_0)v-0.5\sigma^2Tv^2}$

Proof: From (4.398), it is clearly seen that the statement is equivalent with

$$\Re(e^{-ivp}\tilde{c}_T(v)) \leq \frac{\exp(0.5a^2\sigma^2T+as+aTr+0.5\sigma^2Ta+s)}{a(a+1)}, \quad \text{for } v \geq 0 \tag{4.403}$$

This follows since

$$|\Re(e^{-ivp}\tilde{c}_T(v))| \leq |e^{-ivp}\tilde{c}_T(v)| = \tilde{c}_T(v) \tag{4.404}$$

where

$$|\tilde{c}_T(v)| = \left|\frac{e^{-rT}\varphi_T(v-(a+1)i)}{a^2+a-v^2+i(2a+1)v}\right| \tag{4.405}$$

Thus,

$$\begin{aligned} |\varphi_T(v-(a+1)i)| &= |e^{i(s+(r-0.5\sigma^2)T)(v-(a+1)i)-0.5\sigma^2T(v-(a+1)i)^2}| \\ &= e^{(0.5a^2\sigma^2T+as+aTr+0.5\sigma^2Ta+s+rT-0.5\sigma^2Tv)} \end{aligned} \tag{4.406}$$

Substituting (4.406) into (4.405) yields

$$\begin{aligned} |\tilde{c}_T(v)| &= \frac{\exp(0.5a^2\sigma^2T+as+aTr+0.5\sigma^2Ta+s-0.5\sigma^2Tv)}{|a^2+a-v^2+i(2a+1)v|} \\ &\leq \frac{\exp(0.5a^2\sigma^2T+as+aTr+0.5\sigma^2Ta+s-0.5\sigma^2Tv)}{|(v-(a+1)i)||(v-ai)|} \\ &\leq \frac{\exp(0.5a^2\sigma^2T+as+aTr+0.5\sigma^2Ta+s)}{a(a+1)} \end{aligned}$$

This completes the proof.

4.9.3 Binomial Model for the Valuation of European Call Option

Binomial model is an iterative solution that models the price evolution over the whole option validity period. The binomial option-pricing model is based on the assumption of no arbitrage. The assumption of no arbitrage implies that all risk-free investments earn the risk-free rate of return. For some types of options such as the American options, using an iterative model is the only choice since there is no known closed form solution that predicts price over time. Black-Scholes model seems dominated the option pricing, but it is not the only popular model, the Cox-Ross-Rubinstein (CRR) "Binomial" model has a large popularity. The binomial model was first suggested by Cox et al. (1979) in paper "Option Pricing: A Simplified Approach" and assumed that stock price movements are composed of a large number of small binomial movements. The stock and option prices in a general one-step and general two-step trees for binomial model are shown in Figures 4.5 and 4.6 below.

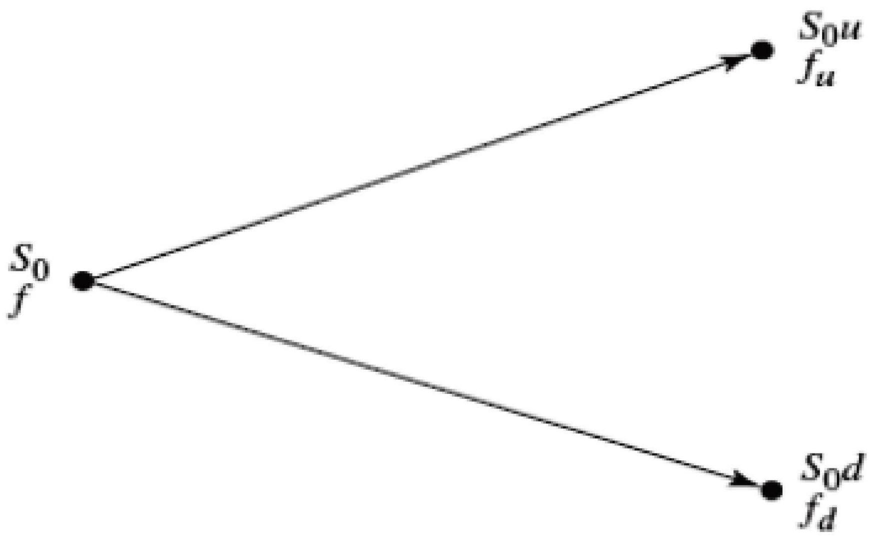

Figure 4.5: Stock and option prices in a general one-step tree.

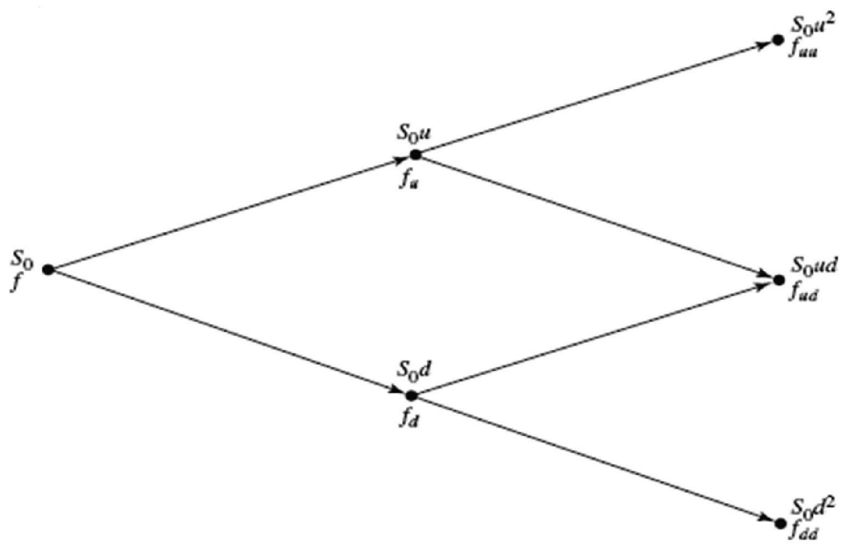

Figure 4.6: Stock and option prices in a general two-step tree.

The following result showed the CRR model for the valuation of European call option.

Theorem 4.9.3

The probability of at least m success in N independent trials, each resulting in a success with probability p and in a failure with probability q is given by

$$\Phi(m; N, p) = \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j} \tag{4.407}$$

Let $\hat{p} = R^{-1}pu$ and $\hat{q} = R^{-1}(1-p)d$, then the CRR model for the valuation of European call option is obtained as

$$f = S_0\Phi(m; N, \hat{p}) - Ke^{-rT}\Phi(m; N, p) \tag{4.408}$$

Proof: After one time period, the stock price can move up to S_0u with probability p or down to S_0d with probability $(1-p)$ as shown in the Figure 4.5. Therefore the corresponding value of the European call option at the first time movement δt is given by

$$f_u = \max(S_0u - K, 0) \tag{4.409}$$

$$f_d = \max(S_0d - K, 0) \tag{4.410}$$

where f_u and f_d are the values of the call option after upward and downward movements respectively. The risk neutral call option price at the present time is

$$f = e^{-r\delta t}[pf_u + (1-p)f_d] \tag{4.411}$$

where the risk neutral probability is given by

$$p = \frac{e^{r\delta t} - d}{u - d} \tag{4.412}$$

with

$$u = e^{\sigma\sqrt{\delta t}} \tag{4.413}$$

$$d = e^{-\sigma\sqrt{\delta t}} \tag{4.414}$$

Now, extend the binomial model to two periods. Let f_{uu} denote the call value at time $2\delta t$ for two consecutive upward stock movements, f_{ud} for one downward and one upward movement and f_{dd} for two consecutive downward movements of the stock price as shown in the Figure 4.6. Then,

$$f_{uu} = \max(S_0 uu - K, 0) \tag{4.415}$$

$$f_{ud} = \max(S_0 ud - K, 0) \tag{4.416}$$

$$f_{dd} = \max(S_0 dd - K, 0) \tag{4.417}$$

The values of the European call options at time δt are

$$f_u = e^{-r\delta t}[pf_{uu} + (1-p)f_{ud}] \tag{4.418}$$

$$f_d = e^{-r\delta t}[pf_{ud} + (1-p)f_{dd}] \tag{4.419}$$

Substituting (4.418) and (4.419) into (4.411) leads to

$$f = e^{-2r\delta t}\left[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right] \tag{4.420}$$

Equation (4.420) is called the current European call value using time $2\delta t$, where the numbers p^2, $2p(1-p)$ and $(1-p)^2$ are the risk neutral probabilities that the underlying asset prices S_0uu, S_0ud and S_0dd respectively attained. The result in (4.420) can be generalized to value an option at $T = N\delta t$ as

$$f = e^{-Nr\delta t} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} f_{u^j d^{N-j}} \tag{4.421}$$

where

$$f_{u^j d^{N-j}} = \max(S_0 u^j d^{N-j} - K, 0) \tag{4.422}$$

and

$$\binom{N}{j} = \frac{N!}{(N-j)!j!} \tag{4.423}$$

is the binomial coefficient. Therefore,

$$f = e^{-Nr\delta t} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} \max(S_0 u^j d^{N-j} - K, 0) \tag{4.424}$$

Assume that m is the smallest integer for which the option's intrinsic value in (4.424) is greater than zero. This implies that $S_0 u^m d^{N-m} \geq K$. Equation (4.424) can be written as

$$\begin{aligned} f = {} & S_0 e^{-Nr\delta t} \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j} u^j d^{N-j} \\ & - K e^{-Nr\delta t} \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j} \end{aligned} \tag{4.425}$$

which gives the present value of the call option. The term $e^{-Nr\delta t}$ is the discounting factor that reduces f to its present value. The first term $\binom{N}{j} p^j (1-p)^{N-j}$ is the binomial probability of jth upward movements to occur after the first N trading periods and $S_0 u^j d^{N-j}$ is the corresponding value of the asset after

jth upward move of the stock price. The second term is the present value of the option's strike price. Setting $R = e^{r\delta t}$ in the first term in (4.425) to get

$$
\begin{aligned}
f &= S_0 R^{-N} \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j} u^j d^{N-j} - K e^{-Nr\delta t} \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j} \\
&= S_0 \sum_{j=m} \binom{N}{j} \left[R^{-1} p u\right]^j \left[R^{-1}(1-p) d\right]^{N-j} \\
&- K e^{-Nr\delta t} \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j}
\end{aligned}
\tag{4.426}
$$

Now, let $\Phi(m; N, p)$ be the binomial distribution function. That is

$$
\Phi(m; N, p) = \sum_{j=m}^{N} \binom{N}{j} p^j (1-p)^{N-j} \tag{4.427}
$$

Equation (4.427) is the probability of at least m success in N independent trials, each resulting in a success with probability p and in a failure with probability $(1-p)$. Then, letting $\hat{p} = R^{-1} p u$, it is clearly seen that $R^{-1}(1-p)d = 1 - \hat{p}$. Consequently it follows from (4.426) that

$$
f = S_0 \Phi(m; N, \hat{p}) - K e^{-rNT} \Phi(m; N, p)
$$

This completes the proof.

Remark 4.9.4

(i) The corresponding value of the European put option can be obtained as

$$
f_p = K e^{-NrT} \Phi(m; N, p) - S_0 \Phi(m; N, \hat{p}) \tag{4.428}
$$

by means of call-put parity (3.77).

(ii) The CRR model contains the Black-Scholes analytical formula as the limiting case as the number of steps tends to infinity.

(iii) For the case of American options, each node must be checked to see whether early exercise is preferable to holding the option for a further time period δt.

4.10 Numerical Experiments

Some numerical experiments under the Mellin transform method, double transform method, Fourier transform method and binomial model are presented below. The sample programs used in generating the tables and figures are based on Matlab codes.

4.10.1 Numerical Experiments under the Mellin Transform Method

Experiment 1

By varying the underlying asset price S_t, consider the performances of the Mellin Transform Method (MTM), Binomial Model (BM) with ($N = 1000$ time steps), Implicit Euler (IE) with (400 steps in both time and the underlying state variable) and Monte Carlo Method (MCM) with (1.0×10^7 Monte Carlo trials) against the Black-Scholes Model (BS) for the valuation of European power put option using the following parameters

$$n = 1, K = \$60, r = 5\%, \sigma = 35\%, T = 5, q = 0, c = 2.$$

The comparative analyzes of the results of the four methods are shown in Table 4.3 below.

Table 4.3: The comparative analyzes of the results of the Black-Scholes Model (BS), Binomial Model (BM), Monte Carlo Method (MCM), Implicit Euler (IE) and the Mellin Transform Method (MTM) for the valuation of European power put option with fixed values of $n = 1, K = \$60, r = 5\%, \sigma = 35\%, T = 5$ and $c = 2$.

S ($)	Black-Scholes Model	Binomial Model	Monte Carlo Method	Implicit Euler	Mellin Transform
10	36.8746	36.8747	36.8739	36.8799	36.8746
20	28.3391	28.3396	28.3425	28.3442	28.3391
30	21.7413	21.7429	21.7363	21.7387	21.7413
40	16.8115	16.8111	16.8076	16.7920	16.8115
50	13.1399	13.1388	13.1438	13.0886	13.1399
60	10.3856	10.3849	10.3912	10.2826	10.3856
70	8.2972	8.2957	8.2937	8.1183	8.2972
80	6.6954	6.6911	6.6941	6.4130	6.6954
90	5.4528	5.4496	5.4542	5.0373	5.4528
100	4.4785	4.4738	4.4817	3.8995	4.4785

Analysis of Experiment 1

From Table 4.3, it is observed that the Mellin transform method, binomial model, Implicit Euler and Monte Carlo method all performed well. The values generated by the Binomial model, Implicit Euler and Monte Carlo method are close to that of Black-Scholes model while the values of the Mellin transform method coincide with that of Black-Scholes model.

Experiment 2

Consider the valuation of European power put options with Forty-Eight months to go until expiration on the "Standard and Poor's 500" index (S&P 500), with the underlying asset price of \$40, strike price of \$100, a continuously compounded risk-free interest rate of 5%, a volatility of 35% and varying constant annual index dividend estimated at $q = \{1\%, 2\%, 3\%, 4\%, 5\%\}$. The price of the European power put options for $n = \{2, 4, 6, 8, 10\}$ using the Mellin transform method is shown in Table 4.4 below.

Table 4.4: Price of European power put option.

n	$q = 0.01$	$q = 0.02$	$q = 0.03$	$q = 0.04$	$q = 0.05$
2	0.93390	1.07390	1.23140	1.40820	1.60600
4	0.00790	0.00980	0.01220	0.01510	0.01870
6	0.00100	0.00130	0.00170	0.00210	0.00270
8	0.00034	0.00044	0.00057	0.00074	0.00096
10	0.00018	0.00023	0.00030	0.00039	0.00050

Analysis of Experiment 2

From Table 4.4, it is observed that the higher the dividend yield, the higher the values of the European power put option.

Experiment 3

Consider the valuation of the American power put option by means of the Mellin Transform Method (MTM) with (a 16-point Gauss-Laguerre quadrature method), 100 time steps and $\epsilon = 0.0001$ for the calculation of the free boundary ($\hat{S}_t$), Accelerated Binomial Model (ABM) with (150 time steps) (Breen (1989)), Binomial Model (BM) with ($N = 150$ time steps) (Cox et al. (1979)), Finite Difference Method (FDM) with (200 steps in both time and the underlying state variable) (Wilmott et al. (1995)) and Recursive Method (RM) with (a four-point extrapolation)(Huang et al. (1996)) varying the volatility $\sigma = \{20\%, 30\%, 40\%\}$, time to expiry $T = \{1, 4, 7\}$ in months, the strike price $K = \{35, 40, 45\}$ in dollars with the following parameters:

$$S_t = \$40, \ q = 0, \ r = 4.88\%, \ n = 1, \ c = 2$$

The comparative analyzes of the results of the five methods are shown in Tables 4.5-4.13. The influences of the volatility and time to expiry on the price of the option by means of the Mellin transform method are shown in Tables 4.14-4.16 and Tables 4.17-4.19 respectively. The results $\hat{S}_t$ for the free boundary of the option were compared with S^* of Balakrishna (1996). Time to expiry is $T =$ 1-month for Tables 4.20-4.22 and $T =$ 7-months for Tables

4.23-4.25.

Table 4.5: Price of American power put option using $T = 0.0833, n = 1$, $r = 4.88\%, q = 0, \sigma = 20\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	0.0061	0.0061	0.0278	0.0062	0.0065
40	0.8517	0.8512	0.9874	0.8543	0.8516
45	4.9200	5.0000	5.0052	5.0020	5.0305

Table 4.6: Price of American power put option using $T = 0.0833, n = 1$, $r = 4.88\%, q = 0, \sigma = 30\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	0.0772	0.0775	0.1216	0.0775	0.0777
40	1.3095	1.3083	1.3860	1.3116	1.3098
45	5.0632	5.0600	5.1016	5.0604	5.0578

Table 4.7: Price of American power put option using $T = 0.0833, n = 1$, $r = 4.88\%, q = 0, \sigma = 40\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	0.2456	0.2454	0.2949	0.2467	0.2468
40	1.7674	1.7658	1.8198	1.7694	1.7681
45	5.2863	5.2875	5.3289	5.2853	5.2860

Table 4.8: Price of American power put option using $T = 0.3333, n = 1$, $r = 4.88\%, q = 0, \sigma = 20\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	0.1994	0.1995	0.2382	0.2004	0.2014
40	1.5752	1.5783	1.6244	1.5873	1.5792
45	4.9253	5.0886	5.1327	5.0954	5.0846

Table 4.9: Price of American power put option using $T = 0.3333, n = 1$, $r = 4.88\%, q = 0, \sigma = 30\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	0.6977	0.6993	0.7300	0.6973	0.6986
40	2.4781	2.4799	2.5068	2.4919	2.4831
45	5.6978	5.7065	5.7193	5.6970	5.7051

Table 4.10: Price of American power put option using $T = 0.3333, n = 1$, $r = 4.88\%, q = 0, \sigma = 40\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	1.3481	1.3505	1.3696	1.3468	1.3470
40	3.3863	3.3835	3.4011	3.3970	3.3879
45	6.5054	6.5103	6.5147	6.5128	6.5095

Table 4.11: Price of American power put option using $T = 0.5833, n = 1$, $r = 4.88\%, q = 0, \sigma = 20\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	0.4331	0.43405	0.4624	0.4337	0.4346
40	1.9856	1.9886	2.0177	1.9987	1.9904
45	5.2844	5.2719	5.2699	5.2631	5.2638

Table 4.12: Price of American power put option using $T = 0.5833, n = 1$, $r = 4.88\%, q = 0, \sigma = 30\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	1.2218	1.2239	1.2407	1.2233	1.2216
40	3.1622	3.1665	3.1819	3.1842	3.1705
45	6.2395	6.2448	6.2477	6.2303	6.2431

Table 4.13: Price of American power put option using $T = 0.5833, n = 1$, $r = 4.88\%, q = 0, \sigma = 40\%, c = 2, S_t = \$40$.

K (\$)	**Accelerated Binomial Model**	**Binomial Model**	**Finite Difference Method**	**Recursive Method**	**Mellin Transform Method**
35	2.1569	2.1602	2.1676	2.1603	2.1568
40	4.3426	4.3426	4.3567	4.3699	4.3543
45	7.3785	7.3897	7.3792	7.3865	7.3840

Table 4.14: Influence of the volatility $\sigma = 20\%, 30\%$ and 40% on the price of American power put option with $T = 0.0833$ via the Mellin transform method.

Strike Price K(\$)	**Time to Expiry** T(yrs)	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
35	0.0833	0.0065	0.0777	0.2468
40	0.0833	0.8516	1.3098	1.7681
45	0.0833	5.0305	5.0578	5.2860

Table 4.15: Influence of the volatility $\sigma = 20\%, 30\%$ and 40% on the price of American power put option with $T = 0.3333$ via the Mellin transform method.

Strike Price K(\$)	**Time to Expiry** T(yrs)	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
35	0.3333	0.2014	0.6986	1.3470
40	0.3333	1.5792	2.4831	3.3879
45	0.3333	5.0846	5.7051	6.5095

Table 4.16: Influence of the volatility $\sigma = 20\%, 30\%$ and 40% on the price of American power put option with $T = 0.5833$ via the Mellin transform method.

Strike Price K(\$)	**Time to Expiry** T(yrs)	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
35	0.5833	0.4346	1.2216	2.1568
40	0.5833	1.9904	3.1705	4.3543
45	0.5833	5.2638	6.2431	7.3840

Table 4.17: Influence of the time to expiry $T = 0.0833, 0.3333$ and 0.5833 on the price of American power put option with $\sigma = 20\%$ via the Mellin transform method.

Strike Price $K(\$)$	**Volatility** σ	$T = 0.0833$	$T = 0.3333$	$T = 0.5833$
35	0.2	0.0065	0.2014	0.4346
40	0.2	0.8516	1.5792	1.9904
45	0.2	5.0305	5.0846	5.2638

Table 4.18: Influence of the time to expiry $T = 0.0833, 0.3333$ and 0.5833 on the price of American power put option with $\sigma = 30\%$ via the Mellin transform method.

Strike Price $K(\$)$	**Volatility** σ	$T = 0.0833$	$T = 0.3333$	$T = 0.5833$
35	0.3	0.0777	0.6986	1.2216
40	0.3	1.3098	2.4831	3.1705
45	0.3	5.0578	5.7051	6.2431

Table 4.19: Influence of the time to expiry $T = 0.0833, 0.3333$ and 0.5833 on the price of American power put option with $\sigma = 40\%$ via the Mellin transform method.

Strike Price $K(\$)$	**Volatility** σ	$T = 0.0833$	$T = 0.3333$	$T = 0.5833$
35	0.4	0.2468	1.3470	2.1568
40	0.4	1.7681	3.3879	4.3543
45	0.4	5.2860	6.5095	7.3840

Table 4.20: Free boundary of American power put option using $T = 0.0833$, $n = 1, r = 4.88\%, q = 0, \sigma = 20\%, c = 2$.

Strike Price $K(\$)$	**Stock Price** $S_t(\$)$	σ	$\hat{S}_t$	S^* **Balakrishna, (1996)**
35	40	0.2	31.7384	31.704
40	40	0.2	36.2725	36.274
45	40	0.2	40.8066	40.808

Table 4.21: Free boundary of American power put option using $T = 0.0833$, $n = 1, r = 4.88\%, q = 0, \sigma = 30\%, c = 2$.

Strike Price $K(\$)$	**Stock Price** $S_t(\$)$	σ	$\hat{S}_t$	S^* **Balakrishna, (1996)**
35	40	0.3	29.7825	29.779
40	40	0.3	34.0370	34.033
45	40	0.3	38.2914	38.287

Table 4.22: Free boundary of American power put option using $T = 0.0833$, $n = 1, r = 4.88\%, q = 0, \sigma = 40\%, c = 2$.

Strike Price $K(\$)$	**Stock Price** $S_t(\$)$	σ	$\hat{S}_t$	S^* **Balakrishna, (1996)**
35	40	0.4	27.8478	27.849
40	40	0.4	31.8260	31.827
45	40	0.4	35.8041	35.805

Table 4.23: Free boundary of American power put option using $T = 0.5833$, $n = 1, r = 4.88\%, q = 0, \sigma = 20\%, c = 2$.

Strike Price $K(\$)$	**Stock Price** $S_t(\$)$	σ	$\hat{S}_t$	S^* **Balakrishna, (1996)**
35	40	0.2	29.0740	29.085
40	40	0.2	33.2280	33.240
45	40	0.2	37.3810	37.395

Table 4.24: Free boundary of American power put option using $T = 0.5833$, $n = 1, r = 4.88\%, q = 0, \sigma = 30\%, c = 2$.

Strike Price $K(\$)$	**Stock Price** $S_t(\$)$	σ	$\hat{S}_t$	S^* **Balakrishna, (1996)**
35	40	0.3	25.4730	25.483
40	40	0.3	29.1120	29.124
45	40	0.3	32.7510	32.764

Table 4.25: Free boundary of American power put option using $T = 0.5833$, $n = 1, r = 4.88\%, q = 0, \sigma = 40\%, c = 2$.

Strike Price $K(\$)$	**Stock Price** $S_t(\$)$	σ	$\hat{S}_t$	S^* **Balakrishna, (1996)**
35	40	0.4	22.1470	22.156
40	40	0.4	25.3106	25.321
45	40	0.4	28.4744	28.486

Analysis of Experiment 3

From literature, the recursive method is a standard alternative method for the valuation of American put option. Thus comparing these other methods with the recursive method, it is observed from Tables 4.5-4.13 that the Mellin transform method is the closest to the recursive method with respect to price as volatility increases. It is observed from Tables 4.14-4.16 that as the volatility increases, the price increases. From Tables 4.17-4.19, it is observed that as the time to expiry increases the price increases. From Tables 4.20-4.22 and Tables 4.23-4.25, it is observed that the values obtained for the free boundary $\hat{S}_t$ are close to that of Balakrishna (1996). Also from Tables 4.20-4.22 and Tables 4.23-4.25, it is observed that the value of the free boundary $\hat{S}_t$ decreases as volatility increases.

Experiment 4

By varying the dividend yield, $q = \{4\%, 10\%\}$ and risk-free interest rate, $r = \{4\%, 10\%\}$, consider the valuation of the American power put option via the Mellin transform method with the following parameters

$$n = 1, c = 2, S_t = \$100, \sigma = 40\%, K = \$100, T = 1, t = 0$$

The free boundary is obtained as $\bar{S}_t = \$63$ for the case when $r > q$, that is $r = 10\%$ and $q = 4\%$. For the case when $r < q$, that is $r = 4\%$ and $q = 10\%$, the free boundary is obtained as $\bar{S}_t = \$32$.

Analysis of Experiment 4

In experiment 4, dividend yields are paid continuously at a rate q. It is observed that increase in risk-free interest rate r and decrease in dividend yield q lead to increase in the value of the free boundary of the American power put option. Similarly, it is observed that decrease in risk-free interest rate r and increase in dividend yield q lead to decrease in the value of the free boundary of the American power put option.

Experiment 5

Assume that the stocks are currently trading at \$10 and \$10 with annual volatilities of $\sigma_1 = 40\%$ and $\sigma_2 = 10\%, 20\%, 30\%$ respectively. The basket contains one unit of the first stock and one unit of the second stock. On January 1, 2015, an investor wants to buy a 1-year put option with a strike price of \$20. The current annualized, continuously compounded interest rate is 3%. Use this data to compute the price of the European basket put option using the Mellin transform in two dimensions with $c_1 = c_2 = 3$, $M = 128$ and binomial (tree) model (Schneggenburger (2002)) varying the correlation coefficients $\rho = \{-0.5, 0.5\}$. The comparative analyzes of the results of the two methods for negative and positive correlation coefficients are shown in the Tables 4.26 and 4.27 below respectively.

The effect of the correlation coefficients on the price of the European basket put option with non-dividend paying stocks via the Mellin transform in two dimensions is displayed in the Figure 4.7 below.

Table 4.26: The comparative analyzes of the results of the double Mellin transform method and binomial (tree) model with negative correlation coefficient.

σ_1	σ_2	ρ	**Binomial (Tree) Model**	**Double Mellin Transform Method**
0.1	0.1	-0.5	1.108	1.104
0.1	0.2	-0.5	1.083	1.082
0.1	0.3	-0.5	1.198	1.198

Table 4.27: The comparative analyzes of the results of the double Mellin transform method and binomial (tree) model with positive correlation coefficient.

σ_1	σ_2	ρ	**Binomial (Tree) Model**	**Double Mellin Transform Method**
0.1	0.1	0.5	1.496	1.494
0.1	0.2	0.5	1.783	1.782
0.1	0.3	0.5	2.101	2.100

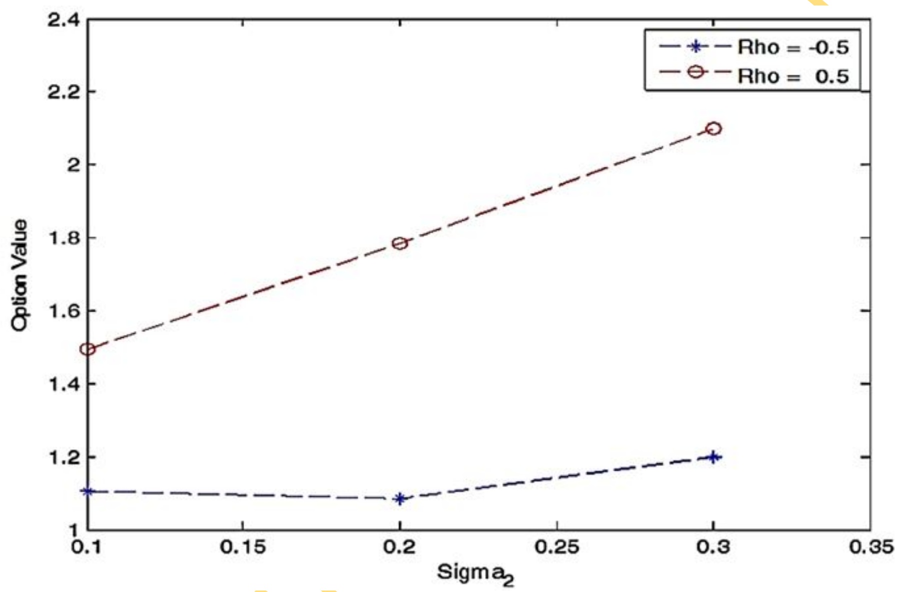

Figure 4.7: Effect of correlation coefficients on the price of European basket put option.

Analysis of Experiment 5

From Tables 4.26, it is observed that when the correlation coefficient is negative ($\rho = -0.5$) the prices of the European basket put option via the binomial model and Mellin transform in two dimensions decrease. From Table 4.27, it is observed that when the correlation coefficient is positive ($\rho = 0.5$) these prices increase. However the prices via the binomial model are greater than that of the Mellin transform in two dimensions in both cases. From Figure 4.7, it is observed that the option's value generated by the Mellin transform in two dimensions increases with the volatility.

Experiment 6

Consider the valuation of European basket put option which pays three-dividend yields using the Triple Mellin Transform Method (TMT) with $c_1 = c_2 = c_3 = 3$, $M = 128$, Monte Carlo Method (MCM) with (1.0×10^4 Monte Carlo trials) (Wan (2002)) and Implied Binomial Model (IBM) with (10 time steps) (Wan (2002)) in the context of Black-Scholes-Merton Model (BSM) with the following parameters:

Time to expiry, $T = 12$ months

Risk-free interest rate, $r = 5\%$

Dividends paying stocks, $q_1 = q_2 = q_3 = 5\%$

Correlation coefficient, $\rho = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$

Underlying asset prices, $S_1 = S_2 = S_3 = 33.33$

Strike price, $K = \{60, 70, 80, 90, 100, 110, 120, 130, 140\}$

Volatilities, $\sigma_1 = \sigma_2 = \sigma_3 = 20\%$

The comparative analyzes of the results of the three methods against the Black-Scholes-Merton model are shown in the Table 4.28 below. The absolute differences to the results from the Black-Scholes-Merton model are shown in Table 4.29 below.

Table 4.28: The comparative analyzes of the results of the three methods against the Black-Scholes-Merton model.

Strike Price, K	**BSM**	**TMT**	**MCM**	**IBM**
60	0.0028	0.0028	0.0028	0.0030
70	0.0652	0.0652	0.0697	0.0717
80	0.5420	0.5420	0.5470	0.5846
90	2.2921	2.2921	2.2884	2.3923
100	6.1744	6.1744	6.1516	6.2738
110	12.3145	12.3145	12.3179	12.3909
120	20.1422	20.1422	20.1567	20.196
130	28.9356	28.9356	28.9516	28.9679
140	38.1788	38.1788	38.1907	38.1849

Table 4.29: The absolute differences to the results from the Black-Scholes-Merton model.

Strike Price, K	**TMT**	**IBM**	**MCM**
60	0.0000	0.0002	0.0000
70	0.0000	0.0065	0.0045
80	0.0000	0.0426	0.0050
90	0.0000	0.1002	0.0037
100	0.0000	0.0994	0.0228
110	0.0000	0.0764	0.0034
120	0.0000	0.0538	0.0145
130	0.0000	0.0323	0.0160
140	0.0000	0.0061	0.1190

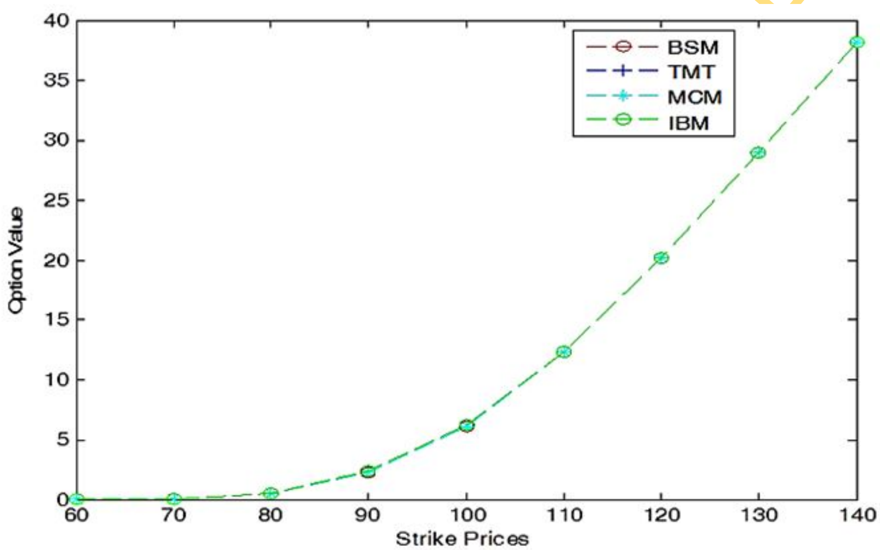

Figure 4.8: The comparative analyzes of the results using Table 4.28.

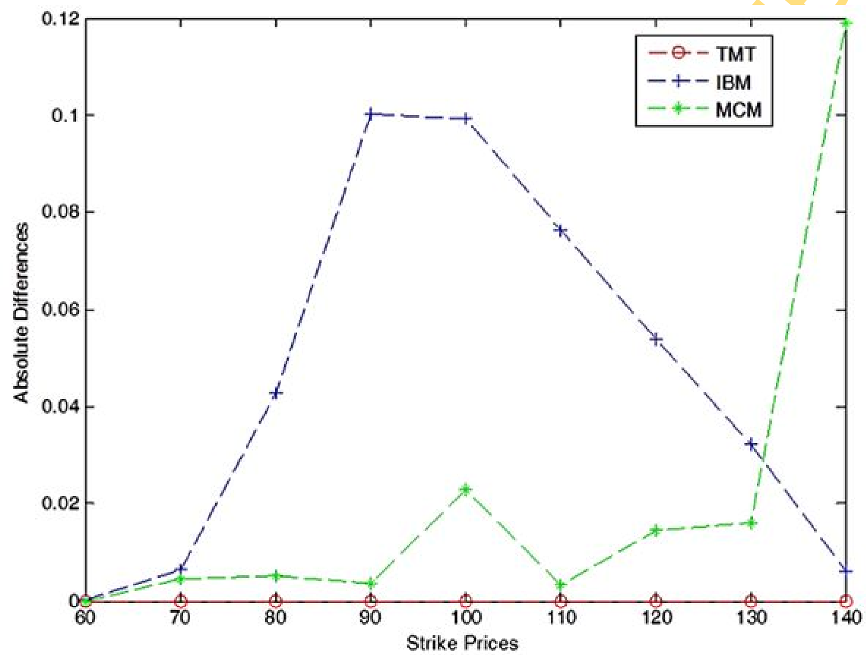

Figure 4.9: The absolute differences to the results from the Black-Scholes-Merton model using Table 4.29.

Analysis of Experiment 6

From Figure 4.8, it is observed that the prices of the European basket put option with three dividend yields generated by the Monte Carlo method and implied binomial model are satisfactory in the sense that they are close to the value obtained by the Black-Scholes model. The value for the triple Mellin transform method coincides with that obtained from the Black-Scholes model. This is so because using the convolution property of the triple Mellin transform, the integral representation model obtained for the price of the European basket put option is the same as the Black-Scholes model. From Figure 4.9, it is observed that there is no significant difference between price generated by the triple Mellin transform method and that of the Black-Scholes model. This confirms the explanation given by Figure 4.8.

4.10.2 Numerical Experiments under the Double Transform Method

Experiment 7

Consider the pricing of Asian option using the following parameters:

$$S_0 = 100, \sigma = 10\%, 20\%, 30\%, 40\%, K = 90, 95, 100, r = 9\%, T = 1$$

and

$$n_f = m_f + 15, n_p = m_p + 15, g_f = g_p = 22.4$$

The accuracy desired and parameters of the Euler algorithm are shown in Table 4.30 below. The parameters of the Euler algorithm and Asian option prices are shown in Table 4.31 below. The comparative analyzes of the results of double numerical inversion, lognormal approximation (Levy (1992)), Crank Nicolson finite difference method with 3000 spatial and time grids (Rogers and Shi (1992)) are shown in Table 4.32 below.

Table 4.30: Accuracy desired and parameters of the Euler algorithm with $S_0 = 100, K = 100, r = 9\%$, $T = 1$.

No. of Decimal Digits	**2**	**3**	**4**	**5**
Volatility, σ	$m_f; m_p$	$m_f; m_p$	$m_f; m_p$	$m_f; m_p$
0.1	15;115	15;115	35;115	35;135
0.2	15;15	15;35	15;55	15;55
0.3	15;15	15;35	15;15	15;15
0.4	15;15	15;15	15;15	15;15

Table 4.31: The parameters of the Euler algorithm and Asian option prices with $S_0 = 100, K = 100, r = 9\%$, $T = 1$.

σ	$m_f; m_p$	15	35	55	75	95	115	135
0.10	15	5.293	4.913	4.904	4.913	4.915	4.915	4.915
0.10	35	5.293	4.913	4.904	4.913	4.915	4.915	4.915
0.10	55	5.293	4.913	4.904	4.913	4.915	4.915	4.915
0.10	75	5.293	4.913	4.904	4.913	4.915	4.915	4.915
0.10	95	5.293	4.913	4.904	4.913	4.915	4.915	4.915
0.20	15	6.776	6.777	6.777	6.777	6.777	6.777	6.777
0.20	35	6.776	6.777	6.777	6.777	6.777	6.777	6.777
0.20	55	6.776	6.777	6.777	6.777	6.777	6.777	6.777
0.20	75	6.776	6.777	6.777	6.777	6.777	6.777	6.777
0.20	95	6.776	6.777	6.777	6.777	6.777	6.777	6.777
0.30	15	8.828	8.829	8.829	8.829	8.829	8.829	8.829
0.30	35	8.828	8.829	8.829	8.829	8.829	8.829	8.829
0.30	55	8.828	8.829	8.829	8.829	8.829	8.829	8.829
0.30	75	8.828	8.829	8.829	8.829	8.829	8.829	8.829
0.30	95	8.828	8.829	8.829	8.829	8.829	8.829	8.829
0.40	15	10.924	10.924	10.924	10.924	10.924	10.924	10.924
0.40	35	10.924	10.924	10.924	10.924	10.924	10.924	10.924
0.40	55	10.924	10.924	10.924	10.924	10.924	10.924	10.924
0.40	75	10.924	10.924	10.924	10.924	10.924	10.924	10.924
0.40	95	10.924	10.924	10.924	10.924	10.924	10.924	10.924

Table 4.32: The comparative analyzes of the results of Asian option pricing models with $S_0 = 100, r = 9\%$, $T = 1$.

σ	K	**Lognormal Approximation**	**Crank Nicolson Finite Difference Method** with 3000 spatial and time grids	**Double Numerical Inversion** $n_f; m_f = 15; 30$ $n_p; m_p = 15; 30$
0.1	90	13.386	13.385	12.534
0.1	95	8.917	8.910	8.511
0.1	100	4.909	4.913	5.293
0.2	90	13.862	13.831	13.737
0.2	95	10.030	9.996	9.928
0.2	100	6.804	6.777	6.776
0.3	90	15.067	14.984	14.983
0.3	95	11.733	11.656	11.655
0.3	100	8.886	8.829	8.828
0.4	90	16.654	16.500	16.500
0.4	95	13.648	13.511	13.510
0.4	100	11.031	10.923	10.924

Analysis of Experiment 7

From Table 4.30, it is observed that as the volatility increases, the values of the parameters m_f and m_p decrease quickly and consequently the computational time required for estimating the option price decreases. Table 4.31 shows how the choice relative to m_f and m_p affects the estimate in the Asian option price. It is observed from Table 4.32 that the value of double numerical inversion agrees with the values of lognormal approximation and Crank Nicolson finite difference method.

4.10.3 Numerical Experiments under the Fourier Transform Method

Experiment 8

Consider the valuation of the European call option with dividend-paying stock via fast Fourier transform method (FFT) and Fourier-Mellin transform method (FMT) with $m = 1$ in the context of Black-Scholes-Merton model (BSM) with the following parameters in Table 4.33 below.

Table 4.33: The parameters.

Variables	**Values**
Underlying asset price, S_t	100
Strike price, K	$80, 90, 100, 110, 120$
Risk-free interest rate, r	5%
Volatility, σ	50%
Dividend yield, q	5%
Time to expiry, T	0.0822
Size of integration grid, N	2^{14}
Integrability, a	2
Fineness, η	5%
Constant, c	1

The option values are shown in Tables 4.34 and 4.35. The absolute error and log absolute error for the FFT and FMT are shown in Figures 4.12 and 4.13, respectively.

Table 4.34: The comparative analyzes of the results of the fast Fourier transform method and Black-Scholes-Merton model.

Strike Price, K	**Fast Fourier Transform Method**	**Black-Scholes-Merton Model**
80	20.2407	20.2459
90	11.7753	11.7794
100	5.6873	5.6906
110	2.2636	2.2663
120	0.7521	0.7544

Table 4.35: The comparative analyzes of the results of the Fourier-Mellin transform method and Black-Scholes-Merton model.

Strike Price, K	**Fourier-Mellin Transform Method**	**Black-Scholes-Merton Model**
80	20.2459	20.2459
90	11.7794	11.7794
100	5.6906	5.6906
110	2.2663	2.2663
120	0.7544	0.7544

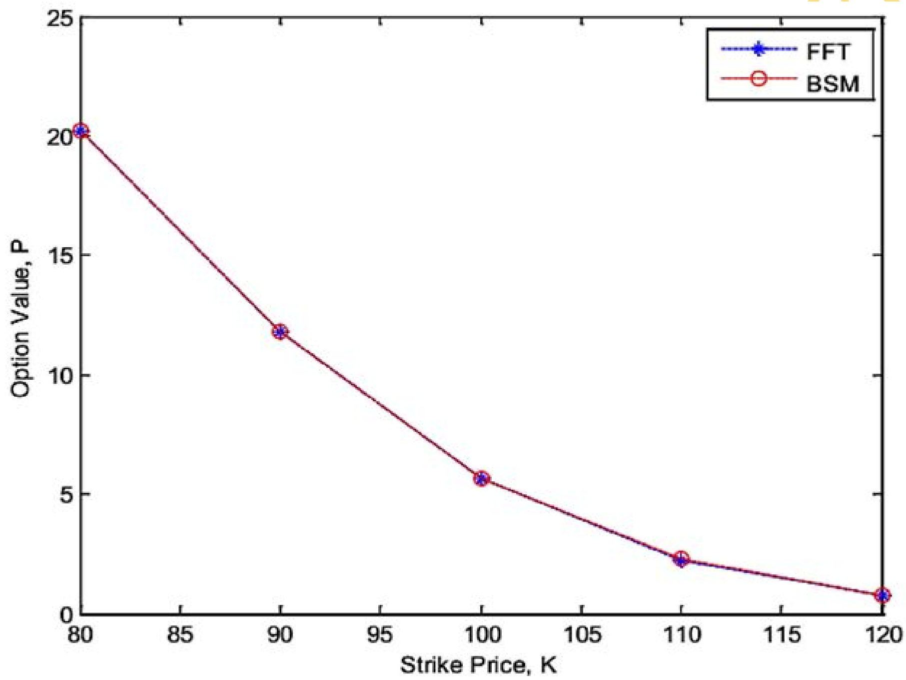

Figure 4.10: The comparative analyzes of the results of the fast Fourier transform method (FFT) and Black-Scholes-Merton model (BSM) using Table 4.34.

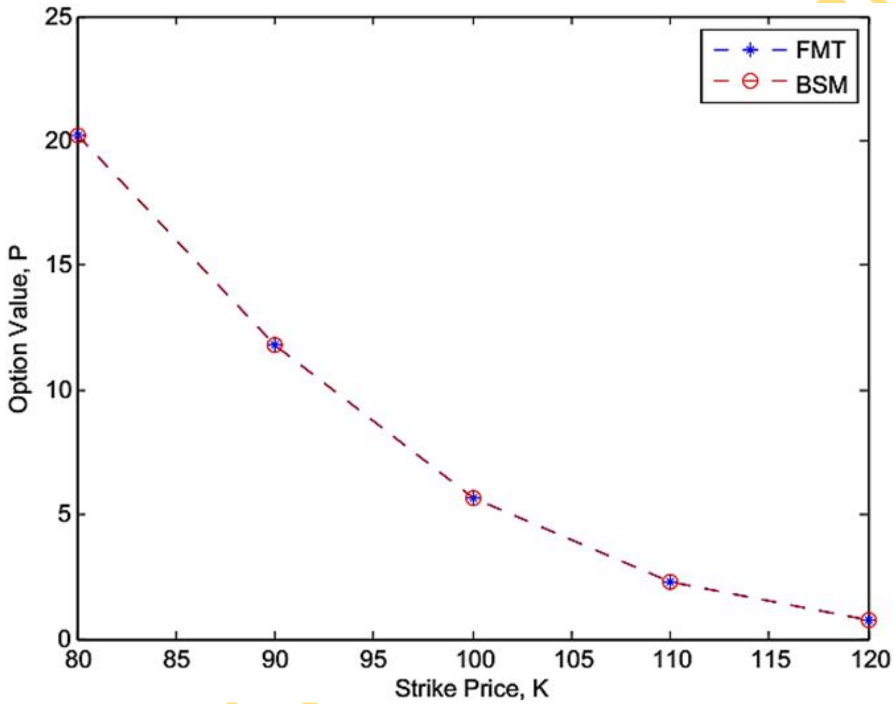

Figure 4.11: The comparative analyzes of the results of the Fourier-Mellin transform method (FMT) and Black-Scholes-Merton model (BSM) using Table 4.35.

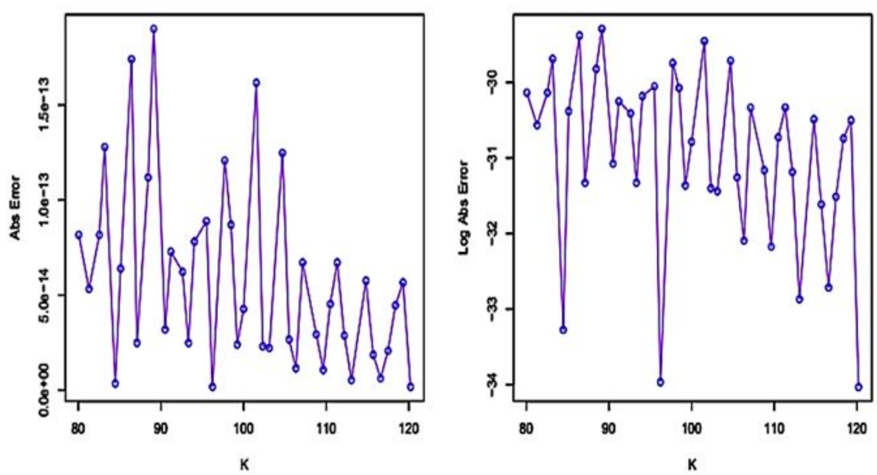

Figure 4.12: The absolute and log absolute European option price errors between fast Fourier transform method (FFT) and Black-Scholes-Merton model (BSM).

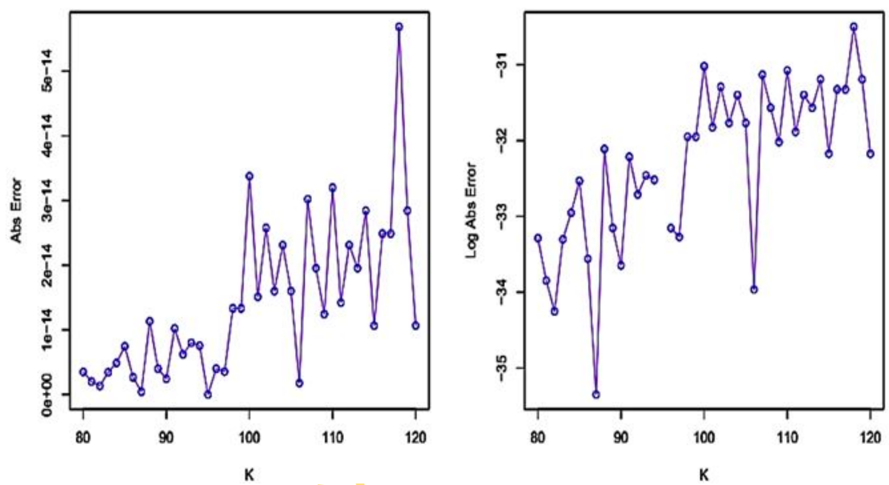

Figure 4.13: The absolute and log absolute European option price errors between Fourier-Mellin transform method (FMT) and Black-Scholes-Merton model (BSM).

Analysis of Experiment 8

From Figures 4.10 and 4.11, it is observed that the fast Fourier transform and Fourier-Mellin transform methods provide a close approximation to the Black-Scholes-Merton model and they both have computational advantages in terms of speed. Figures 4.12 and 4.13 confirm the results obtained from Figures 4.10 and 4.11 respectively.

4.10.4 Numerical Experiments under the Binomial Model

Experiment 9

Consider the valuation of a vanilla option on a stock paying a known dividend yield with the following parameters:

$$S_0 = 50, r = 0.1, T = 0.5, \tau = 0.17, \sigma = 0.25, q = 0.05$$

The result obtained is shown in Table 4.36 below.

Table 4.36: Out of the money, at the money and in the money vanilla options on a stock paying a known dividend yield.

K	E_c	A_c	**E.E.Premium**	E_p	A_p	**E.E.Premium**
30	18.97	20.50	1.53	0.004	0.004	0.00
45	6.06	6.47	0.41	1.37	1.49	0.12
50	3.32	3.42	0.10	3.38	3.78	0.40
55	1.62	1.63	0.01	6.40	7.31	0.91
70	0.11	0.11	0.00	19.19	21.35	2.16

Analysis of Experiment 9

From Table 4.36, it is observed that the American option with dividend paying stock is always worth more than its European counterpart with respect to price. When there is no dividend yield the price of the American call and that of its European call counterpart are the same. When the option is deeply "in the money", it is observed that American option has a high early exercise premium. The premium of both the put and call options decreases as the option goes out of the money. When the option is deeply "out of the money", it is observed that both call and put are worth the same this is because early exercise premium is zero.

Experiment 10

Consider the convergence of binomial model against the "true" Black-Scholes price for vanilla call and put options with

$$S_0 = 45, K = 40, T = 0.5, r = 0.1, \sigma = 0.25$$

The Black-Scholes prices for vanilla call and put options are 7.6200 and 0.6692, respectively. The values of European and American style options via the Cox-Ross-Rubinstein "CRR" model are shown in Table 4.37. The convergence of Cox-Ross-Rubinstein "CRR" model to the Black-Scholes value of the option as N increases is shown in Figure 4.14 below.

Table 4.37: The values of European and American style options via the Cox-Ross-Rubinstein "CRR" model.

N	**European Call**	**American Call**	**European Put**	**American Put**
20	7.6305	7.6305	0.6797	0.7235
40	7.6251	7.6251	0.6742	0.7228
60	7.6219	7.6219	0.6710	0.7199
80	7.6124	7.6124	0.6616	0.7134
100	7.6216	7.6216	0.6707	0.7214
120	7.6181	7.6181	0.6673	0.7182
140	7.6209	7.6209	0.6700	0.7211
160	7.6178	7.6178	0.6670	0.7184
180	7.6211	7.6211	0.6703	0.7213
200	7.6171	7.6171	0.6663	0.7185
300	7.6199	7.6199	0.6691	0.7208
500	7.6204	7.6204	0.6695	0.7211
700	7.6195	7.6195	0.6691	0.7205

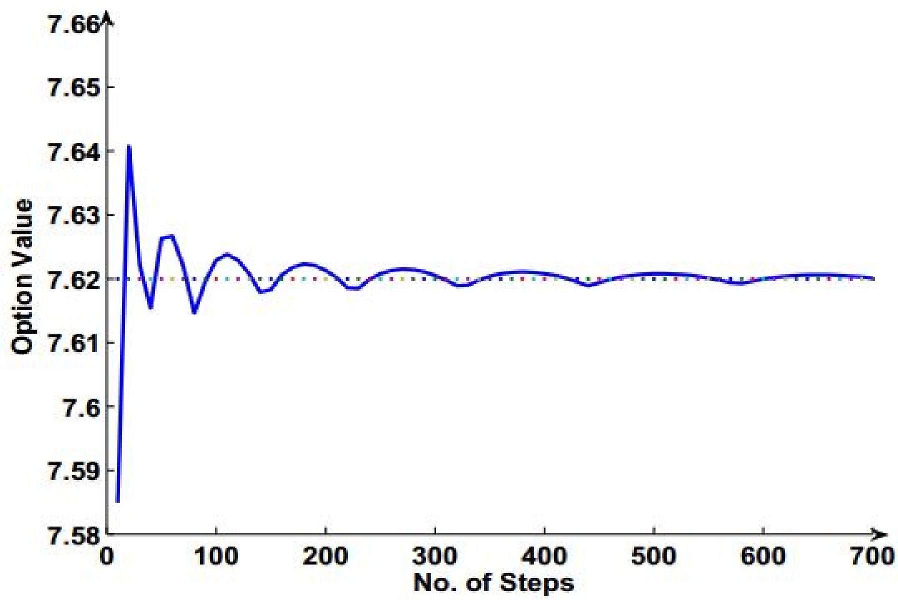

Figure 4.14: Convergence of the European call price for a non-dividend paying stock using "CRR" model to the Black-Scholes value of 7.6200.

Analysis of Experiment 10

From Table 4.37, it is observed that the values of European call and American call options are the same since it is never optimal to exercise an American call option before expiration. As the time step N increases, the value of the American put option increases faster than that of its European counterpart because of the early exercise premium. From Figure 4.14, it is observed that for very large N the option value of Cox-Ross-Rubinstein "CRR" model converges to that of the Black-Scholes model.

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

The valuation of American power put option with non-dividend and dividend yields, respectively, based on the Mellin transform method has been studied extensively in this thesis. Integral representations for the price of the European power put option with non-dividend and dividend yields, respectively was obtained. It was established that the integral representations reduced to the "Black-Scholes-like model" and "Black-Scholes-Merton-like model" for the cases of non-dividend and dividend yields, respectively. For an American power put option on one underlying asset, integral representations for the price and free boundary for both non-dividend and dividend yields, respectively was obtained by means of the Mellin transform method. To emphasize the generality of the results, the equivalence of the integral representation for the price of American power put option with dividend yield to the integral characterizations of Kim (1990) and Carr et al. (1992) for $n = 1$ was shown. By using cosine and sine transforms, the integral representation for the price of American power put option with dividend yield for $n = 1$ was transformed to a form that permits the use of the Gauss-Laguerre quadrature method. Expressions for the price and the free boundary of the perpetual American power put options using the super-contact condition was obtained. The Mellin transform in higher dimensions was used to obtain the expressions for the integral equations for prices of the put options on a basket of multi-dividend paying stocks. For an American option on a basket of multi-dividend paying stocks, an expression for the price and the integral equation for the free boundary was obtained and solved numerically. Other related methods such as double transform method, Fourier transform method and binomial model for options valuation were also considered. To provide a sufficient numerical analysis, the results generated by the Mellin transform method was compared with accelerated binomial model, binomial model and finite difference method for the valuation of American power put option for $n = 1$ in the context of the recursive method. Numerical results showed that the Mellin transform method was the closest to the recursive method with respect to price as volatility increases. The price of the option generated by the Mellin transform method increases for higher values of volatility and time to expiry. Hence the Mellin transform method gives aids in obtaining a closed-form solution for the price of American power put option which have been difficult to obtain through some other methods this is due to its flexibility, efficiency and the robustness.

5.2 Contributions to Knowledge

Contributions to the knowledge of this thesis are outlined below:

(i) The Mellin transform method was used to solve the partial differential equations for the price of power put options namely European and American power put options with non-dividend and dividend yields, respectively.

(ii) The integral representations for the price of the European power put option which pays both non-dividend and dividend yields, respectively was obtained.

(iii) It was shown that the integral representations for the European power put option with non-dividend and dividend yields reduced to the fundamental valuation formula "Black-Scholes-like" and "Black-Scholes-Merton-like" models, respectively by means of the convolution property of the Mellin transform method.

(iv) The integral representations for the price and the optimal exercise boundary (called the free boundary) of the American power put options with non-dividend and dividend yields, respectively was obtained.

(v) The optimal exercise boundary and the analytical valuation formula for the perpetual American power put option with non-dividend and dividend yields, respectively was obtained.

(vi) A closed-form solution for the price of the American power put option with dividend yield for $n = 1$ was obtained.

(vii) The integral representations for the price of put options on a basket of multi-dividend yields using the multidimensional Mellin transform method was obtained.

5.3 Recommendations

Some extensions and modifications of the methodology can be explored by further research. A natural extension is the valuation of American and European power options with dividend yield under jump diffusion processes. In the case of European options, extension may be possible to other price processes such as stochastic volatility and interest rate models. The methodology can be applied to the valuation of path dependent American and four-asset options with more complicated payoffs using univariate Mellin transform method and Mellin transform in four dimensions respectively.

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