STUDIES IN THE THEORY OF MULTIPLIERS WITH APPLICATIONS TO SEMIGROUPS OF OPERATORS

BY

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ABSTRACT

This thesis divides naturally into two parts. The first half is a study of the general theory of multipliers, while the second half deals with applications of the theory of multipliers to the theory of semigroups of operators defined on a Banach space.

We study the multiplier problem for an abstract Hilbert space H, and generalise to H certain important results established for $L_2(G)$-multipliers (Larsen [12], Hewitt and Ross [8]). A significant result of this study is the identification of certain projection operators on $L_2(G)$ which are, in several respects, like the translation operators on $L_2(G)$. We also discuss the restricted multiplier problem for the Banach algebra $L_1(G)$ of all absolutely integrable complex-valued functions defined on a compact group G, and we obtain results which are analogous to those obtained by Brainerd and Edwards [1] for $L_1(G)$, where G is a locally compact abelian group.

In connection with semigroups of operators, we discuss, in the context of various Banach spaces, the representation of the multipliers which arise from semigroups of operators on these Banach spaces. In this respect, we extend the results of Hille and Phillips [10] proved for the circle group (and generalised to compact abelian groups by Olubummo A. and Babalola V.A. [13]) to certain Banach spaces which are

not even function spaces. All these results put together provide a good link between the theory of multipliers for a Banach space and the theory of semigroups of operators on the Banach space.

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CERTIFICATION

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CHAPTER 1

INTRODUCTION

The theory of multipliers has its genesis in classical Fourier series. Multipliers were then referred to as factor sequences. We get an insight into the issue of factor sequences by examining the situation for $L_1(T)$, the Banach algebra, under the usual convolution, of all absolutely integrable complex-valued functions defined on the circle group T. The dual group of T is Z, the additive group of integers. Now, every $f \in L_1(T)$ has associated with it an infinite series

$$\sum_{n\in Z} \hat{f}(n)e_n \tag{1.1}$$

called the Fourier series of f, where for each $n\in Z$ and $t\in T$, $e_n(t) = e^{int}$, and $\hat{f}(n) = \int_T f(t)e^{-int}\,dt$. The complex numbers $\hat{f}(n)$ are called the Fourier coefficients of f. Let $\{\lambda_n\}_{n\in Z}$ be a sequence of complex numbers. Given an $f \in L_1(T)$, whose Fourier series is given by (1.1), we form a new infinite series

$$\sum_{n\in Z} \lambda_n \hat{f}(n)e_n \tag{1.2}$$

It may turn out that this new series is also the Fourier series of some element g of $L_1(T)$, in which case we have

$$\hat{g}(n) = \lambda_n \hat{f}(n) \qquad (n\in Z) \tag{1.3}$$

If, given any $f \in L_1(T)$, there always exists $g \in L_1(T)$ such that $\hat{g}(n) = \lambda_n \hat{f}(n)$ for each $n \in Z$, then the sequence $\{\lambda_n\}$ is called a factor sequence of type $(L_1(T), L_1(T))$. It is clear that the sequence $\{\lambda_n\}$ with $\lambda_n = 1$ for each $n \in Z$ is trivially a factor sequence of type $(L_1(T), L_1(T))$. An interesting problem which then arises is to determine necessary and sufficient conditions for an arbitrary sequence $\{\lambda_n\}$ of complex numbers to be a factor sequence of type $(L_1(T), L_1(T))$. This is the so-called multiplier problem for the Banach algebra $L_1(T)$. Of course one may consider other spaces of functions defined on T, and so define the factor sequence more generally. Denote by $C(T)$ the space of all continuous complex-valued functions on T, and by $L_p(T)$, $1 \leq p < \infty$, the usual space of all p-integrable complex-valued functions on T.

Let $X, Y \in \{C(T), L_p(T); 1 \leq p < \infty\}$. A sequence $\{\lambda_n\}_{n \in Z}$ is called a factor sequence of type (X, Y) if to each $x \in X$ corresponds a (unique) $y \in Y$ such that $\hat{y}(n) = \lambda_n \hat{x}(n)$ for all $n \in Z$ (Edwards [4], Hille and Phillips [10] and Zygmund [20]).

Recent trends in harmonic analysis show that the theory of classical Fourier series has its analogue for complex-valued functions defined on compact abelian groups and even, to some extent, on still more general groups (Rudin [14]). Let G be a compact abelian group, and let $L_1(G)$ denote the Banach algebra, under the usual convolution product, of all complex-valued functions on G which are absolutely integrable with respect to the Haar measure λ on G. Denote by $\hat{G}$ the dual group of G. For each $f \in L_1(G)$, the Fourier transform $\hat{f}$ of f is defined by

$$(1.4) \qquad \hat{f}(\sigma) = \int_G f(a)\, \overline{\chi_\sigma(a)}\, d\lambda(a)$$

where χ_σ is written for the character σ when we wish to consider the latter as a function on G rather than as an element of $\hat{G}$. We call a complex-valued function ϕ on $\hat{G}$ an $(L_1(G), L_1(G))$ - multiplier if to each f in $L_1(G)$ corresponds a (unique) g in $L_1(G)$ such that

$$(1.5) \qquad \hat{g}(\sigma) = \phi(\sigma)\, \hat{f}(\sigma)$$

for each $\sigma \in \hat{G}$. This extends to compact abelian groups, in a natural way, the definition of an $(L_1(T), L_1(T))$ - multiplier stated earlier. We also define an (X, Y)-multiplier, for $X, Y \in \{C(G), L_p(G);\ 1 \leq p < \infty\}$, as a complex-valued function ϕ on $\hat{G}$ such that to each $x \in X$ corresponds a $y \in Y$ satisfying $\hat{y}(\sigma) = \phi(\sigma)\, \hat{x}(\sigma)$, $\sigma \in \hat{G}$.

The facts which are known about the $(L_1(G), L_1(G))$-multipliers (we shall hereinafter refer to $(L_1(G), L_1(G))$-multipliers as multipliers for $L_1(G)$, or simply $L_1(G)$-multipliers) provide a motivation as well as a framework for studying the multipliers for certain other topological spaces or linear spaces which may not even be function spaces. For let ϕ be an $L_1(G)$-multiplier; then given any $f \in L_1(G)$, there exists $g \in L_1(G)$, uniquely defined, such that

$\hat{g}(\sigma) = \phi(\sigma)\ \hat{f}(\sigma)$ for all $\sigma \in \hat{G}$. Thus ϕ gives rise to a mapping T_ϕ, say, of $L_1(G)$ into itself, defined by $T_\phi f = g$. It is known that any such operator T_ϕ on $L_1(G)$ is bounded, linear and commutes with the translation operators on $L_1(G)$, and that, in fact, every bounded linear operator on $L_1(G)$ which commutes with translations is of the form T_ϕ for some $L_1(G)$ - multiplier ϕ. This suggests that an $L_1(G)$-multiplier may be regarded as a bounded linear operator on $L_1(G)$ which commutes with translations. The multipliers for $L_1(G)$, whether as bounded linear operators on $L_1(G)$ or as complex-valued functions on $\hat{G}$, have been thoroughly investigated by Wendel [19], Helson [7] and Edwards [5]. Their results are all contained in the following theorem ([12], Theorem 0.1.1):

1.1 <u>Theorem</u>: Let G be a locally compact abelian group and suppose $T : L_1(G) \to L_1(G)$ is a bounded linear operator. Then the following are equivalent:

(i) T commutes with translations

(ii) $T(f * g) = Tf * g$ for all $f, g \in L_1(G)$, i.e. T commutes with convolution

(iii) There exists a unique function ϕ defined on $\hat{G}$ such that $(Tf)^\wedge = \phi\hat{f}$ for each $f \in L_1(G)$.

(iv) There exists a measure $\mu \in M(G)$, the space of all bounded regular Borel measures on G, such that $(Tf)^\wedge = \hat{\mu}\hat{f}$ for each $f \in L_1(G)$ ($\hat{\mu}$ here denotes the Fourier-Stieltjes transform of μ).

(v) There exists $\mu \in M(G)$ such that $Tf = \mu * f$ for each $f \in L_1(G)$.

A multiplier for $L_1(G)$ is therefore defined also as a bounded linear operator T on $L_1(G)$ which satisfies any one, and hence all, of the five characterisations given in the preceding theorem.

It should be noted that if one wishes to study the multipliers for more general spaces, then certain of the definitions used for the multipliers for $L_1(G)$ may no longer be meaningful and some may be more appropriate than others. For example, if X is a Banach algebra, then neither the concept of an operator which commutes with translations may be meaningful, nor, in the non-commutative case, is the machinery of the Gelfand representation theory available. Here it seems natural to define a multiplier for X as a bounded linear operator from X to X such that $(Tx)y = x(Ty)$ for all x,y in X. If X is commutative and semi-simple, then we could also define a multiplier for X as a complex-valued function ϕ on the regular maximal ideal space of X such that $\phi\hat{x} \in \hat{X}$ whenever $x \in X$. Here $\hat{x}$ denotes the Gelfand transform of x and $\hat{X} = \{\hat{x} : x \in X\}$. On the other hand, if X is a topological linear space of functions on a locally compact abelian group, then the definition of a multiplier as a bounded linear operator which commutes with translations is evidently the most natural one. In whatever context, one seeks characterisations of multipliers similar to those established for $L_1(G)$ in Theorem 1.1. Larsen [12] and Hewitt and Ross [8] contain very good accounts of the theory of multipliers in the context of abstract algebras and topological linear spaces of functions and measures. Larsen, in addition, gives a list of references ([12], p.15) where one can find the various applications of the theory of multipliers in the theory of semigroups of operators, in the study of partial differential equations, in the theory of stochastic processes, in the theory of interpolation and in the general theory of Banach algebras. In chapters 4,5 and 6, we also investigate a link between the theory of multipliers and the theory of semigroups of operators.

In our study of the multipliers for a Hilbert space H in Chapter 2, we use heavily the fact that a Hilbert space is a space with projection operators. We single out certain projections P_i on closed subspaces of H, and show that these projections play exactly the same role as do the translation operators in the case of the multipliers for $L_1(G)$. An important deduction from this result is a new characterisation of the multipliers for $L_2(G)$, the Hilbert space of complex-valued square-integrable functions on a compact group G, as the bounded linear operators on $L_2(G)$ which commute with the projection operators referred to above. We furthermore prove separately, that, in fact, a bounded linear operator on $L_2(G)$ commutes with translations if and only if it commutes with these special projections. We characterise the multipliers for H in another way. Let E be a complete orthonormal set in H. A complex-valued function ϕ on E is called a multiplier for

H if $\phi\hat{x} \in \hat{H}$ whenever $x \in H$, where $\hat{x}$ denotes the Fourier transform of x and $\hat{H}$ is the set of all Fourier transforms $\hat{x}$, for $x \in H$. We show that this definition of a multiplier for H is in order, in the sense that if we define an operator T from H to H by $(Tx)^\wedge = \phi\hat{x}$, $x \in H$, then the operator T is bounded, linear and commutes with the projections P_i. We then show that every multiplier ϕ is bounded, and that conversely, every bounded complex-valued function on E is a multiplier for H. This last result extends quite naturally the situation in $L_2(G)$ ([12], Theorem 4.1.1).

In Chapter 3 we examine a variant of the multiplier problem, raised by Edwards ([4], 16.7.5,(1)), namely the question of restricted multipliers. We consider the situation in $L_1(G)$, the Banach algebra of all the absolutely integrable complex-valued functions on an infinite compact group G. Denote by Σ the dual object of G. The multipliers for $L_1(G)$ are operator-valued functions on Σ, namely the Fourier-Stieltjes transforms of measures in $M(G)$; for each $\mu \in M(G)$ and $\sigma \in \Sigma$, $\hat{\mu}(\sigma)$ is an operator on some finite-dimensional Hilbert space H_σ. Let S be a subset of Σ; we call an operator-valued function ϕ on S a function of type $(L_1(G), L_1(G), S)$ if to each $f \in L_1(G)$ corresponds <u>at least one</u> $g \in L_1(G)$ satisfying $\hat{g}(\sigma) = \phi(\sigma)\hat{f}(\sigma)$ for each $\sigma \in S$. One such function is, trivially, the function ϕ such that for each $\sigma \in \Sigma$, $\phi(\sigma)$ is the identity operator on

H_σ. We then show that a necessary and sufficient condition for an arbitrary function ϕ to be of type $(L_1(G), L_1(G), S)$ is that ϕ is the restriction to S of some Fourier-Stieltjes transform, i.e. some restricted $L_1(G)$-multiplier. Our result in this direction is analogous to that established by Brainerd and Edwards [1] in the case where G is a locally compact abelian group.

In Chapter 4 we discuss the multipliers for the Banach spaces (i) AP(G), the space of all complex-valued almost periodic functions on a locally compact abelian group G, (ii) an abstract Banach algebra $\mathcal{A}$, and (iii) an abstract Hilbert space H, where these multipliers arise from semigroups of operators defined on the Banach space being considered. In the case AP(G), we define a multiplier as a complex-valued function on $\hat{G}$ (the character group of G) such that $\phi\hat{f} \in \widehat{AP(G)}$ whenever $f \in AP(G)$; in the case $\mathcal{A}$, we define a multiplier as a complex-valued function ϕ on $\mathcal{M}$ (the space of all regular maximal ideals of $\mathcal{A}$) such that $\phi\hat{x} \in \hat{\mathcal{A}}$ whenever $x \in \mathcal{A}$; and in the case H we define a multiplier as a complex-valued function ϕ on E(a complete orthonormal set in H) such that $\phi\hat{x} \in \hat{H}$ whenever $x \in H$. Let X denote an arbitrary, but fixed, element of the set $\{AP(G), \mathcal{A}, H\}$, and let $\mathcal{J} = \{T(\xi) : \xi > 0\}$ be a one-parameter semigroup of operators on X such that each operator $T(\xi)$ determines an X-multiplier ϕ_ξ. [$\mathcal{J}$ is called a semigroup of operators because the operators $T(\xi)$, $\xi > 0$, satisfy

(1.6) $$T(\xi_1+\xi_2) = T(\xi_1)\,T(\xi_2)$$

for all $\xi_1, \xi_2 > 0$.] We show that the set $\Phi = \{\phi_\xi : \xi > 0\}$ is also a 'semigroup' of multipliers in the sense that

(1.7) $$\phi_{\xi_1+\xi_2} = \phi_{\xi_1}\phi_{\xi_2}$$

for all $\xi_1, \xi_2 > 0$. By imposing just the condition of weak measurability on $\mathcal{J}$, we obtain an exponential representation for ϕ_ξ, namely that for fixed $\sigma \in \hat{G}$ (taking $X = AP(G)$ for instance), we have

(1.8) $$\phi_\xi(\sigma) = e^{\xi\alpha_\sigma}$$

for some complex number α_σ, and all $\xi > 0$. This is a significant representation of the multipliers ϕ_ξ, in view of the nice properties of the exponential function e^z of the complex variable z. We furthermore show that if, conversely, we consider all the multipliers for $AP(G)$ which have the exponential representation (1.8), and associate with them operators $T(\xi)$, $\xi > 0$, on $AP(G)$ defined by

(1.9) $$[T(\xi)f]^\wedge(\sigma) = \phi_\xi(\sigma)\ \hat{f}(\sigma) \qquad (f \in AP(G))$$

then the collection $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is a semigroup of bounded linear operators on $AP(G)$ which commute with translations, and $\mathcal{J}$ is continuous in the strong operator topology. In the case $X = \mathcal{A}$ or $X = H$, where the setting is abstract, we establish analogous results. We thus forge a link between the multipliers for a Banach space, on the one hand, and the semigroups of operators on that Banach space, on the other hand. It appears that this link was observed for the first time by Hille and Phillips ([10], Theorems 20.3.1 and 20.3.2) when they investigated semigroups of operators on $C(-\pi, \pi)$ and $L_p(-\pi, \pi)$, $1 \leq p < \infty$, the usual Banach spaces of complex-valued continuous functions and p-integrable functions on the circle group. Subsequently, their results have been generalised to compact abelian groups by Olubummọ and Babalọla [13], and quite recently, Professor Olubummọ in a paper entitled 'Semigroups of multipliers associated with semigroups of operators' (to be published soon)* extends to arbitrary infinite compact not necessarily abelian groups the results in [13] proved for compact abelian groups. We also obtain, in Chapter 5, a generalisation of the results in [13] when we consider n-parameter semigroups of operators in place of the one-parameter semigroups of operators used in [13].

In Chapter 6, we consider once more the operators $T(\xi)$, $\xi > 0$, of the form (1.9) above, where these operators are now defined on the usual Banach spaces $C(G)$ and $L_p(G)$, $1 \leq p < \infty$, of complex-valued continuous functions and p-integrable functions on a compact abelian group G. Our main interest here is to investigate the degree of approximation of $T(0)$, the identity operator, by the operators $T(\xi)$, for small values of the parameter ξ. We show that for semigroups $\{T(\xi): \xi > 0\}$ of class $(1, C_1)$, we have a first degree approximation of f by $T(\xi)f$ only if f is a fixed point for each operator $T(\xi)$, $\xi > 0$. Hille and Phillips ([10], 20.6) have already investigated the

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case where G is the circle group. Our result is then a generalisation to compact abelian groups of Hille and Phillips' result.

CHAPTER 2

MULTIPLIERS FOR A HILBERT SPACE H

2.1 Preliminary Definitions

Let H be an abstract Hilbert space. Then H is a complex Banach space whose norm arises from an inner product. Denote the inner product of the vectors x and y in H by $< x,y >$. The following facts concerning the inner product in H are well known ([15], chapter 10) :

$$(2.1.1) \ldots \quad \begin{array}{ll} \text{(i)} & < \alpha x + \beta y, z > = \alpha < x,z > + \beta < y, z > \\ \text{(ii)} & < x,y > = \overline{< y,x >} \\ \text{(iii)} & < x,x > = \|x\|^2 \end{array}$$

for all x, y, z in H and complex numbers α, β. For some index set I, let $E = \{e_i : i \in I\}$ be a complete orthonormal set in H. E may or may not be countable ; in fact E is countable if and only if H is separable ([15], p.259). For each $x \in H$, the complex numbers $< x,e_i >$ are called the Fourier coefficients of x, and the expression

$$(2.1.2) \qquad x = \sum_i < x,e_i > e_i$$

is called the Fourier series of x, with respect to E. We shall always write the Fourier series of elements of H with respect to a fixed complete orthonormal set $E = \{e_i : i \in I\}$. Property (2.1.1) (iii) of the inner product implies

(2.1.3) $$||x||^2 = \sum_i |< x, e_i >|^2$$

for each $x \in H$.

Let $x \in H$, and define a complex-valued function $\hat{x}$ on E by setting $\hat{x}(e_i) = < x, e_i >$ for each $e_i \in E$; $\hat{x}$ is called the Fourier transform of x. For x, y in H, we have $x = y$ iff $\hat{x}(e_i) = \hat{y}(e_i)$ for all $e_i \in E$. The equation (2.1.3) is therefore equivalent to

(2.1.4) $$||x||^2 = \sum_i |\hat{x}(e_i)|^2$$

for each $x \in H$. Denote by $\hat{H}$ the set of all Fourier transforms $\hat{x}$ where $x \in H$.

2.1.1 Definition: Let H be a Hilbert space and let E be a complete orthonormal set in H. A complex-valued function ϕ on E is called a multiplier for H, or simply an H-multiplier, if $\phi\hat{x} \in \hat{H}$ whenever $x \in H$.

Denote by M(H) the set of all H-multipliers. The above definition of an H-multiplier is of course a natural extension of the definition, stated in the introduction, of a multiplier for the Hilbert space $L_2(G)$, where G is a compact abelian group.

2.2 Multipliers for H as bounded functions on E

In this section we shall characterise M(H) as the set of all bounded complex-valued functions on E. This characterisation is well known in the case $L_2(G)$ ([12], Theorem 4.1.1). We start with some definitions.

2.2.1 <u>Definition</u>: Denote by $\ell^2(E)$ the set of all complex-valued functions f defined on E, satisfying

(i) the set $\{e_i \epsilon E : f(e_i) \neq 0\}$ is either empty or countable, and

(ii) $$\sum_i |f(e_i)|^2 < \infty .$$

These functions form a complex linear space with respect to point-wise addition and scalar multiplication. In fact ([15], p.260, problem 8), $\ell^2(E)$ is a Hilbert space if inner product is defined by

$$(2.2.1) \qquad < f,g > \; = \sum_i f(e_i)\, \overline{g(e_i)}$$

for all $f, g \in \ell^2(E)$. Now, for each $x \epsilon H$, the set $\{e_i \epsilon E: \hat{x}(e_i) \neq 0\}$ is either empty or countable ([15], p.253), and $\sum_i |\hat{x}(e_i)|^2 = ||x|| < \infty$

Hence for each $x \in H$, we have $\hat{x} \in \ell^2(E)$, and

$$(2.2.2) \qquad ||\hat{x}||_{\ell^2(E)} = ||x||_H .$$

Moreover ([15], p.260, problem 9), the mapping $x \to \hat{x}$ is an isometric isomorphism of H onto $\ell^2(E)$. It follows that $\hat{H} = \ell^2(E)$.

We shall denote, also, by $\ell^\infty(E)$ the linear space (under point-wise addition and scalar multiplication) of bounded complex-valued functions f on E, with sup norm

(2.2.3) $$||f||_\infty = \sup_i |f(e_i)|.$$

It is clear that $\ell^2(E)$ is contained in $\ell^\infty(E)$.

We now prove the main theorem of this section.

2.2.2 Theorem: Let H be a Hilbert space and let E be a complete orthonormal set in H. Then, M(H) is isomorphic to $\ell^\infty(E)$.

Proof: Let $\phi \in M(H)$; then $\phi\hat{x} \in \hat{H}$ for each $x \in H$. As $\hat{H} = \ell^2(E)$, it follows that $\phi\hat{x} \in \ell^2(E)$ for each $x \in H$. Conversely, suppose $\phi\hat{x} \in \ell^2(E)$ for each $x \in H$. Since $\ell^2(E) = \hat{H}$, this implies that $\phi\hat{x} \in \hat{H}$ for each $x \in H$, i.e. $\phi \in M(H)$. Thus $\phi \in M(H)$ iff $\phi\hat{x} \in \ell^2(E)$ for each $x \in H$. We complete the proof of the theorem by showing that $\phi\hat{x} \in \ell^2(E)$ for each $x \in H$ iff $\phi \in \ell^\infty(E)$. Suppose $\phi \in \ell^\infty(E)$. For each $x \in H$, we have

$$\sum_i |\phi(e_i)\hat{x}(e_i)|^2 \leq \sum_i ||\phi||_\infty^2 \, |\hat{x}(e_i)|^2 = ||\phi||_\infty^2 \, ||x||^2 < \infty$$

Hence $\phi\hat{x} \in \ell^2(E)$. Conversely, let $\phi\hat{x} \in \ell^2(E)$ for each $x \in H$. This must imply that ϕ is bounded, i.e. $\phi \in \ell^\infty(E)$. For if ϕ is not bounded then to each positive integer n corresponds an $e_n \in E$ such that $|\phi(e_n)| > n$. We define a function η on E by

$$\eta(e_i) = \begin{cases} \dfrac{1}{|\phi(e_n)|} & \text{if } e_i = e_n \quad n = 1, 2, 3, \ldots \\ 0 & \text{if } e_i \neq e_n \quad \text{for any } n \end{cases}$$

for each $e_i \in E$. Now, the set $\{e_i \in E : \eta(e_i) \neq 0\}$ is countable, by the definition of η; furthermore,

$$\left\{\sum_i |\eta(e_i)|^2\right\}^{1/2} = \left\{\sum_{n=1}^{\infty} \frac{1}{|\phi(e_n)|^2}\right\}^{1/2} < \left\{\sum_{n=1}^{\infty} \frac{1}{n^2}\right\}^{1/2},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series. It follows that $\eta \in \ell^2(E)$.

As $\ell^2(E) = \hat{H}$, there exists $x \in H$ such that $\hat{x}(e_i) = \eta(e_i)$ for each $e_i \in E$. Thus, if ϕ is not bounded, then there exists $x \in H$ such that $\hat{x}(e_i) = \begin{cases} \frac{1}{|\phi(e_n)|} & \text{if } e_i = e_n \\ 0 & \text{if } e_i \neq e_n \end{cases}$

We then have

$$\sum_i |\phi(e_i)\,\hat{x}(e_i)|^2 = \sum_{n=1}^{\infty} |\phi(e_n)\hat{x}(e_n)|^2 = \sum_{n=1}^{\infty} 1$$

Since $\sum_{n=1}^{\infty} 1$ is a divergent series, it follows that $\phi\hat{x} \notin \ell^2(E)$.

Thus, our assumption that ϕ is not bounded gives rise to an $x \in H$ for which $\phi\hat{x} \notin \ell^2(E)$; this contradicts $\phi\hat{x} \in \ell^2(E)$ for all $x \in H$. Hence $\phi \in \ell^\infty(E)$.

2.3. Multipliers for H as operators commuting with projections

$L_2(G)$ - multipliers have been characterized as bounded linear operators on $L_2(G)$ which commute with translations ([12], Chapter 4). One cannot directly extend this result to an abstract Hilbert space, since the notion of translation is not defined in an abstract Hilbert space. However, in this section, we define a certain family P of projections on closed subspaces of H and show that the multipliers for H are precisely the bounded linear operators on H which commute with every projection in P. Our result provides, in particular, a new characterisation of $L_2(G)$ - multipliers, for a compact group G.

2.3.1 <u>Definition</u>: Let E be the fixed complete orthonormal set in 2.2. For each $e_i \in E$, let M_i denote the closed subspace of H generated by e_i; that is, the closure of the set of all scalar multiples of e_i, and let P_i denote the projection on M_i. Set $P = \{P_i \ : i \in I\}$. An operator T on H commutes with a projection P_i if $T(P_i x) = P_i(Tx)$ for all x in H. If T commutes with every P_i in P, then we say T commutes with P.

We have the following lemmas:

2.3.2 <u>Lemma</u>: For each $i \in I$, let $M_i^{\perp}$ denote the orthogonal complement of M_i in H, and let P_i be the projection on M_i. An operator T on H commutes with P_i iff both M_i and $M_i^{\perp}$ are invariant under T.

<u>Proof</u>: This lemma follows from combining the results of Theorem C and Theorem E on p.275 of [15].

2.3.3 <u>Lemma</u> : Let T be an operator on H which commutes with P. If $i, j \in I$ and $i \neq j$, then $< Te_i, Te_j > = < Te_i, e_j > = < e_i, Te_j > = 0$.

Proof: Let $i \in I$. Since T commutes with P_i, lemma 2.3.2 implies that M_i is invariant under T. Hence $\{e_i, Te_i\} \subseteq M_i$. Similarly $\{e_j, Te_j\} \subseteq M_j$ for $j \in I$. If $i \neq j$, the orthogonality of M_i and M_j implies that $\langle Te_i, Te_j\rangle = \langle Te_i, e_j\rangle = \langle e_i, Te_j\rangle = 0$.

2.3.4 Lemma: Let $\phi \in M(H)$. The operator $T : H \to H$ defined by

$$(2.3.1) \qquad (Tx)^\wedge = \phi\hat{x} \qquad (x \in H)$$

is linear, bounded and commutes with P.

Proof: Let $x, y \in H$ and let α, β be scalars

$$[T(\alpha x + \beta y)]^\wedge = \phi(\alpha x + \beta y)^\wedge = \phi(\alpha\hat{x}) + \phi(\beta\hat{y})$$

$$= \alpha\phi\hat{x} + \beta\phi\hat{y} = \alpha(Tx)^\wedge + \beta(Ty)^\wedge = [\alpha Tx + \beta Ty]^\wedge$$

By the uniqueness of the Fourier transform, we have

$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$. Hence T is a linear operator on H.

To show that T is bounded, we apply the closed graph theorem ([15], p.238). Let $\{x_n\}$ be a sequence in H such that

$$\lim_{n\to\infty} \|x_n - x\| = 0, \text{ and } \lim_{n\to\infty} \|Tx_n - y\| = 0 .$$

$$\|y - Tx\| = \|(y - Tx)^\wedge\|_{\ell^2(E)} \qquad [\text{by } (2.2.2)]$$

$$= \|\hat{y} - (Tx)^\wedge\|_{\ell^2(E)}$$

$$\leq \|\hat{y} - (Tx_n)^\wedge\|_{\ell^2(E)} + \|(Tx_n)^\wedge - (Tx)^\wedge\|_{\ell^2(E)}$$

$$= \|(y - Tx_n)^\wedge\|_{\ell^2(E)} + \|[T(x_n - x)]^\wedge\|_{\ell^2(E)}$$

$$= \|y - Tx_n\| + \|\phi(x_n - x)^\wedge\|_{\ell^2(E)} \qquad [\text{by } (2.2.2) \text{ and } (2.3.1)]$$

$$\leq \|y - Tx_n\| + \|\phi\|_\infty \|(x_n - x)^\wedge\|_{\ell^2(E)}$$

$$= \|y - Tx_n\| + \|\phi\|_\infty \|x_n - x\| \qquad (\text{by } (2.2.2))$$

Since $\phi \in \mathcal{E}^{\infty}(E)$, by Theorem 2.2.2, and both $||y - Tx_n||$ and $||x_n - x||$ converge to zero, it follows that $||y - Tx|| = 0$. Hence $y = Tx$. By the closed graph theorem, T is a bounded operator on H.

Finally we show that T commutes with every P_i in P. In view of lemma 2.3.2, it suffices to show that for each $i \in I$, both M_i and $M_i^{\perp}$ are invariant under T. Let $i \in I$.

$$< Te_i, e_j > = (Te_i)^{\wedge}(e_j) = \phi(e_j)\, \hat{e}_i(e_j) = \begin{cases} \phi(e_i) & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Hence $Te_i = \phi(e_i)e_i$, a scalar multiple of e_i, for each $e_i \in E$. Now if $x \in M_i$, it is either $x = \alpha_i e_i$ for some scalar α_i, or $x = \lim_n x_n$ where each x_n is a scalar multiple of e_i. If $x = \alpha_i e_i$, then linearity of T implies $Tx = T\alpha_i e_i = \alpha_i Te_i = \alpha_i \phi(e_i)e_i$ is a scalar multiple of e_i. Similarly, each Tx_n is a scalar multiple of e_i. Hence, in the other case $x = \lim_n x_n$, continuity of T implies $Tx = T(\lim_n x_n) = \lim_n (Tx_n)$ is the limit of a sequence of scalar multiples of e_i. Since M_i is a closed subspace of H, it follows that $Tx \in M_i$ if $x \in M_i$. That is, M_i is invariant under T. Now let $y \in M_i^{\perp}$; then $\hat{y}(e_i) = 0$. Hence $(Ty)^{\wedge}(e_i) = \phi(e_i)\, \hat{y}(e_i) = 0$. The Fourier series of Ty then reduces to $Ty = \sum_{j \neq i} < Ty, e_j > e_j$.

It follows that $Ty \in M_i^{\perp}$. Hence $M_i^{\perp}$ is also invariant under T. This concludes the proof of the lemma.

2.3.5 Definition: Let T be an operator on H, i.e. an operator on H to itself. If there exists $\phi \in M(H)$ such that $(Tx)^\wedge = \phi\hat{x}$ for each $x \in H$, then we shall call T a multiplier operator on H.

2.3.6 Lemma. Let T be a bounded linear operator on H which commutes with P. There exists $\phi \in M(H)$ such that

(i) $(Tx)^\wedge = \phi\hat{x}$ for each $x \in H$, and

(ii) $\|\phi\|_\infty = \|T\|$.

Proof: Let T be a bounded linear operator on H which commutes with P. We associate with T a complex-valued function ϕ_T on E defined by

(2.3.2) $$\phi_T(e_i) = < Te_i, e_i >$$

for each $e_i \in E$. For each $x = \sum_k < x, e_k > e_k \in H$ and $e_i \in E$, we have

$$\begin{aligned}
(Tx)^\wedge(e_i) &= < Tx, e_i > \\
&= < T\sum_k < x, e_k > e_k , e_i > \\
&= < \sum_k < x, e_k > Te_k, e_i > \\
&= \sum_k < x, e_k > < Te_k, e_i > \\
&= < x, e_i > < Te_i, e_i > \qquad \text{(by lemma 2.3.3)} \\
&= \phi_T(e_i)\, \hat{x}(e_i) .
\end{aligned}$$

Thus for each $x \in H$ and $e_i \in E$, we have

(2.3.3) $$(Tx)^\wedge(e_i) = \phi_T(e_i)\,\hat{x}(e_i).$$

Hence $(Tx)^\wedge = \phi_T \hat{x}$ for each $x \in H$. As T is an operator on H, $Tx \in H$ for each $x \in H$. Therefore $\phi_T \hat{x} = (Tx)^\wedge \in \hat{H}$ for each $x \in H$. It follows that $\phi_T \in M(H)$.

We show that $\|\phi_T\|_\infty = \|T\|$. For each $e_i \in E$,

$$\|Te_i\|^2 = \sum_j |(Te_i)^\wedge(e_j)|^2 \qquad \text{(by (2.1.4))}$$

$$= \sum_j |< Te_i, e_j >|^2$$

$$= |< Te_i, e_i >|^2 \qquad \text{(by lemma 2.3.3)}$$

$$= |\phi_T(e_i)|^2.$$

Hence $|\phi_T(e_i)| = \|Te_i\|$ for each $e_i \in E$. It follows that

$$\|\phi_T\|_\infty = \sup_{e_i \in E} |\phi_T(e_i)|$$

$$= \sup_{e_i \in E} \|Te_i\|$$

$$\leq \sup_{e_i \in E} \|T\|\,\|e_i\|$$

$$= \|T\|.$$

i.e.

(2.3.4) $$\|\phi_T\|_\infty \leq \|T\|.$$

But

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

$$= \sup_{\|x\|=1} \left\{ \sum_i |(Tx)^\wedge(e_i)|^2 \right\}^{1/2}$$

$$= \sup_{\|x\|=1} \left\{ \sum_i |\phi_T(e_i)\ \hat{x}(e_i)|^2 \right\}^{1/2} \qquad \text{[by (2.3.3)]}$$

$$\leq \sup_{\|x\|=1} \left\{ \|\phi_T\|_\infty^2 \sum_i |\hat{x}(e_i)|^2 \right\}^{1/2} \qquad [\phi_T \in \ell^\infty(E) \text{ by Theorem 2.2.2}]$$

$$= \sup_{\|x\|=1} \|\phi_T\|_\infty \|x\|$$

$$= \|\phi_T\|_\infty .$$

Hence

(2.3.5) $$\|T\| \leq \|\phi_T\| .$$

(2.3.4) and (2.3.5) imply that $\|T\| = \|\phi_T\|_\infty$.

Let us denote by $B_P(H)$ the set of all bounded linear operators on H which commute with P. Elements of $B_P(H)$ are multiplier operators on H, by lemma 2.3.6. Considering this fact along with Lemma 2.3.4 we have the following theorem:

2.3.7 Theorem: $B_P(H)$ is isometrically isomorphic to $\ell^\infty(E)$.

Proof: By lemmas 2.3.4 and 2.3.6(i), the correspondence $T \longleftrightarrow \phi_T$ defined by

(2.3.5) $$\phi_T(e_i) = \langle Te_i, e_i \rangle \qquad (e_i \in E)$$

is an isomorphism of $B_P(H)$ onto $M(H)$. But $M(H)$ is isomorphic to $\ell^\infty(E)$, by Theorem 2.2.2. [In fact, if $M(H)$ is given the sup norm, then Theorem 2.2.2 actually implies $M(H) = \ell^\infty(E)$]. Hence $B_P(H)$ is isomorphic to $\ell^\infty(E)$.

Lemma 2.3.6(ii) provides the isometry.

2.4 Real-valued multipliers and self-adjoint operators

Every operator T on H gives rise to a unique operator T^* on H, called the adjoint of T, satisfying

$$(2.4.1) \qquad < Tx, y > = < x, T^* y >$$

for all x and y in H ([15], section 56). T is called a self-adjoint operator on H if

$$(2.4.2) \qquad < Tx, y > = < x, Ty >$$

for all x, y in H. Self-adjoint operators on H are closely linked with real numbers, as is evident in the following lemma:

2.4.1 Lemma: An operator T on H is self-adjoint iff for each $x \in H$, $< Tx, x >$ is a real number.

Proof: This lemma is Theorem D, p.268 of [15].

Denote by $B_{Ps}(H)$ the subset of $B_P(H)$ comprising the self-adjoint operators and by $M_R(H)$ the subset of $M(H)$ consisting of the real-valued functions. We have the following theorem:

2.4.2 Theorem: $B_Ps(H)$ is isomorphic to $M_R(H)$.

Proof: As in the proof of Theorem 2.3.7, every $T \in B_P(H)$ corresponds to a $\phi_T \in M(H)$, where $\phi_T(e_i) = < Te_i, e_i >$ for each $e_i \epsilon E$. The proof will be complete if we show that $T \in B_P(H)$ is self-adjoint iff ϕ_T is real-valued. If T is self-adjoint, then lemma 2.4.1 implies that $\phi_T(e_i) = < Te_i, e_i >$ is real for each $e_i \epsilon E$. Conversely, if $\phi_T(e_i)$ is real for each $e_i \epsilon E$, then

for each $x = \sum_k < x, e_k > e_k \in H$,

$$< Tx, x > = < T \sum_k < x, e_k > e_k, \sum_i < x, e_i > e_i >$$

$$= < \sum_k < x, e_k > Te_k, \sum_i < x, e_i > e_i >$$

$$= \sum_{i,k} < x, e_k > < \overline{x, e_i} > < Te_k, e_i >$$

$$= \sum_i < x, e_i > < \overline{x, e_i} > < Te_i, e_i > \qquad \text{(lemma 2.3.3)}$$

$$= \sum_i |< x, e_i >|^2 \phi(e_i)$$

is real, and by lemma 2.4.1 T is self-adjoint.

2.5 Multiplier operators of finite norm

The function space $\ell^2(E)$ has been defined in 2.2.1. We denote by $\ell^2_R(E)$ the subspace of $\ell^2(E)$ consisting of real-valued functions only. $\ell^2(E)$ and $\ell^2_R(E)$ are subspaces of $\ell^\infty(E)$. We show in this section that $\ell^2(E)$ and $\ell^2_R(E)$ are isometrically isomorphic to certain well known subspaces of $B_p(H)$.

2.5.1 Definition: Let T be a bounded linear operator on a Hilbert space H. With T and two complete orthonormal sets $E_1 = \{\xi_\alpha : \alpha \in \Omega_1\}$ and $E_2 = \{\eta_\beta : \beta \in \Omega_2\}$ we associate a number $N(T; \xi, \eta)$ defined by

$$(2.5.1) \qquad N(T; \xi, \eta) = \left\{\sum_{\alpha,\beta} < T\xi_\alpha , \eta_\beta > < \eta_\beta, T\xi_\alpha >\right\}^{1/2}$$

M. H. Stone ([16], p.66) shows that $N(T; \xi, \eta)$ is independent of the orthonormal sets used to define it. Thus the number $N(T) = N(T; \xi, \eta)$ is a characteristic of the operator T. T is said to have the norm $N(T)$, and T is said to be of finite norm if $N(T) < \infty$. The norm $N(T)$ is different from the usual operator norm $\|T\|$ defined in [15], p. 220. In fact, there are bounded linear operators on H which are not of finite norm, for example the identity operator on H. The class $\mathcal{F}$ of all bounded linear operators on H which are of finite norm is closed under the operations of scalar multiplication, addition, and formation of the adjoint ([16], Theorem 2.31).

Suppose in defining $N(T)$, we take $E_1 = E_2 = E$, where $E = \{e_i : i \in I\}$. From (2.5.1) we shall have

$$N(T) = \left\{ \sum_{i,j} < Te_i, e_j > < e_j, Te_i > \right\}^{1/2} = \left\{ \sum_{i,j} |< Te_i, e_j >|^2 \right\}^{1/2}$$

If T commutes with P, then $< Te_i, e_j > = 0$ for $i \neq j$ (lemma 2.3.3), and so

$$(2.5.2) \qquad N(T) = \left\{ \sum_{i} |< Te_i, e_i >|^2 \right\}^{1/2}$$

If $\mathcal{F}$ is given the norm $N(T)$, then we have the following theorem:

2.5.2 <u>Theorem</u>: $\ell^2(E)$ is isometrically isomorphic to the subspace of $B_P(H)$ comprising operators which are of finite norm.

<u>Proof:</u> By Theorem 2.3.7 there is an isomorphism of $B_P(H)$ onto $\ell^\infty(E)$. Let $T \in B_P(H)$ correspond to $\phi_T \in \ell^\infty(E)$ under the isomorphism; then $\phi_T(e_i) = < Te_i, e_i >$ for each $e_i \in E$. The operator T is of finite norm iff $\phi_T \in \ell^2(E)$, since

$$N(T) = \left\{ \sum_{i} |< Te_i, e_i >|^2 \right\}^{1/2} = \left\{ \sum_{i} |\phi_T(e_i)|^2 \right\}^{1/2}$$

2.5.3 <u>Corollary</u> $\ell^2_R(E)$ is isometrically isomorphic to the subspace of $B_P(H)$ comprising operators which are of finite norm and are self-adjoint.

<u>Proof:</u> We combine the results of Theorem 2.5.2 and lemma 2.4.2 .

2.6. The case $H = L_2(G)$

Let G be a compact abelian group and let $L_2(G)$ denote as usual the Banach space of all (equivalence classes of) complex-valued measurable functions on G which are square-integrable with respect to the Haar measure λ on G. With inner product defined by

$$(2.6.1) \qquad \langle f, g \rangle = \int_G f(a)\, \overline{g(a)}\, d\lambda(a)$$

for all $f, g \in L_2(G)$, $L_2(G)$ is a Hilbert space. Denote by $\hat{G}$ the character group of G. For the sake of clarity, we shall always write χ_σ for a continuous character of G when it is considered as a function on G, and σ for the same character when it is considered as an element of the character group $\hat{G}$.

$\hat{G}$ is a complete orthonormal set in $L_2(G)$, and so every $f \in L_2(G)$ has a convergent Fourier series

$$(2.6.2) \qquad f = \sum_{\sigma \in X} \hat{f}(\sigma)\chi_\sigma$$

where $\hat{f}$, the Fourier transform of f, is defined by

$$(2.6.3) \qquad \hat{f}(\sigma) = \int_G f(a)\, \overline{\chi_\sigma(a)}\, d\lambda(a)$$

for each $\sigma \in \hat{G}$.

For each $\sigma \in \hat{G}$, we denote by M_σ the closed subspace of $L_2(G)$ generated by χ_σ, and by P_σ the projection on M_σ. An operator T on $L_2(G)$ commutes with P_σ iff M_σ and $M_\sigma^\perp$ are invariant under T (lemma 2.3.2 in the case $H = L_2(G)$).

Let $f \in L_2(G)$ and let $a \in G$. The translate of f by a, denoted f_a, is the function on G defined by

$$f_a(x) = f(xa) \tag{2.6.4}$$

for all $x \in G$. Translations $f \to f_a$ (as well as projections P_σ) are bounded linear operators on $L_2(G)$. An operator T on $L_2(G)$ commutes with translations iff $Tf_a = (Tf)_a$ for all $f \in L_2(G)$ and all $a \in G$.

We have the following theorem:

2.6.1 Theorem Let G be a compact abelian group with character group $\hat{G}$, and let T be a bounded linear operator on $L_2(G)$. Then T commutes with translations iff T commutes with each projection P_σ, $\sigma \in \hat{G}$.

Proof: Suppose T commutes with translations. Then ([12], Theorem 4.1.1) there exists $\phi \in \mathcal{E}^\infty(\hat{G})$ such that $(Tf)^\wedge = \phi\hat{f}$ for each $f \in L_2(G)$. Let σ_0 be an arbitrary but fixed element of $\hat{G}$. We show that T commutes with P_{σ_0}. Now, for any $f = \sum_{\sigma} \hat{f}(\sigma)\chi_\sigma$ in $L_2(G)$, we have $P_{\sigma_0}f = \hat{f}(\sigma_0)\chi_{\sigma_0}$. Thus, for each $\sigma \in \hat{G}$, $f \in L_2(G)$, we have

$$\left[P_{\sigma_0}(Tf)\right]^\wedge(\sigma) = \left[(Tf)^\wedge(\sigma_0)\chi_{\sigma_0}\right]^\wedge(\sigma) = (Tf)^\wedge(\sigma_0)\hat{\chi}_{\sigma_0}(\sigma) =$$

$$= \begin{cases} \phi(\sigma_0)\ f(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases}$$

and

$$\left[T(P_{\sigma_0} f)\right]^\wedge(\sigma) = \left[T(\hat{f}(\sigma_0)\chi_{\sigma_0})\right]^\wedge(\sigma) = \hat{f}(\sigma_0)(T\chi_{\sigma_0})^\wedge(\sigma) = \hat{f}(\sigma_0)\phi(\sigma_0)\hat{\chi}_{\sigma_0}(\sigma)$$

$$= \begin{cases} \hat{f}(\sigma_0)\ \phi(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 , \end{cases}$$

since $\hat{\chi}_{\sigma_0}(\sigma) = 1$ or 0 according as $\sigma = \sigma_0$ or not.

It follows that for each $f \in L_2(G)$, we have

$\left[P_{\sigma_0}(Tf)\right]^\wedge(\sigma) = \left[T(P_{\sigma_0} f)\right]^\wedge(\sigma)$ for each $\sigma \in \hat{G}$. By the uniqueness of the Fourier transform,

$$P_{\sigma_0}(Tf) = T(P_{\sigma_0} f) , \quad \text{for each } f \in L_2(G) .$$

Hence T commutes with P_{σ_0}. As σ_0 was arbitrarily picked from among the elements of $\hat{G}$, it follows that T commutes with P_σ for each $\sigma \in \hat{G}$.

Conversely suppose T commutes with P_σ, for each $\sigma \in \hat{G}$, then M_σ and $M_\sigma^\perp$ are both invariant under T, for each $\sigma \in \hat{G}$. In particular $T\chi_\sigma \in M_\sigma$ for $\sigma \in \hat{G}$. So, either

(2.6.5) $$T\chi_\sigma = \alpha(\sigma)\chi_\sigma$$

for some complex number $\alpha(\sigma)$, or

(2.6.6) $$TX_\sigma = \lim_{n\to\infty} (\alpha_n(\sigma)X_\sigma),$$

a limit, in the sense of $L_2(G)$- norm, of a sequence $\{\alpha_n(\sigma)X_\sigma\}$ of scalar multiples of X_σ. Let $a\epsilon G$ and $f \epsilon L_2(G)$ be arbitrary.

$$Tf_a = T\left(\sum_\sigma \hat{f}(\sigma)X_\sigma\right)_a = T \sum_\sigma \hat{f}(\sigma)(X_\sigma)_a = T \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, X_\sigma$$

$$= \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, TX_\sigma .$$ That is,

(2.6.7) $$Tf_a = \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, TX_\sigma$$

for all $a\epsilon G$, $f \epsilon L_2(G)$. If TX_σ is of the form (2.6.5), then we have from (2.6.7) the following chain of equalities

$$Tf_a = \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, TX_\sigma = \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, \alpha(\sigma)X_\sigma$$

$$= \sum_\sigma \hat{f}(\sigma)\alpha(\sigma)(X_\sigma)_a = \left(\sum_\sigma \hat{f}(\sigma)\alpha(\sigma)X_\sigma\right)_a$$

$$= \left(\sum_\sigma \hat{f}(\sigma)\, TX_\sigma\right)_a = \left(T \sum_\sigma \hat{f}(\sigma)\, X_\sigma\right)_a = (Tf)_a$$

If TX_σ is of the form (2.6.6), we similarly have

$$Tf_a = \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, TX_\sigma = \sum_\sigma \hat{f}(\sigma)\, \overline{X_\sigma(a)}\, \lim_n(\alpha_n(\sigma)X_\sigma)$$

$$= \sum_\sigma \hat{f}(\sigma) \lim_n \left[\alpha_n(\sigma)(X_\sigma)_a\right] = \sum_\sigma \hat{f}(\sigma) \lim_n \alpha_n(\sigma)(X_\sigma)_a$$

$$= \left(\sum_\sigma \hat{f}(\sigma)\lim_n \alpha_n(\sigma)X_\sigma\right)_a = \left(\sum_\sigma \hat{f}(\sigma)TX_\sigma\right)_a = (Tf)_a .$$

In either case, $Tf_a = (Tf)_a$. Hence T commutes with translations.

We now consider Theorem 2.6.1 in the case where G is a compact, not necessarily abelian, group. In definitions and notations, we follow Hewitt and Ross [8], where any undefined terms concerning harmonic analysis on compact non-abelian groups, used in this section, will be found.

Let G be an infinite compact (not necessarily abelian) group with dual object Σ, and let $L_2(G)$ denote, as before, the Hilbert space of complex-valued measurable functions on G which are square-integrable with respect to the Haar measure λ on G. For each $\sigma \in \Sigma$, let the representation $U^{(\sigma)} \in \sigma$ have the representation space H_σ, and let d_σ be the (finite) dimension of the Hilbert space H_σ. Further, let

(2.6.8) $$\{\xi_1^{(\sigma)}, \xi_2^{(\sigma)}, \ldots, \xi_{d_\sigma}^{\sigma}\}$$

be a fixed (but arbitrarily chosen) orthonormal basis in H_σ. Coordinate functions $u_{jk}^{(\sigma)}$, for $j, k = 1, 2, \ldots, d_\sigma$, are defined on G by

(2.6.9) $$u_{jk}^{(\sigma)}(x) = \langle U_x^{(\sigma)}\xi_k^{(\sigma)}, \xi_j^{(\sigma)} \rangle .$$

In this same basis, the $\overline{u}_{jk}^{(\sigma)}$'s are coordinate functions for the conjugate representation $\overline{U}^{(\sigma)} \in \overline{\sigma}$ ([8], (27.28)). The set $\{d_\sigma^{\frac{1}{2}} u_{jk}^{(\sigma)} : \sigma \in \Sigma, j,k = 1, 2, \ldots, d_\sigma\}$ is a complete orthonormal set in $L_2(G)$, and for each $f \in L_2(G)$, we have

$$(2.6.10) \qquad f = \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < f, u_{jk}^{(\sigma)} > u_{jk}^{(\sigma)}$$

where

$$(2.6.11) \qquad < f, u_{jk}^{(\sigma)} > = \int_G \bar{u}_{jk}^{(\sigma)} f \, d\lambda$$

and the series in (2.6.10) converges in $L_2(G)$-norm. For $f \in L_2(G)$, let $\hat{f}(\sigma)$ be the operator on H_σ defined by

$$(2.6.12) \qquad < \hat{f}(\sigma)\xi, \eta > = \int_G < \bar{U}_x^{(\sigma)}\xi, \eta > f(x) \, d\lambda(x)$$

for all $\xi, \eta \in H_\sigma$. Denote by $B(H_\sigma)$ the set of all bounded linear operators on H_σ. The involution algebra $\underset{\sigma\in\Sigma}{\mathrm{P}} B(H_\sigma)$ is denoted by $\mathfrak{E}(\Sigma)$ ([8], (28.24)). In $\mathfrak{E}(\Sigma)$, scalar multiplication, addition, multiplication, and the adjoint of an element are defined coordinate-wise. Let $E = (E_\sigma)$ be an element of $\mathfrak{E}(\Sigma)$. For $1 \leq p < \infty$, we define

$$(2.6.13) \qquad \|E\|_p = \left\{ \sum_{\sigma\in\Sigma} d_\sigma \|E_\sigma\|_{\phi_p}^p \right\}^{\frac{1}{p}}$$

and

$$(2.6.14) \qquad \|E\|_\infty = \sup\{ \|E_\sigma\|_{\phi_\infty} : \sigma \in \Sigma \}$$

where $\|.\|_{\phi_p}$ are the operator norms of [8], (D.37) and (D.36.e).

For $1 \leq p \leq \infty$, $\mathfrak{E}_p(\Sigma)$ is defined as the set of all $E \in \mathfrak{E}(\Sigma)$ for which $||E||_\rho < \infty$. Since G is compact, $L_2(G)$ is contained in $M(G)$, the set of all bounded regular complex-valued Borel measures on G. It follows from (28.36) of [8] that $\hat{f} \in \mathfrak{C}_\infty(\Sigma)$ for each $f \in L_2(G)$. Following [8], (28.35), we shall use a fixed collection $\{U^{(\sigma)} : \sigma\in\Sigma\}$ of representations, picking one from each equivalence class $\sigma\in\Sigma$, so that $\hat{f}$, defined by (2.6.12), is fixed for each $f \in L_2(G)$. From (2.6.11) we then have

$$< f, u_{jk}^{(\sigma)} > = \int_G \bar{u}_{jk}^{(\sigma)}(x)f(x)\, d\lambda(x) = < \hat{f}(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > .$$

Thus for each $f \in L_2(G)$, we have

$$(2.6.15) \qquad f = \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > u_{jk}^{(\sigma)}$$

For $\sigma\in\Sigma$, let M_σ denote the closed subspace of $L_2(G)$ generated by the set $\{u_{jk}^{(\sigma)} : j,k = 1, 2, \ldots, d_\sigma\}$, i.e. the closure of the set of all finite complex linear combinations of coordinate functions $u_{jk}^{(\sigma)}$. $M_\sigma = \overline{\mathcal{L}_\sigma}(G)$ ([8], (27.7)). Denote by P_σ the projection on M_σ. [These definitions of P_σ and M_σ reduce to the definitions given earlier for P_σ and M_σ, if G is abelian, since then $d_\sigma = 1$ for each $\sigma \in \Sigma$]. For each $\sigma \in \Sigma$, P_σ is a bounded linear operator on $L_2(G)$; thus for $f = \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > u_{jk}^{(\sigma)}$ in $L_2(G)$, we have

$$(2.6.16) \qquad P_\sigma f = \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > u_{jk}^{(\sigma)}$$

2.6.2 : <u>Lemma</u> Let $f \in L_2(G)$ and let σ_0 be arbitrary but fixed in Σ. Then, for each $\sigma \in \Sigma$, we have

$$(P_{\sigma_0} f)^{\wedge}(\sigma) = \begin{cases} \hat{f}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases} \tag{2.6.17}$$

<u>Proof</u>: Let $\sigma_0, \sigma \in \Sigma$ be as in the lemma. By the orthogonality relations for coordinate functions ([9], (27.15)), $(u_{jk}^{(\sigma_0)})^{\wedge}(\sigma) = 0$ if $\sigma \neq \sigma_0$. Hence, if $f \in L_2(G)$,

$$(P_{\sigma_0} f)^{\wedge}(\sigma) = \left\{ \sum_{j,k=1}^{d_{\sigma_0}} d_{\sigma_0} < \hat{f}(\sigma_0)\xi_k^{(\sigma_0)}, \xi_j^{(\sigma_0)} > u_{jk}^{(\sigma_0)} \right\}^{\wedge}(\sigma) \text{ [using(2.6.16)]}$$

$$= \sum_{j,k=1}^{d_{\sigma_0}} d_{\sigma_0} < \hat{f}(\sigma_0)\xi_k^{(\sigma_0)}, \xi_j^{(\sigma_0)} > (u_{jk}^{(\sigma_0)})^{\wedge}(\sigma)$$

$$= \begin{cases} \sum_{j,k=1}^{d_{\sigma_0}} d_{\sigma_0} < \hat{f}(\sigma_0)\xi_k^{(\sigma_0)}, \xi_j^{(\sigma_0)} > (u_{jk}^{(\sigma_0)})^{\wedge}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases}$$

$$= \begin{cases} \hat{f}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases}$$

2.6.3 <u>Lemma</u>: Let $B(H_\sigma)$ be the set of all bounded linear operators on H_σ. Then

$$B(H_\sigma) = \{\hat{f}(\sigma) : f \in \mathcal{L}_\sigma(G)\}$$

for each $\sigma \in \Sigma$.

Proof: This lemma is Theorem (28.39) (i) of [8].

2.6.4 Definition: Let $f \in L_2(G)$ and let $a \in G$. The right-translate of f by a, denoted f_a, is the function on G defined by

$$(2.6.18) \qquad f_a(x) = f(xa)$$

for each $x \in G$. Similarly we define ${}_af$, the left-translate of f, by

$$(2.6.19) \qquad {}_af(x) = f(ax)$$

for each $x \in G$.

We shall henceforth consider right-translations only, in our lemmas and theorems, since it is clear that analogous statements can be made if left-translations are also considered. Of course if G is abelian, then left-translation is the same as right-translation.

2.6.5 Lemma: Let $a \in G$ and let $f \in L_2(G)$. Then

$$(2.6.20) \qquad (f_a)^\wedge(\sigma) = \hat{f}(\sigma)\, \overline{U}^{(\sigma)}_{a^{-1}}$$

for each $\sigma \in \Sigma$.

Proof: For all $\xi, \eta \in H_\sigma$,

$$< \hat{f}(\sigma)\overline{U}^{(\sigma)}_{a^{-1}}\xi, \eta > \; = \; < \hat{f}(\sigma)(\overline{U}^{(\sigma)}_{a^{-1}}\xi), \eta >$$

$$= \int_G < \overline{U}^{(\sigma)}_x (\overline{U}^{(\sigma)}_{a^{-1}}\xi), \eta > f(x)d\lambda(x)$$

$$= \int_G < \overline{U}^{(\sigma)}_{xa^{-1}}\xi, \eta > f(x)\, d\lambda(x)$$

$$= \int_G < \overline{U}^{(\sigma)}_{x}\xi , \eta > f(xa)\, d\lambda(x)$$

$$= \int_G < \overline{U}^{(\sigma)}_{x}\xi, \eta > f_a(x)\, d\lambda(x)$$

$$= < (f_a)^{\wedge}(\sigma)\xi, \eta >$$

Hence $(f_a)^{\wedge}(\sigma) = \hat{f}(\sigma)\overline{U}^{(\sigma)}_{a^{-1}}$, for each $\sigma \in \Sigma$.

Using this lemma and (2.6.15), we have

(2.6.21) $$f_a = \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\overline{U}^{(\sigma)}_{a^{-1}}\xi^{(\sigma)}_k, \xi^{(\sigma)}_j > u^{(\sigma)}_{jk}$$

2.6.5 <u>Lemma</u>: Let G be a compact group with dual object Σ, and let T be a bounded linear operator on $L_2(G)$ which commutes with right-translations, i.e. $Tf_a = (Tf)_a$ for all $a \in G$ and $f \in L_2(G)$. Then there exists $E \in \mathfrak{C}_\infty(\Sigma)$ such that for each $f \in L_2(G)$, we have

(2.6.22) $$(Tf)^{\wedge}(\sigma) = E_\sigma\hat{f}(\sigma)$$

for all $\sigma \in \Sigma$.

<u>Proof</u>: Since G is compact, $L_2(G)$ is a Banach algebra under convolution $*$, where

(2.6.23) $$(f * g)(x) = \int_G f(xy^{-1})g(y)d\lambda(y)$$

for all $x \in G$ and $f, g \in L_2(G)$. Thus if T is an operator on $L_2(G)$, then $f * g$, $Tf * g$ and $T(f * g)$ all belong to $L_2(G)$ for all $f, g \in L_2(G)$. Now, for any $h \in L_2(G)$,

$$\begin{aligned}
< Tf * g, h > &= \int_G (Tf * g)(x)\, \overline{h(x)}\, d\lambda(x) \\
&= \int_G \left[\int_G (Tf)(xy^{-1}) g(y) d\lambda(y) \right] \overline{h(x)}\, d\lambda(x) \\
&= \int_G g(y) \left[\int_G (Tf)_{y^{-1}}(x)\, \overline{h(x)}\, d\lambda(x) \right] d\lambda(y) \qquad \text{(By Fubini [8], p.156)} \\
&= \int_G g(y) < (Tf)_{y^{-1}} , h > d\lambda(y) \\
&= \int_G g(y) < Tf_{y^{-1}}, h > d\lambda(y) \qquad \text{[T commutes with right-translations]} \\
&= \int_G g(y) < f_{y^{-1}}, T^* h > d\lambda(y) \\
&= \int_G g(y) \left[\int_G f_{y^{-1}}(x)\, \overline{(T^* h)(x)}\, d\lambda(x) \right] d\lambda(y) \\
&= \int_G \left[\int_G f(xy^{-1}) g(y) d\lambda(y) \right] \overline{(T^* h)(x)}\, d\lambda(x) \qquad \text{(By Fubini)} \\
&= \int_G (f * g)(x)\, \overline{(T^* h)(x)}\, d\lambda(x) \\
&= < f * g , T^* h > \\
&= < T(f * g), h > .
\end{aligned}$$

Hence $T(f * g) = Tf * g$ for all f, g in $L_2(G)$.

We associate with T an operator-valued function $E : \Sigma \to \mathcal{C}(\Sigma)$ defined in the following way. Let $\sigma \in \Sigma$, by lemma 2.6.3 there exists $e_\sigma \in \mathcal{L}_\sigma(G)$ such that $\hat{e}_\sigma(\sigma) = I_{d_\sigma}$, the identity operator on H_σ. We define E_σ, the value of E at σ, by

$$E_\sigma = (Te_\sigma)\hat{}(\sigma) \tag{2.6.24}$$

Observe that for each $u_{jk}^{(\sigma)} \in \mathcal{L}_\sigma(G)$, $(u_{jk}^{(\sigma)})\hat{}(\sigma) = \hat{e}_\sigma(\sigma)(u_{jk}^{(\sigma)})\hat{}(\sigma)$ implies that $u_{jk}^{(\sigma)} = e_\sigma * u_{jk}^{(\sigma)}$. Let σ_0 be arbitrary but fixed in Σ. For each $\sigma \in \Sigma$, we have

$$(Tu_{jk}^{(\sigma_0)})\hat{}(\sigma) = \left[T(e_{\sigma_0} * u_{jk}^{(\sigma_0)})\right]\hat{}(\sigma) = (Te_{\sigma_0} * u_{jk}^{(\sigma_0)})\hat{}(\sigma)$$

$$= (Te_{\sigma_0})\hat{}(\sigma)(u_{jk}^{(\sigma_0)})\hat{}(\sigma) = \begin{cases} E_{\sigma_0}(u_{jk}^{(\sigma_0)})\hat{}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases}$$

since $(u_{jk}^{(\sigma_0)})\hat{}(\sigma) = 0$ if $\sigma \neq \sigma_0$. That is,

$$(Tu_{jk}^{(\sigma_0)})\hat{}(\sigma) = \begin{cases} E_{\sigma_0}(u_{jk}^{(\sigma_0)})\hat{}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases} \tag{2.6.25}$$

for all $\sigma \in \Sigma$ and $j, k = 1, 2, \ldots, d_{\sigma_0}$. Let $f \in L_2(G)$, and let σ_0 be arbitrary but fixed in Σ;

$$(Tf)\hat{}(\sigma_0) = \left[\sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > Tu_{jk}^{(\sigma)}\right]\hat{}(\sigma_0)$$

$$= \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > (Tu_{jk}^{(\sigma)})\hat{}(\sigma_0)$$

$$= \sum_{j,k=1}^{d_{\sigma_0}} d_{\sigma_0} < \hat{f}(\sigma_0)\xi_k^{(\sigma_0)} , \xi_j^{(\sigma_0)} > E_{\sigma_0}(u_{jk}^{(\sigma_0)})^\wedge(\sigma_0) \quad \text{[by (2.6.25)]}$$

$$= E_{\sigma_0} \sum_{j,k=1}^{d_{\sigma_0}} d_{\sigma_0} < \hat{f}(\sigma_0)\xi_k^{(\sigma_0)} , \xi_j^{(\sigma_0)} > (u_{jk}^{(\sigma_0)})^\wedge(\sigma_0)$$

$$= E_{\sigma_0} \hat{f}(\sigma_0).$$

Hence $(Tf)^\wedge(\sigma_0) = E_{\sigma_0}\hat{f}(\sigma_0)$ for all $f \in L_2(G)$. As σ_0 could be any element of Σ, it follows that $(Tf)^\wedge(\sigma) = E_\sigma \hat{f}(\sigma)$ for all $f \in L_2(G)$ and all $\sigma \in \Sigma$. This implies that $(Tf)^\wedge = E\hat{f}$ for all $f \in L_2(G)$. As the range of T is contained in $L_2(G)$, we have $E\hat{f} = (Tf)^\wedge \in \widehat{L_2(G)}$, for each $f \in L_2(G)$. E is therefore an $L_2(G)$-multiplier in the sense of Hewitt and Ross [8], (35.1). It follows ([8], (35.16) (g)) that $E \in \mathfrak{C}_\infty(\Sigma)$.

This concludes the proof of the lemma.

* * 2.6.6 <u>Theorem:</u> Let G be a compact group with dual object Σ, and let T be a bounded linear operator on $L_2(G)$. T commutes with right-translations iff T commutes with the projection P_σ, for each $\sigma \in \Sigma$.

<u>Proof:</u> Suppose T commutes with right-translations. By lemma 2.6.5, there exists $E \in \mathfrak{C}_\infty(\Sigma)$ such that $(Tf)^\wedge = E\hat{f}$ for each $f \in L_2(G)$. We show that T commutes with P_σ for each $\sigma \in \Sigma$. As in the proof of Lemma 2.6.5, let σ_0 be fixed (but arbitrary in Σ), and let $f \in L_2(G)$.

** If part of the theorem is false, It will be corrected in due course by the author.

$$\left[T(P_{\sigma_0} f)\right]^{\wedge}(\sigma) = E_\sigma (P_{\sigma_0} f)^{\wedge}(\sigma)$$

$$= \begin{cases} E_{\sigma_0} \hat{f}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases} \quad \text{(by lemma 2.6.2)}$$

$$= \begin{cases} (Tf)^{\wedge}(\sigma_0) & \text{if } \sigma = \sigma_0 \\ 0 & \text{if } \sigma \neq \sigma_0 \end{cases}$$

$$= \left[P_{\sigma_0}(Tf)\right]^{\wedge}(\sigma) \quad \text{(lemma 2.6.2)}$$

for each $\sigma \in \Sigma$. By the uniqueness of the Fourier transform, we have $T(P_{\sigma_0} f) = P_{\sigma_0}(Tf)$. Hence T commutes with P_{σ_0}. It follows that T commutes with P_σ for each $\sigma \in \Sigma$.

Conversely, suppose T commutes with P_σ for each $\sigma \in \Sigma$. Then M_σ is invariant under T, for each $\sigma \in \Sigma$. In particular, $Tu_{jk}^{(\sigma)} \in M_\sigma$ for each coordinate function $u_{jk}^{(\sigma)} \in \mathcal{L}_\sigma(G)$. Therefore, either

(2.6.26) $$Tu_{jk}^{(\sigma)} = \sum_{r,s=1}^{d_\sigma} \alpha_{rs}^{(\sigma)} u_{rs}^{(\sigma)}$$

for some complex numbers $\alpha_{rs}^{(\sigma)}$, $r, s = 1, 2, \ldots, d_\sigma$, or

(2.6.27) $$Tu_{jk}^{(\sigma)} = \lim_n u_n$$

where limit is taken in the sense of $L_2(G)$ - norm, and each u_n is of the form (2.6.26), say

(2.6.28) $$u_n = \sum_{r,s=1}^{d_\sigma} \alpha_{rsn}^{(\sigma)} u_{rs}^{(\sigma)}$$

Let $f = \sum_{\sigma'\in\Sigma} \sum_{i,m=1}^{d_{\sigma'}} d_{\sigma'} < \hat{f}(\sigma')\xi_m^{(\sigma')}, \xi_i^{(\sigma')} > u_{im}^{(\sigma')} \in L_2(G).$

$$Tf = \sum_{\sigma'\in\Sigma} \sum_{i,m=1}^{d_{\sigma'}} d_{\sigma'} < \hat{f}(\sigma')\xi_m^{(\sigma')}, \xi_i^{(\sigma')} > Tu_{im}^{(\sigma')}$$

$$= \begin{cases} \sum_{\sigma'\in\Sigma} \sum_{i,m,r,s=1}^{d_{\sigma'}} d_{\sigma'} < \hat{f}(\sigma')\xi_m^{(\sigma')}, \xi_i^{(\sigma')} > \alpha_{rs}^{(\sigma')} u_{rs}^{(\sigma')} & \text{if } (2.6.26) \\ \sum_{\sigma'\in\Sigma} \sum_{i,m,r,s=1}^{d_{\sigma'}} d_{\sigma'} < \hat{f}(\sigma')\xi_m^{(\sigma')}, \xi_i^{(\sigma')} > \lim_n \alpha_{rsn}^{(\sigma')} u_{rs}^{(\sigma')} & \text{if } (2.6.27) \end{cases}$$

Since $< u_{rs}^{(\sigma')}, u_{jk}^{(\sigma)} > = \frac{1}{d_\sigma} \delta_{\sigma\sigma'} \delta_{rj} \delta_{sk}$ ([8], (27.19)), it follows that for each $\sigma\in\Sigma$, and $j, k \in \{1, 2, \ldots, d_\sigma\}$, we have

$$< Tf, u_{jk}^{(\sigma)} > = \begin{cases} \sum_{i,m=1}^{d_\sigma} < \hat{f}(\sigma)\xi_m^{(\sigma)}, \xi_i^{(\sigma)} > \alpha_{jk}^{(\sigma)} & \text{if } (2.6.26) \\ \sum_{i,m=1}^{d_\sigma} < \hat{f}(\sigma)\xi_m^{(\sigma)}, \xi_i^{(\sigma)} > \lim_n \alpha_{jkn}^{(\sigma)} & \text{if } (2.6.27) \end{cases}$$

That is, for each $\sigma\in\Sigma$, and $j,k \in \{1, 2, \ldots, d_\sigma\}$, we have

$$(2.6.29) \quad < (Tf)^\wedge(\sigma)\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > = \begin{cases} \sum_{i,m=1}^{d_\sigma} < \hat{f}(\sigma)\xi_m^{(\sigma)}, \xi_i^{(\sigma)} > \alpha_{jk}^{(\sigma)} & \text{if } (2.6.26) \\ \sum_{i,m=1}^{d_\sigma} < \hat{f}(\sigma)\xi_m^{(\sigma)}, \xi_i^{(\sigma)} > \lim_n \alpha_{jkn}^{(\sigma)} & \text{if } (2.6.27) \end{cases}$$

It follows that for $a \in G$, and $f \in L_2(G)$, we have

$$Tf_a = T \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\overline{U}_{a^{-1}}^{(\sigma)}\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > u_{jk}^{(\sigma)} \qquad \text{[by (2.6.21)]}$$

$$= \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)\, \overline{U}_{a^{-1}}^{(\sigma)}\, \xi_k^{(\sigma)}, \xi_j^{(\sigma)} > Tu_{jk}^{(\sigma)} \qquad \text{(T bounded and linear)}$$

$$= \begin{cases} \sum_{\sigma\in\Sigma} \sum_{j,k,r,s=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)(\overline{U}_{a^{-1}}^{(\sigma)}\xi_k^{(\sigma)}), \xi_j^{(\sigma)} > \alpha_{rs}^{(\sigma)} u_{rs}^{(\sigma)} & \text{if } (2.6.26) \\ \sum_{\sigma\in\Sigma} \sum_{j,k,r,s=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)(\overline{U}_{a^{-1}}^{(\sigma)}\xi_k^{(\sigma)}), \xi_j^{(\sigma)} > \lim_n \alpha_{rsn}^{(\sigma)} u_{rs}^{(\sigma)} & \text{if } (2.6.27) \end{cases}$$

$$= \begin{cases} \sum_{\sigma\in\Sigma} \sum_{i,m,j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)(\overline{U}_{a^{-1}}^{(\sigma)}\xi_m^{(\sigma)}), \xi_i^{(\sigma)} > \alpha_{jk}^{(\sigma)} u_{jk}^{(\sigma)} & \text{if } (2.6.26) \\ \sum_{\sigma\in\Sigma} \sum_{i,m,j,k=1}^{d_\sigma} d_\sigma < \hat{f}(\sigma)(\overline{U}_{a^{-1}}^{(\sigma)}\xi_m^{(\sigma)}), \xi_i^{(\sigma)} > \lim_n \alpha_{jkn}^{(\sigma)} u_{jk}^{(\sigma)} & \text{if } (2.6.27) \end{cases}$$

$$= \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < (Tf)^\wedge(\sigma)(\overline{U}_{a^{-1}}^{(\sigma)}\xi_k^{(\sigma)}), \xi_j^{(\sigma)} > u_{jk}^{(\sigma)} \qquad \text{[by (2.6.29)]}$$

$$= \sum_{\sigma\in\Sigma} \sum_{j,k=1}^{d_\sigma} d_\sigma < (Tf)^\wedge(\sigma)\overline{U}_{a^{-1}}^{(\sigma)}\xi_k^{(\sigma)}, \xi_j^{(\sigma)} > u_{jk}^{(\sigma)}$$

$$= (Tf)_a$$

i.e. $Tf_a = (Tf)_a$ for all $a \in G$ and all $f \in L_2(G)$. Hence T commutes with right-translations.

CHAPTER 3

RESTRICTED MULTIPLIERS FOR $L_1(G)$

3.1 Preliminaries

We now turn our attention to the problem raised by Edwards [4], 16.7.5 concerning restricted multipliers. Let G be an infinite compact, not necessarily abelian, group with dual object Σ, and let $L_1(G)$ denote as usual the Banach algebra of all absolutely integrable complex-valued functions on G. As was indicated in the introduction, the $L_1(G)$- multipliers are precisely the Fourier-Stieltjes transforms of measures in $M(G)$ ([8], (35.9), where for each $\mu \in M(G)$ and $\sigma \in \Sigma$, $\hat{\mu}(\sigma)$ is an operator (rather than a complex number) on the finite-dimensional Hilbert space H_σ defined in 2.6. Let $B(H_\sigma)$ be the set of all bounded linear operators on H_σ, and, as in 2.6, let $\mathfrak{C}(\Sigma)$ be the operator algebra $\mathbf{P}_{\sigma\in\Sigma} B(H_\sigma)$. For a given subset S of Σ, we form the subalgebra $\mathfrak{C}(S) = \mathbf{P}_{\sigma\in S} B(H_\sigma)$ of $\mathfrak{C}(\Sigma)$, where the norms in $\mathfrak{C}(S)$ are simply the norms $\|\cdot\|_p$, $1 \leq p < \infty$, of $\mathfrak{C}(\Sigma)$, restricted to S([8], (35.6)).

3.1.1 Definition: An $\Xi = (E_\sigma)$ in $\mathfrak{C}(S)$ is called a function of type $(L_1(G), L_1(G), S)$ if to each $f \in L_1(G)$ corresponds at least one $g \in L_1(G)$ such that

$$\hat{g}(\sigma) = E_\sigma \hat{f}(\sigma) \qquad (3.1.1)$$

for each $\sigma \in S$.

Of course, for $X, Y \in \{C(G), L_p(G);\ 1 \leq p < \infty\}$, one can similarly define E to be a function of type (X, Y, S) if to each $x \in X$ corresponds at least one $y \in Y$ satisfying $\hat{y}(\sigma) = E_\sigma \hat{x}(\sigma)$, $\sigma \in S$.

3.1.2 Remark: It is clear that an $L_1(G)$- multiplier, or its restriction to S, is a function of type $(L_1(G), L_1(G), S)$ for any subset S of Σ. In fact, the original multiplier problem for $L_1(G)$ corresponds exactly to the case in which $S = \Sigma$. The non-trivial problem is in deciding whether, conversely, every function of type $(L_1(G), L_1(G), S)$ is the restriction to S of some $L_1(G)$-multiplier. This problem has been solved in the case where G is a locally compact abelian group by Brained and Edwards ([1], Part II, Theorem 3.3). In this chapter, we propose to solve the problem for the case where G is a compact, but not necessarily abelian, group.

3.2. Restricted $L_1(G)$- multipliers

Let G be an infinite compact group with dual object Σ, and let M(G) denote the Banach algebra of all bounded regular Borel measures on G. For $\mu \in M(G)$ and $\sigma \in \Sigma$, $\hat{\mu}(\sigma)$ is defined by

$$(3.2.1) \qquad < \hat{\mu}(\sigma)\xi, \eta > = \int_G < \overline{U}_a^{(\sigma)}\xi , \eta > d\mu(a)$$

for all $\xi, \eta \in H_\sigma$. By [8], (28.36), $\hat{\mu} \in \mathfrak{C}_\infty(\Sigma)$ for each $\mu \in M(G)$. Let λ denote the Haar measure on G. If $\mu \in M(G)$ is absolutely continuous with respect to λ so that $d\mu = f\, d\lambda$ for some $f \in L_1(G)$, we write $\hat{f}$ for $\hat{\mu}$ and call $\hat{f}$ the Fourier transform of f. Thus

$$(3.2.2) \qquad \langle \hat{f}(\sigma)\xi, \eta \rangle = \int_G \langle \overline{U}_a^{(\sigma)}\xi, \eta \rangle f(a)d\lambda(a)$$

for all $\xi, \eta \in H_\sigma$ and $f \in L_1(G)$. If $\mu \in M(G)$ and $f \in L_1(G)$, we define $\mu * f$, the convolution of μ and f, by

$$(3.2.3) \qquad (\mu * f)(a) = \int_G f(ax^{-1})d\mu(x)$$

for each $a \in G$. By [8], (20.5), $\mu * f \in L_1(G)$ for all $\mu \in M(G)$ and $f \in L_1(G)$.

We shall need the following lemmas:

3.2.1 __Lemma:__ If E is a function of type $(L_1(G), L_1(G), S)$, then E is bounded, i.e. $\|E\|_\infty < \infty$.

__Proof:__ Suppose that E is not bounded, i.e. $\|E\|_\infty = \sup_{\sigma \in S} \|E_\sigma\|_{\phi_\infty}$ is not finite. Then for any number m, however large, there is a $\sigma_m \in S$ such that $\|E_{\sigma_m}\|_{\phi_\infty} > m$. Let $t \in L_1(G)$ be such that $\hat{t}(\sigma_m)$ is the identity operator on H_{σ_m} ([8], Theorem (28.39)(i)); then $E_{\sigma_m}\hat{t}(\sigma_m) = E_{\sigma_m}$. Now, since E is of type $(L_1(G), L_1(G), S)$, there will exist $g \in L_1(G)$ such that $\hat{g}(\sigma) = E_\sigma \hat{t}(\sigma)$ for all $\sigma \in S$. In particular, $\hat{g}(\sigma_m) = E_{\sigma_m}\hat{t}(\sigma_m) = E_{\sigma_m}$ and so, $\|\hat{g}\|_\infty \geq \|\hat{g}(\sigma_m)\|_{\phi_\infty} = \|E_{\sigma_m}\|_{\phi_\infty} > m$, for any number m. This is impossible, since by [8] Theorem (28.36)(i), $\hat{g} \in \mathfrak{C}_\infty(\Sigma)$. Hence E is bounded.

3.2.2 <u>Lemma</u>: For $a \in G$ and $f \in L_1(G)$, denote by f_a the right-translate of f by $a \in G$. Then,

$$(3.2.4) \qquad (f_a)^\wedge(\sigma) = \hat{f}(\sigma)\overline{U}_{a^{-1}}^{(\sigma)}$$

for each $\sigma \in \Sigma$.

<u>Proof</u>: The proof of this lemma is the same as the proof of lemma 2.6.5.

We shall require a property (C) defined in [1], Part I, Definition 2.8.1. Let G be a compact group, and let J_0 be a closed subspace of $L_1(G)$. J_0 is said to satisfy condition (C) if whenever $\{j_\alpha\}$ is a norm-bounded directed family of elements of J_0 such that $\lim_\alpha j_\alpha = j$ for the topology $\sigma(L_1(G), C_0(G))$, then $j \in J_0$. [Here, $\sigma(M(G), C(G))$ denotes the weak topology on $M(G)$, viewed as the space of continuous linear functionals on $C(G)$ ([1], p. 291).] Brainerd and Edwards show that ([1], p.305) if G is compact and J_0 is a closed subspace of $L_1(G)$ such that $L_1(G) * J_0 \subset J_0$, then J_0 satisfies condition (C). In view of this, we have the following lemma:

3.2.3 <u>Lemma</u>: Let G be a compact group with dual object Σ, and let S be a subset of Σ. Define $J = \{f \in L_1(G): \hat{f}(\sigma) = 0$ for each $\sigma \in S\}$. Then J satisfies condition (C).

<u>Proof</u>: Since G is compact, we only need to show that J is a closed subspace of $L_1(G)$ satisfying $L_1(G) * J \subset J$. It is clear that J is

a subspace of $L_1(G)$. Let $\{j_n\}$ be a sequence in J such that $\lim_n \|j_n - j\|_1 = 0$ for some $j \in L_1(G)$. We show that $j \in J$. For each $\sigma \in S$, we have

$$\|\hat{j}(\sigma)\|_{B(H_\sigma)} = \|\hat{j}(\sigma) - \hat{j}_n(\sigma)\|_{B(H_\sigma)} \quad (\text{since } \hat{j}_n(\sigma) = 0)$$

$$= \|(j - j_n)^\wedge(\sigma)\|_{\phi_\infty} \qquad ([8],\ p.42)$$

$$\leq \|(j - j_n)^\wedge\|_\infty \qquad ([8],\ (28.34))$$

$$\leq \|j - j_n\|_1 \qquad ([8],\ (28.36))$$

Since $\lim_n \|j - j_n\|_1 = 0$, it follows that $\hat{j}(\sigma) = 0$ for each $\sigma \in S$. Hence $j \in J$. This means that J is closed. Finally, let $f \in L_1(G)$ and $j \in J$ be arbitrary. For each $\sigma \in S$, $(f * j)^\wedge(\sigma) = \hat{f}(\sigma)\hat{j}(\sigma) = 0$ since $\hat{j}(\sigma) = 0$. It follows that $f * j \in J$. Hence $L_1(G) * J \subset J$.

Now, let J be as in lemma 3.2.3. We form the quotient space $L_1(G)/_J$ in the usual manner. Denote by $[f]$ the coset modulo J of the element $f \in L_1(G)$, i.e. $[f] = \{f + j : j \in J\}$. For $[f],[g] \in L_1(G)/_J$ and complex number α, define

$$[f] + [g] = [f + g]$$

$$\alpha[f] = [\alpha f],$$

and define norm in $L_1(G)/_J$ by

$$\|[f]\| = \inf_{j \in J} \|f + j\|_1 .$$

Since J is closed, $L_1(G)/_J$ is a Banach space. We define right-translation in $L_1(G)/_J$ by

$$[f]_a = \{g_a \in L_1(G) : g \in [f]\}$$

for all $a \in G$ and $[f] \in L_1(G)/_J$. This is well-defined, as the next lemma shows.

3.2.4 <u>Lemma</u>: Let $[f] \in L_1(G)/_J$ and let $a \in G$. Then

$$[f]_a = [f_a]$$

<u>Proof</u>: We show that $g \in [f]_a$ if and only if $g \in [f_a]$. Suppose $g \in [f]_a$; then $g = (f + j)_a$ for some $j \in J$. By lemma 3.2.2, $(j_a)^\wedge(\sigma) = \hat{j}(\sigma)\, \overline{U}^{(\sigma)}_{a^{-1}}$. Since $\hat{j}(\sigma) = 0$ for each $\sigma \in S$, it follows that $(j_a)^\wedge(\sigma) = 0$ for each $\sigma \in S$, implying $j_a \in J$ for each $a \in G$. In short, J is invariant under right-translations. Thus $g = (f + j)_a = f_a + j_a = f_a + j'$ for some $j' \in J$, i.e. $g = f_a + j'$. This implies $g \in [f_a]$.

Conversely, suppose $g \in [f_a]$; then $g = f_a + j''$ for some j'' in J. Now, $g_{a^{-1}} = (f_a + j'')_{a^{-1}} = (f_a)_{a^{-1}} + (j'')_{a^{-1}} = f + j''_{a^{-1}} = f + j'''$ for some $j''' \in J$, by the invariance of J under right-translations. Hence, $g = (g_{a^{-1}})_a = (f + j''')_a \in [f]_a$. This completes the proof of the lemma.

Thus, for each $a \in G$, the mapping $[f] \to [f]_a$ of $L_1(G)/_J$ into itself is well-defined. An operator $T : L_1(G) \to L_1(G)/_J$ is said to commute with right-translations if $Tf_a = (Tf)_a$ for all $a \in G$ and $f \in L_1(G)$.

The next lemma is a consequence of [1], Part I, Theorem 2.9.

3.2.5 <u>Lemma:</u> Let S be a subset of Σ, and let $J = \{f \in L_1(G): \hat{f}(\sigma) = 0$ for each $\sigma \in S\}$. If $T : L_1(G) \to L_1(G)/_J$ is a continuous linear operator commuting with right-translations, then there exists $\mu \in M(G)$ such that

$$(3.2.5) \qquad Tf = [\mu * f]$$

for each $f \in L_1(G)$.

<u>Proof:</u> By lemma 3.2.3, $L_1(G) * J \subset J$ and J satisfies condition (C). The theorem now follows from [1], Part I, Theorem 2.9.

We are now in a position to prove the main theorem.

3.2.6 <u>Theorem:</u> Let G be a compact group with dual object Σ, and let S be a subset of Σ. The element E of $\phi(S)$ is a function of type $(L_1(G), L_1(G), S)$ if and only if there exists $\mu \in M(G)$ such that

$$(3.2.6) \qquad E_\sigma = \hat{\mu}(\sigma)$$

for each $\sigma \in S$.

<u>Proof:</u> Given $S \subset \Sigma$, we define

$$J = \{f \in L_1(G) : \hat{f}(\sigma) = 0 \text{ for all } \sigma \in S\}$$

and form the quotient space $L_1(G)/_J$ as above. Suppose E is a function of type $(L_1(G), L_1(G), S)$. We associate with E an operator $T : L_1(G) \to L_1(G)/_J$ defined as follows. Let $f \in L_1(G)$; there exists $g \in L_1(G)$ such that $\hat{g}(\sigma) = E_\sigma \hat{f}(\sigma)$ for each $\sigma \in S$. We define $Tf = [g]$, and check immediately that T is well-defined. Let

$g_1, g_2 \in L_1(G)$ be such that $Tf = [g_1]$ and $Tf = [g_2]$. This implies $\hat{g}_1(\sigma) = E_\sigma \hat{f}(\sigma) = \hat{g}_2(\sigma)$ for each $\sigma \in S$. Hence $(g_1 - g_2)^\wedge(\sigma) = 0$ for each $\sigma \in S$. Thus $g_1 - g_2 \in J$. It follows that $g_1 \in [g_2]$ and $g_2 \in [g_1]$, implying $[g_1] = [g_2]$. Hence if $f, g \in L_1(G)$ are such that $\hat{g}(\sigma) = E_\sigma \hat{f}(\sigma)$ for each $\sigma \in S$, it is in order to put $Tf = [g]$.

We show that T is linear, continuous and commutes with right-translations. Let $f_1, f_2 \in L_1(G)$ and let α_1, α_2 be complex numbers. We have $Tf_1 = [g_1]$ and $Tf_2 = [g_2]$ for some $g_1, g_2 \in L_1(G)$ satisfying $\hat{g}_1(\sigma) = E_\sigma \hat{f}_1(\sigma)$ and $\hat{g}_2(\sigma) = E_\sigma \hat{f}_2(\sigma)$ for each $\sigma \in S$. Now, if $\sigma \in S$, then $(\alpha_1 g_1 + \alpha_2 g_2)^\wedge(\sigma) = \alpha_1 \hat{g}_1(\sigma) + \alpha_2 \hat{g}_2(\sigma) = \alpha_1 E_\sigma \hat{f}_1(\sigma) + \alpha_2 E_\sigma \hat{f}_2(\sigma)$ $= E_\sigma(\alpha_1 f_1 + \alpha_2 f_2)^\wedge(\sigma)$. Hence $T(\alpha_1 f_1 + \alpha_2 f_2) = [\alpha_1 g_1 + \alpha_2 g_2]$, by the action of T. Thus $T(\alpha_1 f_1 + \alpha_2 f_2) = [\alpha_1 g_1 + \alpha_2 g_2] = \alpha_1 [g_1] + \alpha_2 [g_2] = \alpha_1 Tf_1 + \alpha_2 Tf_2$, for all $f_1, f_2 \in L_1(G)$ and all complex numbers α_1, α_2. It follows that T is a linear operator.

To show that T is continuous we apply the closed graph theorem ([15], p. 238). Let $\{f_n\}$ be a sequence in $L_1(G)$ such that $f_n \to f$ and $Tf_n \to [g]$, in norm; we show that $Tf = [g]$. First, consider f_1; there will , ingeneral, be several g's in $L_1(G)$ (depending on f_1) such that $\hat{g}(\sigma) = E_\sigma \hat{f}_1(\sigma)$ for each $\sigma \in S$. These g's all belong to the same coset modulo J; for if g' and g'' in $L_1(G)$ are such that for each $\sigma \in S$, $(g')^\wedge(\sigma) = E_\sigma \hat{f}_1(\sigma)$ and

$(g'')^\wedge(\sigma) = E_\sigma \hat{f}_1(\sigma)$, then $(g' - g'')^\wedge(\sigma) = 0$ for each $\sigma \in S$. This implies $g' - g'' \in J$, and consequently g' and g'' belong to the same coset modulo J. Denote by g_1 an arbitrary but fixed member of the coset $\{g \in L_1(G): \hat{g}(\sigma) = E_\sigma \hat{f}_1(\sigma), \sigma \in S\}$, so that we may write $Tf_1 = [g_1]$. We do the same for f_2, f_3, f_4, and so on. We then have a sequence $\{g_n\}$ in $L_1(G)$ such that for each n, $g_n(\sigma) = E_\sigma \hat{f}_n(\sigma)$, $\sigma \in S$. Now, $||[g-g_n]||=||[g]-[g_n]|| = ||[g]- Tf_n|| \to 0$ as $n \to \infty$. This means that $\alpha_n = \inf_{j \in J}||g-g_n+j||_1 \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, $\exists$ N such that $\alpha_n < \frac{\varepsilon}{2}$ if $n > N$. Let n_0 be a fixed integer greater than N. By the definition of inf, $\alpha_{n_0} \leq ||g-g_{n_0} + j||_1$ for all $j \in J$, and if m is a positive integer, there will exist $j_m^{(n_0)} \in J$ such that $||g-g_{n_0} + j_m^{(n_0)}||_1 < \alpha_{n_0} + \frac{1}{m}$. Let M be a positive integer such that $\frac{1}{m} < \frac{\varepsilon}{2}$ for all $m > M$. Then, if $m > M$, we shall have

$$||g-g_{n_0} + j_m^{(n_0)}||_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} .$$

Consider the sequence $\{j_n^{(n)}\}$ in J formed by the elements of the diagonal of the matrix $(j_m^{(n)})$, where m, n = 1, 2, 3, ... If $n > \max(M, N)$, then $||g-g_n+ j_n^{(n)}||_1 < \varepsilon$. Hence $||g-g_n+ j_n^{(n)}||_1 \to 0$ as $n \to \infty$. Moreover, since $j_n^{(n)} \in J$ for all n, then for each $\sigma \in S$, we have $\hat{g}_n(\sigma) = \hat{g}_n(\sigma) - (j_n^{(n)})^\wedge(\sigma)$. But for each $\sigma \in S$,

$$||\hat{g}(\sigma)-E_\sigma \hat{f}(\sigma)||_{B(H_\sigma)} \leq ||\hat{g}(\sigma)-\hat{g}_n(\sigma)||_{B(H_\sigma)} + ||\hat{g}_n(\sigma)-E_\sigma \hat{f}(\sigma)||_{B(H_\sigma)}$$

$$= \|\hat{g}(\sigma) - \hat{g}_n(\sigma) + (j_n^{(n)})^\wedge(\sigma)\|_{B(H_\sigma)} + \|E_\sigma \hat{f}_n(\sigma) - E_\sigma \hat{f}(\sigma)\|_{B(H_\sigma)}$$

$$= \|(g-g_n + j_n^{(n)})^\wedge(\sigma)\|_{\phi_\infty} + \|E_\sigma(f_n - f)^\wedge(\sigma)\|_{\phi_\infty} \qquad ([8], \text{ D.42})$$

$$\leq \|(g-g_n+j_n^{(n)})^\wedge\|_\infty + \|E(f_n - f)^\wedge\|_\infty \qquad ([8], (28.34))$$

$$\leq \|g-g_n + j_n^{(n)}\|_1 + \|E\|_\infty \|f_n - f\|_1 \qquad \text{[by [8], (28.36), and lemma 3.2.1]}$$

Since $\|E\|_\infty < \infty$ by lemma 3.2.1, and as $n \to \infty$, $\|g-g_n + j_n^{(n)}\|_1 \to 0$ and $\|f_n - f\|_1 \to 0$, it follows that $\hat{g}(\sigma) - E_\sigma \hat{f}(\sigma) = 0$, for each $\sigma \in S$. That is, $\hat{g}(\sigma) = E_\sigma \hat{f}(\sigma)$ for each $\sigma \in S$. By the definition of T, we have $Tf = [g]$. Thus T has a closed graph. By the closed graph theorem, T is continuous.

To show that T commutes with right-translations, we show that $Tf_a = (Tf)_a$ for all $a \in G$ and $f \in L_1(G)$. Let f be arbitrary in $L_1(G)$ and let $Tf = [g]$ for some $g \in L_1(G)$ satisfying $\hat{g}(\sigma) = E_\sigma \hat{f}(\sigma)$ for each $\sigma \in S$. Now,

$(g_a)^\wedge(\sigma) = \hat{g}(\sigma)\overline{U}_{a^{-1}}^{(\sigma)} = E_\sigma \hat{f}(\sigma)\overline{U}_{a^{-1}}^{(\sigma)} = E_\sigma(f_a)^\wedge(\sigma)$ for each $\sigma \in S$ and $a \in G$. It follows that $Tf_a = [g_a]$, $a \in G$. Thus $Tf_a = [g_a] = [g]_a = (Tf)_a$ for all $a \in G$, $f \in L_1(G)$. So, $T : L_1(G) \to L_1(G)/_J$ is a continuous linear operator which commutes with right-translations. By lemma 3.2.5, there exists $\mu \in M(G)$ such that $Tf = [\mu * f]$ for each f in $L_1(G)$. By the definition of T, we must have $(\mu * f)^\wedge(\sigma) = E_\sigma \hat{f}(\sigma)$ for each $\sigma \in S$, i.e.

$\hat{\mu}(\sigma)\hat{f}(\sigma) = E_\sigma \hat{f}(\sigma)$ for each $\sigma \in S$ and $f \in L_1(G)$.

For each $\sigma \in S$, let $t_\sigma \in L_1(G)$ be such that $\hat{t}_\sigma(\sigma) = I_{d_\sigma}$, the identity operator on H_σ([8], (28.39)(i)). Then, $E_\sigma(t_\sigma)^\wedge(\sigma) = \hat{\mu}(\sigma)(t_\sigma)^\wedge(\sigma)$ for each $\sigma \in S$. It follows that $E_\sigma = \hat{\mu}(\sigma)$ for each $\sigma \in S$, i.e. E is the restriction of $\hat{\mu}$ to S. We have thus proved that a function E of type $(L_1(G), L_1(G), S)$ is the restriction to S of an $L_1(G)$-multiplier. The converse holds trivially, by Remark 3.1.2.

3.2.10 Remark: We wish to compare our result, Theorem 3.2.9, proved for compact not necessarily abelian groups, with the exact analogue for locally compact abelian groups, proved by Brainerd and Edwards ([1], Part II, Theorem 3.3). We feel that ours is slightly more general than their result, in the sense that we do not require the condition $S \subset \overline{\text{Int } S}$ (as they do), nor any condition on S for that matter.

CHAPTER 4

ONE - PARAMETER SEMIGROUPS OF OPERATORS ON A BANACH SPACE X

4.1 Preliminaries

Let X be a complex Banach space and let $B(X)$ be the complex Banach algebra of all bounded linear operators on X. For $0 < \xi < \infty$, let $T(\xi)$ be an operator in $B(X)$. The collection $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is said to be a semigroup of operators on X if

(4.1.1) $$T(\xi_1 + \xi_2) = T(\xi_1)\, T(\xi_2)$$

for all $\xi_1, \xi_2 > 0$, i.e. $T(\xi_1 + \xi_2)x = T(\xi_1)\,[T(\xi_2)x]$ for all $x \in X$ and $\xi_1, \xi_2 > 0$. As X may carry the weak, strong or uniform operator topology, the continuity or measurability of the operators $T(\xi)$ is defined relative to the topology on X. Let X^* denote the space of all continuous linear functional on X. $\mathcal{J}$ is said to be weakly measurable if $\psi(T(\xi)x)$ is Lebesgue measurable for all $x \in X$ and $\psi \in X^*$; $\mathcal{J}$ is said to be strongly continuous if

(4.1.2) $$\lim_{\xi \to \xi_0} \|T(\xi)x - T(\xi_0)x\| = 0$$

for each $x \in X$ and all $\xi_0 > 0$. Furthermore, $\mathcal{J}$ is said to be uniformly continuous if

(4.1.3) $$\lim_{\xi \to \xi_0} \|T(\xi) - T(\xi_0)\| = 0$$

for all $\xi_0 > 0$.

The <u>infinitesimal operator</u> A_0 of $\mathcal{J}$ is defined by

$$A_0 x = \lim_{\xi \to 0^+} \frac{1}{\xi}(T(\xi)x - x) \tag{4.1.4}$$

for all $x \in X$ for which this limit exists. A_0 is in general an unbounded operator on X. Denote the domain of A_0 by $D(A_0)$; then $D(A_0)$ is dense in $X_0 = \{T(\xi)x : x \in X, \xi > 0\}$. Moreover, A_0 is in general not closed; its closure A, when it exists is called the <u>infinitesimal generator</u> of $\mathcal{J}$. The infinitesimal generator plays a fundamental role in the theory of semigroups of operators. For example if $\mathcal{J}$ is uniformly continuous, then ([10], p.278) a unique infinitesimal generator A exists, A is bounded, and we have $T(\xi) = e^{\xi A}$, where $e^{\xi A}$ is the usual exponential function $\sum_{k=0}^{\infty} \frac{(\xi A)^k}{k!}$. In the case where $\mathcal{J}$ is strongly continuous, a unique infinitesimal generator A also exists, but A is now an unbounded linear operator on X whose domain D(A) is merely dense in X, and therefore the symbol $e^{\xi A}$ must be redefined. Both cases have significant applications, but the case of strong continuity is by far the more interesting to the analyst, since it offers more difficult problems and calls for more refined analysis.

As pointed out in the introduction the sort of multipliers we consider are such that give rise to strongly continuous semigroups of operators. Let $\mathcal{J} = \{T(\xi) : \xi > 0\}$ be a strongly continuous semigroup of operators on X, and let A be its infinitesimal generator.

Furthermore let $\omega_0 = \inf_{\xi>0} \frac{1}{\xi} \log\|T(\xi)\|$ be the type of $\mathcal{J}$ ([10],p.306). For any complex number λ with $\mathrm{Re}\lambda > \omega_0$, let $R(\lambda : A)$ denote the resolvent of A([10], chapter XI). Since $\mathcal{J}$ is strongly continuous, there exists $\omega_1 > \omega_0$ ([10], p. 342), such that for λ with $\mathrm{Re}\lambda > \omega_1$, we have

$$(4.1.5) \qquad R(\lambda: R)x = \int_0^\infty e^{-\xi\lambda} T(\xi)x \, d\xi$$

for each $x \in X_0$. A comprehensive account of semigroups of operators on Banach spaces is found in Hille and Phillips [10], where all undefined terms used in this thesis, in connection with such semigroups, are explained.

Before we take up one by one the Banach spaces X mentioned in the introduction, we wish to state a modification to our definition of a multiplier, to cover the multipliers which involve only subsets of X, and not the whole of X. If F is a subset of X, we denote by $\hat{F}$ the set of all Fourier transforms $\hat{f}$ where $f \in F$. Whether X is AP(G) or an abstract commutative Banach algebra, or an abstract Hilbert space, the Fourier transform (or an equivalent transform) of an $f \in F$ is well defined. Let F_1 and F_2 be subsets of X. A complex-valued function ϕ defined on $\hat{G}$ (taking X = AP(G)) is called an (F_1, F_2)-multiplier ([12], [8]) if $\phi\hat{f} \in \hat{F}_2$ for each $f \in F_1$. The former definition of a multiplier for X then corresponds to the case $F_1 = F_2 = X$.

In the proofs of our results in this section, we repeatedly make use of the following lemma:

4.1.1 Lemma Let $\phi(\xi)$ be a complex-valued Lebesgue measurable function of $\xi \in (0, \infty)$. If $\phi(\xi)$ satisfies

(4.1.6) $$\phi(\xi_1 + \xi_2) = \phi(\xi_1)\phi(\xi_2)$$

for all $\xi_1, \xi_2 \in (0, \infty)$, then either $\phi(\xi)$ is identically zero, or there is a complex number ν such that $\phi(\xi) = e^{\xi\nu}$ for all $\xi > 0$.

Proof: See corollary to Theorem 4.17.3 of [10].

4.2 The case X = AP(G)

Let G be a locally compact abelian group with character group $\hat{G}$. A complex-valued function f on G is called almost periodic if it is the uniform limit (in the norm of $C(G)$) of a sequence of trigonometric polynomials on G. We denote by $AP(G)$ the linear space, under pointwise addition and scalar multiplication, of all almost periodic functions on G. If G is compact then $AP(G)$ coincides with $C(G)$. In defining convolution and Fourier transforms in $AP(G)$, we follow Helgason [6]. For $f, g \in AP(G)$,

(4.2.1) $$(f * g)(a) = M_s\{f(as^{-1})g(s)\}$$

and

(4.2.2) $$\hat{f}(\sigma) = M_s\{f(s)\,\overline{\chi_\sigma(s)}\}$$

where M is the linear form on $AP(G)$ defined by von Neumann ([18], p.451), and we set $\widehat{AP(G)} = \{\hat{f} : f \in AP(G)\}$. The regular maximal ideals of the Banach algebra $AP(G)$ are of the form M_σ, $\sigma \in \hat{G}$, where $M_\sigma = \{f \in AP(G) : \hat{f}(\sigma) = 0\}$; thus the Fourier transform is unique if and only if $AP(G)$ is semi-simple. Helgason makes use of a compactification G_c of G introduced by van Kampen [17]. Let $\hat{G}_d$ denote the group $\hat{G}$ with the discrete topology, and set $G_c = \widehat{(\hat{G}_d)}$, the compact character group of $\hat{G}_d$ (Pontryagin's duality theorem). Helgason then shows that $AP(G)$ is isomorphic to $C(G_c)$; moreover, the isomorphism carries over to their conjugate spaces, so that $[AP(G)]^* = [C(G_c)]^* = M(G_c)$, the space of all bounded regular complex-valued Borel measures on G_c. The $(AP(G), \widehat{AP(G)})$ - multipliers are isomorphic to the bounded linear operators on $AP(G)$ which commute with translations ([6], p. 57), where a bounded linear operator T on $AP(G)$ which commutes with translations corresponds to an $(AP(G), \widehat{AP(G)})$ - multiplier ϕ under the isomorphism if and only if $(Tf)^\wedge = \phi\hat{f}$ for each $f \in AP(G)$.

The following are the main theorems in this section:

4.2.1 <u>Theorem</u>: For each $\xi > 0$, let $T(\xi)$ be a bounded linear operator on $AP(G)$ which commutes with translations, and let $\mathcal{J} = \{T(\xi) : \xi > 0\}$ be a semigroup of operators. Then $\mathcal{J}$ determines a semigroup $\{\phi_\xi : \xi > 0\}$ of $(AP(G), \widehat{AP(G)})$ - multipliers such that

(i) for each $\xi > 0$, $\phi_\xi \hat{f} = [T(\xi)f]^\wedge$ for all $f \in AP(G)$,

and

(ii) $\phi_{\xi_1+\xi_2}(\sigma) = \phi_{\xi_1}(\sigma)\phi_{\xi_2}(\sigma)$ for all $\xi_1, \xi_2 > 0$ and $\sigma \in \hat{G}$.

If, in addition, $\mathcal{J}$ is weakly measurable, then there exists a subset $\hat{G}_0$ of $\hat{G}$, and a mapping $\nu : \sigma \to \nu(\sigma)$ of $\hat{G}_0$ into K, the field of complex numbers, such that

$$\phi_\xi(\sigma) = \begin{cases} e^{\xi\nu(\sigma)} & \text{if } \sigma \in \hat{G}_0 \\ 0 & \text{if } \sigma \notin \hat{G}_0 \end{cases}$$

<u>Proof</u>: Since the bounded linear operator $T(\xi)$, $\xi > 0$, on AP(G) commutes with translation, there exists an (AP(G), AP(G))-multiplier ϕ_ξ([6], Theorem 1) such that $[T(\xi)f]^\wedge = \phi_\xi \hat{f}$ for each $f \in AP(G)$. Statement (ii) of the theorem follows from the semigroup property of $\mathcal{J}$, since for all $\xi_1, \xi_2 > 0$ and $f \in AP(G)$ we have

$$\phi_{\xi_1+\xi_2}\hat{f} = [T(\xi_1+\xi_2)f]^\wedge = [T(\xi_1)(T(\xi_2)f)]^\wedge = \phi_{\xi_1}[T(\xi_2)f]^\wedge = \phi_{\xi_1}\phi_{\xi_2}\hat{f}$$

Suppose now that $\mathcal{J}$ is weakly measurable; then for each $f \in AP(G)$, $\psi \in [AP(G)]^*$, the mapping $\xi \to \psi(T(\xi)f)$ is Lebesgue measurable. In particular, if for each $\sigma \in \hat{G}$, we define $\psi_\sigma \in [AP(G)]^*$ by

$$\psi_\sigma(f) = \hat{f}(\sigma) \qquad (f \in AP(G))$$

then the mapping $\xi \to \psi_\sigma(T(\xi)\chi_\sigma) = [T(\xi)\chi_\sigma]^\wedge(\sigma) = \phi_\xi(\sigma)\hat{\chi}_\sigma(\sigma) = \phi_\xi(\sigma)$ is measurable, for each $\sigma \in \hat{G}$. That is, for each $\sigma \in \hat{G}$, the function $\phi_\xi(\sigma)$ is measurable. Since also, $\phi_{\xi_1+\xi_2}(\sigma) = \phi_{\xi_1}(\sigma)\phi_{\xi_2}(\sigma)$ for all $\xi_1, \xi_2 > 0$, lemma 4.1.1 implies that for each $\sigma \in \hat{G}$, either $\phi_\xi(\sigma)$ is identically zero or $\phi_\xi(\sigma) = e^{\xi\nu(\sigma)}$ for some complex number $\nu(\sigma)$. Now set $\hat{G}_0 = \{\sigma \in \hat{G} : \phi_\xi(\sigma)$ is not identically zero$\}$, and the proof is complete.

The next theorem is a converse to Theorem 3.2.1. We suppose now that $\hat{G}_0$ is a fixed subset of $\hat{G}$ and ν is a complex-valued function defined on $\hat{G}_0$. For each $\xi > 0$, we define a function ϕ_ξ on $\hat{G}$ by

$$(4.2.3) \qquad \phi_\xi(\sigma) = \begin{cases} e^{\xi\nu(\sigma)} & \text{if } \sigma \in \hat{G}_0 \\ 0 & \text{if } \sigma \notin \hat{G}_0 \end{cases},$$

and assume that ϕ_ξ is an $(AP(G), AP(G))$ - multiplier. Then we have the following:

4.2.2 <u>Theorem:</u> For each $\xi > 0$, define a mapping $T(\xi)$ of $AP(G)$ into itself by

$$(4.2.4) \qquad [T(\xi)f]^\wedge = \phi_\xi \hat{f}$$

for each $f \in AP(G)$. Then,

(i) $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is a semigroup of bounded linear operators on $AP(G)$, the elements of which commute with translations. Moreover, $\mathcal{J}$ is strongly continuous.

(ii) Let A_0, with domain $D(A_0)$, denote the infinitesimal operator of $\mathcal{J}$. Then, for each $f \in D(A_0)$ and $\sigma \notin \hat{G}_0$, we have $\hat{f}(\sigma) = 0$; moreover, $(A_0 f)^\wedge = \nu\hat{f}$ for all $f \in D(A_0)$, which implies that ν is a $(D(A_0), AP(G))$ - multiplier.

(iii) If $\mathcal{J}$ is of class (A), then $\hat{G}_0 = \hat{G}$; moreover, if A denotes the infinitesimal generator of $\mathcal{J}$, then for each f in D(A), the domain of A, we have $(Af)^\wedge = \nu\hat{f}$ (ν is a $(D(A), AP(G))$ - multiplier), and $D(A) = \{f \in AP(G) : \nu\hat{f} \in \widehat{AP(G)}\}$.

Proof(i) That, for each $\xi > 0$, $T(\xi)$ is a bounded linear operator commuting with translations follows from Helgason [6], p.57. The semigroup property is immediate from the definition of $T(\xi)$. We show that $\mathcal{J}$ is strongly continuous. First, suppose that t is a trigonometric polynomial on G, say t is of the form

$t = \alpha_1\chi_{\sigma_1} + \alpha_2\chi_{\sigma_2} + \dots + \alpha_n\chi_{\sigma_n}$. The orthogonality relations of the characters in $\hat{G}$ imply that

$$[T(\xi)t]^\wedge(\sigma) = \begin{cases} \alpha_k e^{\xi\nu(\sigma_k)} & \text{if } \sigma = \sigma_k,\ k=1,2,\dots,n; \\ 0 & \text{otherwise} \end{cases}$$

The Fourier series of $T(\xi)t$ therefore reduces to

$$T(\xi)t = \alpha_1 e^{\xi\nu(\sigma_1)}\chi_{\sigma_1} + \dots + \alpha_n e^{\xi\nu(\sigma_n)}\chi_{\sigma_n}$$

We then have, for all $\xi_0 > 0$,

$$\|T(\xi)t - T(\xi_0)t\| = \|(\alpha_1 e^{\xi\nu(\sigma_1)}\chi_{\sigma_1} - \alpha_1 e^{\xi_0\nu(\sigma_1)}\chi_{\sigma_1}) + \dots + (\alpha_n e^{\xi\nu(\sigma_n)}\chi_{\sigma_n} - \alpha_n e^{\xi_0\nu(\sigma_n)}\chi_{\sigma_n})\|$$

$$\leq |\alpha_1|\,|e^{\xi\nu(\sigma_1)} - e^{\xi_0\nu(\sigma_1)}| + \dots + |\alpha_n|\,|e^{\xi\nu(\sigma_n)} - e^{\xi_0\nu(\sigma_n)}|$$

$$\to 0$$

as $\xi \to \xi_0$. Each character χ_σ is strongly measurable. Since $T(\xi)t$ is a finite linear combination of characters, $T(\xi)t$ is also strongly measurable ([10], Theorem 3.5.4). Suppose now that f is arbitrary in AP(G), and let $\varepsilon > 0$ be given. By the definition of AP(G), there exists a trigonometric polynomial t on G such that $\|f - t\| < \varepsilon$. Since $\|T(\xi)f - T(\xi)t\| \leq \|T(\xi)\|\cdot\|f-t\| < \|T(\xi)\|\cdot\varepsilon$, and $T(\xi)t$ is strongly measurable, it follows by [10] Theorem 3.5.4, that $T(\xi)f$ is strongly measurable for all $f \in AP(G)$. Hence, by [10] Theorem 10.2.3, $\mathcal{J}$ is strongly continuous. This completes the proof of (i).

(ii) Let $f \in D(A_0)$; then there exists $g = A_0 f$ in AP(G) such that $\lim_{\xi\to 0^+} \frac{1}{\xi}[T(\xi)f - f] = g$, in norm. For each $\sigma \in \hat{G}$,

$\lim_{\xi\to 0^+} \frac{1}{\xi}[(T(\xi)f)^\wedge(\sigma) - \hat{f}(\sigma)] = \hat{g}(\sigma)$, i.e. $\lim_{\xi\to 0^+} \frac{1}{\xi}(\phi_\xi(\sigma) - 1)\hat{f}(\sigma) = \hat{g}(\sigma)$.

If $\sigma \notin \hat{G}_0$, then $\phi_\xi(\sigma) = 0$, by (4.2.3); we then have $\lim_{\xi\to 0^+} \frac{1}{\xi}\hat{f}(\sigma) = \hat{g}(\sigma)$, which implies $\hat{f}(\sigma) = 0$. If $\sigma \in \hat{G}_0$, then $\phi_\xi(\sigma) = e^{\xi\nu(\sigma)}$; we then have, for $f \in D(A_0)$,

$$\lim_{\xi\to 0^+} \frac{1}{\xi}(e^{\xi\nu(\sigma)} - 1)\hat{f}(\sigma) = \hat{g}(\sigma) = (A_0 f)^\wedge(\sigma). \qquad \text{Now,}$$

$$\lim_{\xi\to 0^+} \frac{1}{\xi}(e^{\xi\nu(\sigma)} - 1) = \lim_{\xi\to 0^+} \frac{1}{\xi}\left(1 + \xi\nu(\sigma) + \frac{(\xi\nu(\sigma))^2}{2!} + \ldots - 1\right) = \nu(\sigma).$$

Hence $(A_0 f)^\wedge(\sigma) = \nu(\sigma)\hat{f}(\sigma)$ for all $\sigma \in \hat{G}_0$, $f \in D(A_0)$, which proves (ii).

(iii) Suppose that $\mathcal{J}$ is of class (A), with infinitesimal generator A. Then $X_0 = \{T(\xi)f : f \in AP(G), \xi > 0\}$ is dense in AP(G), and $D(A_0)$

is dense in $AP(G)$. Suppose there exists σ_0 in $\hat{G}$ such that $\sigma_0 \notin \hat{G}_0$; choose $f \in AP(G)$ such that $\hat{f}(\sigma_0) \neq 0$. Now, given $\varepsilon > 0$, there exists $f_0 \in D(A_0)$ such that $\|f-f_0\| < \varepsilon$. Since $\hat{f}_0(\sigma_0) = 0$ (by (ii) of the theorem), and $|\hat{f}(\sigma_0) - \hat{f}_0(\sigma_0)| \leq \|f-f_0\| < \varepsilon$, it follows that $\hat{f}(\sigma_0) = 0$, a contradiction. Hence $\hat{G}_0$ is the whole of $\hat{G}$.

Finally, let ω_0 be the type of $\mathcal{J}$. Then there exists $\omega_1 > \omega_0$ (see (4.1.5)) such that $R(\lambda:A)f = \int_0^\infty e^{-\xi\lambda}T(\xi)f\, d\xi$, for all $f \in X_0$ and $Re\lambda > \omega_1$. Since for each $\sigma \in \hat{G}$, the mapping $f \to \hat{f}(\sigma)$ is a bounded linear functional on $AP(G)$, we have, for all $f \in X_0$,

$$[R(\lambda:A)f]^\wedge(\sigma) = \int_0^\infty e^{-\xi\lambda}[T(\xi)f]^\wedge(\sigma)d\xi$$

$$= \int_0^\infty e^{-\xi\lambda}e^{\xi\nu(\sigma)}\hat{f}(\sigma)d\xi$$

$$= (\lambda - \nu(\sigma))^{-1}\hat{f}(\sigma)$$

for each $\sigma \in \hat{G}$. Since X_0 is dense in $AP(G)$, we have

$$(4.2.5) \qquad [R(\lambda:A)f]^\wedge(\sigma) = (\lambda - \nu(\sigma))^{-1}\hat{f}(\sigma)$$

for all $f \in AP(G)$, $Re\lambda > \omega_1$. Let the complex number λ (with $Re\lambda > \omega_1$) be fixed, and suppose that $f \in D(A)$. There exists $g \in AP(G)$ such that $f = R(\lambda:A)g$, since $D(A) = \{R(\lambda:A)f : f \in AP(G)\}$. We then have, for each $\sigma \in \hat{G}$,

$$(Af)^\wedge(\sigma) = [\lambda R(\lambda:A)g - g]^\wedge(\sigma)$$

$$= \lambda(\lambda-\nu(\sigma))^{-1}\hat{g}(\sigma) - \hat{g}(\sigma) \qquad \text{(by (4.2.5))}$$

$$= \nu(\sigma)(\lambda - \nu(\sigma))^{-1} \hat{g}(\sigma)$$

$$= \nu(\sigma)\hat{f}(\sigma) \qquad \text{(by (4.2.5))}.$$

Thus if $f \in D(A)$, then $\nu\hat{f} = (Af)^\wedge \in \widehat{AP(G)}$. Conversely, suppose that f is an element of $AP(G)$ such that $\nu\,\hat{f} \in \widehat{AP(G)}$. This means that there exists $h \in AP(G)$ such that $\nu(\sigma)\,\hat{f}(\sigma) = \hat{h}(\sigma)$ for all $\sigma \in \hat{G}$. Then, the function $g = \lambda f - h$ belongs to $AP(G)$ and, for each $\sigma \in \hat{G}$, we have

$$[R(\lambda:A)g]^\wedge(\sigma) = (\lambda-\nu(\sigma))^{-1}\,\hat{g}(\sigma) \qquad \text{(by (4.2.5))}$$

$$= (\lambda - \nu(\sigma))^{-1}(\lambda\hat{f}(\sigma) - \hat{h}(\sigma))$$

$$= (\lambda - \nu(\sigma))^{-1}(\lambda\hat{f}(\sigma) - \nu(\sigma)\hat{f}(\sigma))$$

$$= \hat{f}(\sigma).$$

Hence $R(\lambda : A)g = f$, and so $f \in D(A)$.

This concludes the proof of the theorem.

4.2.3 Remark: The results in Theorems 4.2.1 and 4.2.2 specialise to those in [13] for $C(G)$, in the case where G is a compact abelian group.

4.3 The case $X = \mathcal{A}$, an abstract Banach algebra

It is interesting to find that even in the setting of an abstract Banach algebra, the minimal conditions on the algebra which enable us to apply the machinery of the Gelfand representation, are all that we need to prove results analogous to the results of section 4.2 proved for the function space $AP(G)$. We state a few facts concerning the

Gelfand apparatus. Let $\mathcal{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be the space of regular maximal ideals of $\mathcal{A}$. Given $x \in \mathcal{A}$, let $\hat{x}$ be the function defined on $\mathfrak{M}$ by

$$(4.3.1) \qquad \hat{x}(M) = \phi(x) \qquad (M \in \mathfrak{M})$$

where ϕ is the bounded linear functional on $\mathcal{A}$ such that $\phi^{-1}(0) = M$; $\hat{x}$ is called the Gelfand transform of x, and we set $\hat{\mathcal{A}} = \{\hat{x} : x \in \mathcal{A}\}$. Given the Gelfand topology, M is a locally compact Hausdorff space, and $\hat{\mathcal{A}}$ separates the points of $\mathfrak{M}$. For $x, y \in \mathcal{A}$, $x = y$ iff $\hat{x}(M) = \hat{y}(M)$ for all $M \in \mathfrak{M}$; moreover, $\hat{x}(M) = 0$ iff $x \in M$. If $\mathcal{A}$ is a supremum norm algebra, i.e. an algebra for which $\|x\| = \|\hat{x}\|_\infty$ ([12], p.30), then $\mathcal{A}$ is semi-simple ([11], p.39), and so the mapping $x \to \hat{x}$ is an isometric isomorphism of $\mathcal{A}$ onto the subalgebra $\hat{\mathcal{A}}$ of $C(\mathfrak{M})$, the set of all continuous complex-valued functions on $\mathfrak{M}$, with uniform norm. The reader is referred to Chapter 3 of [11] for other details of the Gelfand theory, which are made use of in this section, but are not explicitly stated.

4.3.1 <u>Definition</u>: Let $\mathcal{A}$ be a commutative supremum norm Banach algebra, and let $\mathfrak{M}$ be the space of regular maximal ideals of $\mathcal{A}$. A complex-valued function ϕ on $\mathfrak{M}$ is called an $(\mathcal{A}, \mathcal{A})$-multiplier if $\phi\hat{x} \in \hat{\mathcal{A}}$ for each $x \in \mathcal{A}$.

If $\mathcal{A}$ has a unit element, then the set of $(\mathcal{A}, \mathcal{A})$- multipliers is precisely $\hat{\mathcal{A}}$. For let e be the unit element of $\mathcal{A}$, so that $xe = x$ for each $x \in \mathcal{A}$. Then, $\hat{x}\,\hat{e} = \hat{x}$ implies $\hat{e} = 1$, the identity function on $\mathfrak{M}$. If ϕ is an $(\mathcal{A}, \mathcal{A})$-multiplier, then $\phi = \phi\hat{e} = \hat{y}$ for some $y \in H$, i.e. every $(\mathcal{A}, \mathcal{A})$-multiplier is of the form $\hat{y}$ for some $y \in \mathcal{A}$. Of course every $\hat{y}$, $y \in \mathcal{A}$, is an $(\mathcal{A},\mathcal{A})$-multiplier since $\hat{y}\,\hat{x} = (yx)^\wedge \in \hat{\mathcal{A}}$ for all $x \in \mathcal{A}$. Hence, the set of multipliers for a commutative Banach algebra $\mathcal{A}$ with unit element is precisely $\hat{\mathcal{A}}$. The theory of multipliers for such an algebra is not of much interest, since a lot is known already about $\hat{\mathcal{A}}$. For this reason, in what follows, we shall always tacitly assume that $\mathcal{A}$ does not have a unit element.

4.3.2 <u>Definition</u>: An operator T on $\mathcal{A}$ is said to commute with the multiplication in $\mathcal{A}$ if

(4.3.2) $$T(xy) = (Tx)y = x(Ty)$$

for all x, y in $\mathcal{A}$.

We have the following lemmas:

4.3.3 <u>Lemma</u>: Let $\mathcal{A}$ be a commutative supremum norm Banach algebra with maximal ideal space $\mathfrak{M}$, and let T be an operator on $\mathcal{A}$ which commutes with the multiplication in $\mathcal{A}$. Then there exists a unique $\phi \in C_{bd}(\mathfrak{M})$, the subspace of $C(\mathfrak{M})$ comprising the bounded functions, such that

(i) $(Tx)^\wedge = \phi\hat{x}$ for each $x \in X$

and

(ii) $\qquad ||\phi||_{\infty} = ||T||$

Proof: This lemma is a combination of Theorems 1.2.2 and 1.2.3 of [12].

4.3.4 Lemma: Let $\mathcal{A}$ be a commutative supremum norm Banach algebra with maximal ideal space $\mathfrak{M}$, and let ϕ be an $(\mathcal{A}, \mathcal{A})$- multiplier Then, the mapping $T : \mathcal{A} \to \mathcal{A}$ defined by

$$(4.3.3) \qquad (Tx)^{\wedge} = \phi\hat{x} \qquad (x \in \mathcal{A})$$

is bounded, linear and commutes with the multiplication in $\mathcal{A}$.

Proof: The fact that $\mathcal{A}$ is a sup norm algebra implies that $\mathcal{A}$ is semi-simple ([12], p. 39). Now, for $x, y \in \mathcal{A}$,

$$[T(xy)]^{\wedge} = \phi(xy)^{\wedge} = \phi\hat{x}\hat{y} = (\phi\hat{x})\hat{y} = (Tx)^{\wedge}\,\hat{y} = [(Tx)y]^{\wedge} .$$

Since $\mathcal{A}$ is semi-simple, the Fourier transform is unique, and we have $T(xy) = (Tx)y$ for all x, y in $\mathcal{A}$. Moreover, since $\mathcal{A}$ is commutative, $T(xy) = T(yx) = (Ty)x = x(Ty)$. Hence T commutes with the multiplication in $\mathcal{A}$. Now, the semi-simplicity of $\mathcal{A}$ implies that $\mathcal{A}$ is without order ([12], p.29). It follows by Theorem 1.1.1 of [12] that T is linear and bounded.

We are now in a position to prove the main theorems in this section.

4.3.5 Theorem: Let $\mathcal{A}$ be a commutative supremum norm Banach algebra and let $\mathcal{M}$ be the space of regular maximal ideals of $\mathcal{A}$. If $\mathcal{J} = \{T(\xi): \xi > 0\}$ is a semigroup of operators on $\mathcal{A}$ which commute with the multiplication in $\mathcal{A}$, then $\mathcal{J}$ determines a collection $\Phi = \{\phi_\xi : \xi > 0\}$ of functions in $C_{bd}(\mathcal{M})$ such that

(i) $[T(\xi)x]^\wedge = \phi_\xi \hat{x}$ for each $x \in \mathcal{A}$,

(ii) $\|T(\xi)\| = \|\phi_\xi\|_\infty$, $\xi > 0$, and

(iii) $\phi_{\xi_1+\xi_2} = \phi_{\xi_1}\phi_{\xi_2}$ for all $\xi_1, \xi_2 > 0$.

If, in addition, $\mathcal{J}$ is weakly measurable, then there exists a subset $\mathcal{M}_0$ of $\mathcal{M}$ and a mapping $\nu : M \to \nu(M)$ of $\mathcal{M}_0$ into the field K, of complex numbers, such that for each $\xi > 0$,

$$\phi_\xi(M) = \begin{cases} e^{\xi\nu(M)} & \text{if } M \in \mathcal{M}_0 \\ 0 & \text{if } M \notin \mathcal{M}_0 \end{cases}$$

Proof: (i) and (ii) follow from lemma 4.3.3 , and the semigroup property of $\mathcal{J}$ ensures that (iii) holds;

Now, suppose $\mathcal{J}$ is weakly measurable. Then for each $\psi \in \mathcal{A}^*$, the space of all continuous linear functionals on $\mathcal{A}$, and each $x \in \mathcal{A}$, $\xi \to \psi(T(\xi)x)$ is Lebesgue measurable. In particular, for each $M \in \mathcal{M}$, we choose an $x \notin M$, and define $\psi_M \in \mathcal{A}^*$ by $\psi_M(a) = \hat{a}(M)$, $a \in \mathcal{A}$. We then have

$$\phi_\xi(M) = \frac{1}{\hat{x}(M)}(T(\xi)x)^\wedge(M) = \frac{1}{\hat{x}(M)}\psi_M(T(\xi)x).$$ It follows

that $\phi_\xi(M)$ is measurable for each $M \in \mathfrak{M}$. Since also, $\phi_{\xi_1+\xi_2}(M) = \phi_{\xi_1}(M)\,\phi_{\xi_2}(M)$, it follows by lemma 4.1.1 that, for each $M \in \mathfrak{M}$, either $\phi_\xi(M)$ is identically zero or $\phi_\xi(M) = e^{\xi\nu(M)}$ for some complex number $\nu(M)$. Set $\mathfrak{M}_0 = \{M \in \mathfrak{M} : \phi_\xi(M) \neq 0\}$ and the proof is complete.

4.3.6 Theorem: For a fixed subset $\mathfrak{M}_0$ of $\mathfrak{M}$, let $\nu : M \to \nu(M)$ be a mapping of $\mathfrak{M}_0$ into K. Assume that the function ϕ_ξ, $\xi > 0$, defined on $\mathfrak{M}$ by

$$\phi_\xi(M) = \begin{cases} e^{\xi\nu(M)} & \text{if } M \in \mathfrak{M}_0 \\ 0 & \text{if } M \notin \mathfrak{M}_0 \end{cases}$$

is an $(\mathcal{A}, \mathcal{A})$-multiplier. Now define a mapping $T(\xi)$, $\xi > 0$, of $\mathcal{A}$ into itself, by

$$[T(\xi)x]^\wedge = \phi_\xi \hat{x} \qquad (x \in \mathcal{A}) .$$

Then, (i) $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is a semigroup of bounded linear operators on $\mathcal{A}$, the elements of which commute with the multiplication in $\mathcal{A}$. Furthermore, $\mathcal{J}$ is strongly continuous for all $\xi > 0$.

(ii) Let A_0 denote the infinitesimal operator of $\mathcal{J}$ and $D(A_0)$ the domain of A_0. Then, for each $x \in D(A_0)$ and $M \notin \mathfrak{M}_0$, we have $\hat{x}(M) = 0$; thus $D(A_0)$ is contained in every maximal ideal not in $\mathfrak{M}_0$. Moreover, $(A_0x)^\wedge = \nu\hat{x}$ for all $x \in D(A_0)$; thus ν is a $(D(A_0), \mathcal{A})$-multiplier.

(iii) If $\mathcal{J}$ is of class (A), with infinitesimal generator A, then $\mathcal{M}_0 = \mathcal{M}$, $D(A) = \{x \in \mathcal{A} : \nu\hat{x} \in \hat{\mathcal{A}}\}$, i.e. ν is a $(D(A), \mathcal{A})$-multiplier , moreover, $(Ax)^\wedge = \nu\hat{x}$ for all $x \in D(A)$.

Proof (i) That, for each $\xi > 0$, $T(\xi)$ is a bounded linear operator on $\mathcal{A}$ which commutes with the multiplication in $\mathcal{A}$, follows from lemma 4.3.4. The semigroup property of $\mathcal{J}$ is immediate from the definition of $T(\xi)$. We now show that $T(\xi)$ is strongly continuous for $\xi > 0$. For fixed $\xi_0 > 0$ and all $x \in \mathcal{A}$,

$$\|T(\xi)x - T(\xi_0)x\| = \|[T(\xi)x - T(\xi_0)x]^\wedge\|_\infty \quad (\mathcal{A} \text{ is supremum norm algebra})$$

$$= \|(T(\xi)x)^\wedge - (T(\xi_0)x)^\wedge\|_\infty$$

$$= \sup_{M \in \mathcal{M}} \left| e^{\xi\nu(M)}\hat{x}(M) - e^{\xi_0\nu(M)}\hat{x}(M) \right|$$

$$\leq \|\hat{x}\|_\infty \sup_M \left| e^{\xi\nu(M)} - e^{\xi_0\nu(M)} \right|$$

$$= \|x\| \sup_M \left| e^{\xi_0\nu(M)} \right| \left| e^{(\xi-\xi_0)\nu(M)} - 1 \right|$$

Now, $|e^z - 1| \leq |z| e^{|z|}$ for any complex number z. Hence,

$$\left| e^{(\xi-\xi_0)\nu(M)} - 1 \right| \leq |(\xi-\xi_0)\nu(M)| \; e^{|(\xi-\xi_0)\nu(M)|}$$

$$= |\xi-\xi_0| \, |\nu(M)| \, e^{|\xi-\xi_0| \, |\nu(M)|} ,$$

and $$\left| e^{\xi_0\nu(M)} \right| \leq \left| e^{\xi_0\nu(M)} - 1 \right| + 1$$

$$\leq \xi_0 |\nu(M)| \; e^{\xi_0|\nu(M)|} + 1 .$$

Moreover, the assumption that each ϕ_ξ , $\xi > 0$, is an $(\mathcal{A}, \mathcal{A})$-multiplier, implies that for each $\xi > 0$, $\phi_\xi \in C_{bd}(\mathcal{M})$ ([12], chapter 1).

Consequently, $\sup_M |e^{\xi\nu(M)}| < \infty$, for all $\xi > 0$, which implies, in particular, that $\sup_M |e^{\nu(M)}| < \infty$. It follows that there exists a real number n_0 such that $|\nu(M)| < n_0$ for all $M \in \mathfrak{M}$. We then have $\|T(\xi)x - T(\xi_0)x\| \leq \|x\| \sup_M |e^{\xi_0\nu(M)}||e^{(\xi-\xi_0)\nu(M)} - 1|$

$$\leq \|x\| \sup_M \left\{(\xi_0|\nu(M)|e^{\xi_0|\nu(M)|} + 1)(|\xi-\xi_0||\nu(M)|e^{|\xi-\xi_0||\nu(M)|})\right\}$$

$$\leq \|x\|(\xi_0 n_0 e^{\xi_0 n_0} + 1)(|\xi-\xi_0|n_0 e^{n_0|\xi-\xi_0|})$$

$$\to 0 \quad \text{as} \quad \xi \to \xi_0 .$$

Hence, $\lim_{\xi\to\xi_0} \|T(\xi)x - T(\xi_0)x\| = 0$ for all $x \in \mathcal{A}$ and $\xi_0 > 0$.

This implies that $T(\xi)$ is strongly continuous for all $\xi > 0$.

(ii) Suppose $x \in D(A_0)$; then there exists $y = A_0 x \in \mathcal{A}$ such that $\lim_{\xi\to 0^+} \frac{1}{\xi}[T(\xi)x - x] = y$, in norm. For each $M \in \mathfrak{M}$,

$\lim_{\xi\to 0^+} \frac{1}{\xi}[(T(\xi)x)\hat{}(M) - \hat{x}(M)] = \hat{y}(M)$, i.e. $\lim_{\xi\to 0^+} \frac{1}{\xi}(\phi_\xi(M) - 1)\hat{x}(M) = \hat{y}(M)$.

As in the proof of Thoerem 4.2.2 (ii), the definition of ϕ_ξ implies that if $M \notin \mathfrak{M}_0$, then $\hat{x}(M) = 0$, and if $M \in \mathfrak{M}_0$, then $(A_0 x)\hat{}(M) = \nu(M)\,\hat{x}(M)$.

(iii) Let $\mathcal{J}$ be of class (A), with infinitesimal generator A. Then the set $\mathcal{A}_0 = \{T(\xi)x : x \in \mathcal{A}, \xi > 0\}$ is dense in $\mathcal{A}$ and $D(A_0)$ is dense in $\mathcal{A}$. Suppose there exists M_0 in $\mathfrak{M}$ such that $M_0 \notin \mathfrak{M}_0$. By the definition of $T(\xi)$, we shall have $(T(\xi)x)\hat{}(M_0) = 0$ for all $x \in \mathcal{A}$, $\xi > 0$. Hence $T(\xi)x \in M_0$ for all $x \in \mathcal{A}$ and all $\xi > 0$. This implies $\mathcal{A}_0 \subseteq M_0$.

As $\mathcal{A}_0$ is dense in $\mathcal{A}$ and M_0 is closed (every maximal ideal is closed), it follows that $\mathcal{A} \subseteq M_0$. This is impossible, for M_0 is a maximal ideal of $\mathcal{A}$. Hence $\mathcal{M}_0$ is the whole of $\mathcal{M}$.

We give an outline only, of the rest of the proof, since we follow precisely the same steps as in the last part of the proof of Theorem 4.2.2. Let ω_0 be the type of $\mathcal{J}$. Then there exists $\omega_1 > \omega_0$ such that

$$R(\lambda : A)x = \int_0^\infty e^{-\xi\lambda} T(\xi)x \, d\xi \quad \text{for all } x \in \mathcal{A}_0, \ \mathrm{Re}\lambda > \omega_1.$$

Thus, for all $x \in \mathcal{A}_0$,

$$[R(\lambda:A)x]^\wedge(M) = \int_0^\infty e^{-\xi\lambda}[T(\xi)x]^\wedge(M)d\xi = (\lambda - \nu(M))^{-1}\hat{x}(M)$$

for each $M \in \mathcal{M}$. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$,

$[R(\lambda:A)x]^\wedge(M) = (\lambda - \nu(M))^{-1} \hat{x}(M)$ for all $x \in \mathcal{A}$, $\mathrm{Re}\lambda > \omega_1$.

Let λ (with $\mathrm{Re}\lambda > \omega_1$) be fixed, and suppose that $x \in D(A)$. Then $x = R(\lambda : A)y$ for some $y \in \mathcal{A}$, and so, for each $M \in \mathcal{M}$,

$$(Ax)^\wedge(M) = [\lambda R(\lambda : A)y - y]^\wedge(M) = \nu(M)\, \hat{x}(M).$$

Thus $x \in D(A)$ implies $\nu\hat{x} \in \hat{\mathcal{A}}$. Conversely, suppose $x \in \mathcal{A}$ is such that $\hat{h} = \nu\hat{x} \in \hat{\mathcal{A}}$. Then, $y = \lambda x - h \in \mathcal{A}$, and we have $R(\lambda : A)y = x$. This implies that $x \in D(A)$.

4.4 <u>The case where $X = H$, an abstract Hilbert space</u>

The multipliers for an abstract Hilbert space H have been discussed, in some detail, in Chapter 2. We now have the following theorems:

4.4.1 <u>Theorem</u>: Let H be a Hilbert space and let $E = \{e_i : i \in I\}$ be a complete orthonormal set in H. If $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is a semigroup of bounded linear operators on H which commute with the projections P_i, $i \in I$, then $\mathcal{J}$ determines a collection $\{\phi_\xi : \xi > 0\} \subseteq \ell_\infty(E)$ of (H, H)-multipliers such that

(i) $[T(\xi)x]^\wedge = \phi_\xi \hat{x}$ for each $x \in H$, $\xi > 0$,

(ii) $\|T(\xi)\| = \|\phi_\xi\|_\infty$, $\xi > 0$,

and

(iii) $\phi_{\xi_1+\xi_2} = \phi_{\xi_1}\phi_{\xi_2}$ for all $\xi_1, \xi_2 > 0$.

If, in addition, $\mathcal{J}$ is weakly measurable, then there exists a subset E_0 of E and a mapping $\nu : e_i \to \nu(e_i)$ of E_0 into K, such that for each $\xi > 0$

$$\phi_\xi(e_i) = \begin{cases} e^{\xi\nu(e_i)} & \text{if } e_i \in E_0 \\ 0 & \text{if } e_i \notin E_0 \end{cases}$$

<u>Proof</u>: (i) and (ii) follow from Lemma 2.3.6 , and (iii) follows easily from the semigroup property of $\mathcal{J}$.

Now, suppose $\mathcal{J}$ is weakly measurable. Then for each $\psi \in H^*$, the space of all continuous linear functionals on H, and for each $x \in H$, $\xi \to \psi(T(\xi)x)$ is Lebesgue measurable. Now each $e_i \in E$ determines a $\psi_i \in H^*$, defined by $\psi_i(x) = \hat{x}(e_i)$, $x \in H$. Thus for each $e_i \in E$ and $\xi > 0$, $\phi_\xi(e_i) = (T(\xi)e_i)^\wedge(e_i) = \psi_i(T(\xi)e_i)$ is measurable.

Since also, $\phi_{\xi_1+\xi_2}(e_i) = \phi_{\xi_1}(e_i)\,\phi_{\xi_2}(e_i)$, it follows by lemma 4.1.1 that, for each $e_i \in E$, either $\phi_\xi(e_i)$ is identically zero, or $\phi_\xi(e_i) = e^{\xi\nu(e_i)}$ for some complex number $\nu(e_i)$. We now set $E_0 = \{e_i \in E : \phi_\xi(e_i) \neq 0\}$ and the proof is complete.

4.4.2 <u>Theorem</u>: For a fixed subset E_0 of E, let $\nu : e_i \to \nu(e_i)$ be a mapping of E_0 into K. Assume that the function ϕ_ξ, $\xi > 0$, defined on E by

$$\phi_\xi(e_i) = \begin{cases} e^{\xi\nu(e_i)} & \text{if } e_i \in E_0 \\ 0 & \text{if } e_i \notin E_0 \end{cases}$$

is an (H, H) - multiplier. Now define a mapping $T(\xi)$, $\xi > 0$, of H into itself, by

$$[T(\xi)x]^\wedge = \phi_\xi \hat{x} \qquad (x \in H).$$

Then, (i) $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is a semigroup of bounded linear operators on H, the elements of which commute with the projections P_i, $i \in I$. Furthermore, $\mathcal{J}$ is strongly continuous for all $\xi > 0$.

(ii) Let A_0 denote the infinitesimal operator of $\mathcal{J}$ and $D(A_0)$ the domain of A_0. Then, for each $x \in D(A_0)$ and $e_i \notin E_0$, we have $\hat{x}(e_i) = 0$. Moreover, $(A_0 x)^\wedge = \nu\hat{x}$ for all $x \in D(A_0)$; thus ν is a $(D(A_0), H)$ - multiplier.

(iii) If $\mathcal{J}$ is of class (A), with infinitesimal generator A, then $E_0 = E$, $D(A) = \{x \in H : \nu\hat{x} \in \hat{H}\}$, i.e. ν is a $(D(A), H)$- multiplier and moreover, $(Ax)^\wedge = \nu\hat{x}$ for all $x \in D(A)$.

Proof: (i) By lemma 2.3.4, $T(\xi)$, for each $\xi > 0$, is a bounded linear operator on H which commutes with the projections P_i, $i\epsilon I$. The semigroup property of $\mathcal{J}$ is immediate from the definition of $T(\xi)$. We show that $T(\xi)$ is strongly continuous for $\xi > 0$. To this end, we show that $\lim_{\xi\to\xi_0} \|T(\xi)x - T(\xi_0)x\| = 0$ for all $x\epsilon H$ and $\xi_0 > 0$. Now, for arbitrary but fixed $e_0 \epsilon E$,

$$(T(\xi)e_0)^\wedge(e_i) = \begin{cases} e^{\xi\nu(e_0)} & \text{if } e_i = e_0 \\ 0 & \text{if } e_i \neq e_0 , \end{cases}$$

by the definition of $T(\xi)$. Hence $T(\xi)e_0 = e^{\xi\nu(e_0)}e_0$, and for all $\xi_0 > 0$, we have

$$\lim_{\xi\to\xi_0} \|T(\xi)e_0 - T(\xi_0)e_0\| = \lim_{\xi\to\xi_0} \|(e^{\xi\nu(e_0)} - e^{\xi_0\nu(e_0)})e_0\|$$

$$= \lim_{\xi\to\xi_0} |e^{\xi\nu(e_0)} - e^{\xi_0\nu(e_0)}|$$

$$= 0 ,$$

since $e^{\xi\nu(e_0)}$ is continuous in ξ. It follows that for each $e_i \epsilon E$, we have $\lim_{\xi\to\xi_0} \|T(\xi)e_i - T(\xi_0)e_i\| = 0$, for all $\xi_0 > 0$. If x is a finite linear combination of elements of E, say

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_N e_N$$

then, for $\xi_0 > 0$, we have

$$\lim_{\xi\to\xi_0} \|T(\xi)x - T(\xi_0)x\| = \lim_{\xi\to\xi_0} \| \sum_{i=1}^{N} \alpha_i\{T(\xi)e_i - T(\xi_0)e_i\}\|$$

$$\leq \lim_{\xi \to \xi_0} \sum_{i=1}^{N} |\alpha_i| \|T(\xi)e_i - T(\xi_0)e_i\|$$

$$= \sum_{i=1}^{N} |\alpha_i| \lim_{\xi \to \xi_0} \|T(\xi)e_i - T(\xi_0)e_i\|$$

Since $\lim_{\xi \to \xi_0} \|T(\xi)e_i - T(\xi_0)e_i\| = 0$ for each $i = 1, 2, \ldots, N$, it follows that $\lim_{\xi \to \xi_0} \|T(\xi)x - T(\xi_0)x\| = 0$, for all $\xi_0 > 0$. Suppose now that x is an arbitrary vector in H. The set $S_x = \{e_i \in E : < x, e_i > \neq 0\}$ is countable ([15], p.253). If we arrange the elements of S_x in a definite order, say

$$S_x = \{e_1, e_2, e_3, \ldots, e_k, \ldots\}$$

then we may express the Fourier series of x as $x = \sum_{k=1}^{\infty} < x, e_k > e_k$; moreover, by the theory of absolutely convergent series, $\|x\|^2 = \sum_{k=1}^{\infty} |< x, e_k >|^2$ is independent of the order in which the elements of S_x have been arranged. Let $x_N = \sum_{k=1}^{N} < x, e_k > e_k$; then $\|x - x_N\| \to 0$ as $N \to \infty$. Now, for $\xi_0 > 0$,

$$\|T(\xi)x - T(\xi_0)x\| \leq \|T(\xi)x - T(\xi)x_N\| + \|T(\xi)x_N - T(\xi_0)x_N\| + \|T(\xi_0)x_N - T(\xi_0)x\|$$

$$\leq \|T(\xi)\| \cdot \|x - x_N\| + \|T(\xi)x_N - T(\xi_0)x_N\| + \|T(\xi_0)\| \cdot \|x_N - x\|$$

by the linearity and boundedness of $T(\xi)$, $\xi > 0$. As $\xi \to \xi_0$, we have

$$\lim_{\xi \to \xi_0} \|T(\xi)x - T(\xi_0)x\| \leq \|T(\xi_0)\| \cdot \|x - x_N\| + \lim_{\xi \to \xi_0} \|T(\xi)x_N - T(\xi_0)x_N\| + \|T(\xi_0)\| \cdot \|x_N - x\|$$

$$= 2\|T(\xi_0)\| \cdot \|x_N - x\|$$

since by an earlier argument, $\lim_{\xi\to\xi_0} ||T(\xi)x_N - T(\xi_0)x_N|| = 0$. But $T(\xi)$ is bounded, and $\lim_{N\to\infty} ||x_N - x|| = 0$. It follows that $\lim_{\xi\to\xi_0} ||T(\xi)x - T(\xi_0)x|| = 0$, for all $\xi_0 > 0$. This shows that $\mathcal{J}$ is strongly continuous.

(ii) Suppose $x \in D(A_0)$; then there exists $y = A_0 x \in H$ such that $\lim_{\xi\to 0^+} \frac{1}{\xi}[T(\xi)x - x] = y$, in norm. For each $e_i \in E$, $\lim_{\xi\to 0+} \frac{1}{\xi}[(T(\xi)x)\hat{}(e_i) - \hat{x}(e_i)] = \hat{y}(e_i)$,

i.e. $\lim_{\xi\to 0^+} \frac{1}{\xi} (\phi_\xi(e_i) - 1)\hat{x}(e_i) = \hat{y}(e_i)$. As in the proof of Theorem 4.2.2 (ii), the definition of ϕ_ξ implies that if $e_i \notin E_0$, then $\hat{x}(e_i) = 0$, and if $e_i \in E_0$, then $(A_0 x)\hat{}(e_i) = \nu(e_i)\, \hat{x}(e_i)$.

(iii) Let $\mathcal{J}$ be of class (A), with infinitesimal generator A. Then $H_0 = \{T(\xi)x : x \in H, \xi > 0\}$ is dense in H. Suppose there exists $e_0 \in E$ such that $e_0 \notin E_0$. By the definition of $T(\xi)$, we shall have $< T(\xi)x, e_0 > = 0$ for all $x \in H$ and all $\xi > 0$, i.e. $< H_0, e_0 > = 0$. As H_0 is dense in H, this implies $< H, e_0 > = 0$. Hence $e_0 = 0$, which cannot be, since all the elements of E are non-zero. Hence E_0 is the whole of E.

We give an outline only, in the rest of this proof, for the same reasons given in the last part of the proof of Theorem 4.3.6. Let ω_0 be the type of $\mathcal{J}$; then there exists $\omega_1 > \omega_0$ such that

$R(\lambda : A)x = \int_0^\infty e^{-\xi\lambda}\, T(\xi)x\, d\xi$ for all $x \in H_0$, $Re\lambda > \omega_1$. Thus, for all $x \in H_0$, $[R(\lambda:A)x]^\wedge(e_i) = \int_0^\infty e^{-\xi\lambda}[T(\xi)x]^\wedge(e_i)\, d\xi = (\lambda - \nu(e_i))^{-1}\hat{x}(e_i)$ for each $e_i \in E$. Since H_0 is dense in H, we have $[R(\lambda:A)x]^\wedge(e_i) = (\lambda - \nu(e_i))^{-1}\hat{x}(e_i)$ for all $x \in H$, $Re\lambda > \omega_1$. Let λ (with $Re\lambda > \omega_1$) be fixed, and suppose that $x \in D(A)$. Then $x = R(\lambda : A)y$ for some $y \in H$, and so, for each $e_i \in E$, $(Ax)^\wedge(e_i) = [\lambda R(\lambda:A)y - y]^\wedge(e_i) = \nu(e_i)\,\hat{x}(e_i)$. Thus $x \in D(A)$ implies $\nu\hat{x} \in \hat{H}$. Conversely, suppose $x \in H$ is such that $\hat{h} = \nu\hat{x} \in \hat{H}$. Then, $y = \lambda x - h \in H$, and we have $R(\lambda:A)y = x$. This implies that $x \in D(A)$. Thus $D(A) = \{x \in H : \nu\hat{x} \in \hat{H}\}$. This concludes the proof of the theorem.

CHAPTER 5

n-PARAMETER SEMIGROUPS OF OPERATORS ON $C(G)$ OR $L_p(G)$, $1 \leq p < \infty$

Let K be the field of complex numbers, and let K_n be the set of all n-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i \in K$ for each i. With the operations of component wise addition and scalar multiplication, K_n is a linear space. For $\alpha, \beta \in K_n$, the inner product $\alpha.\beta$ of α and β is given by $\alpha.\beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots \alpha_n\beta_n$. Suppose $\alpha \in K_n$ is such that each component α_i is real. Then $\alpha \in R_n$, the n-dimensional Euclidean space. Denote the unit vectors in R_n by $e_1, e_2, \ldots, e_n$ where

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

with 1 in the i^{th}-place and zero elsewhere. Then, for each $\xi \in R_n$, $\xi = \xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_n e_n$, for some real numbers ξ_i, $i = 1,2,\ldots,n$. Denote by R_n^+ the cone $\{\xi \in R_n : \xi_i \geq 0\}$ excluding the origin $(0, 0, \ldots, 0)$. The set R_n^+ is a positive cone but not open.

Let G be a compact abelian group, and let U be an arbitrary, but fixed, member of the usual set $\{C(G), L_p(G) : 1 \leq p < \infty\}$ of complex-valued continuous functions and p-integrable functions on G. Suppose there corresponds to each $\xi \in R_n^+$ an operator $T(\xi)$ in $B(U)$. The collection $\mathcal{J} = \{T(\xi) : \xi \in R_n^+\}$ is called an n-parameter semigroup of operators on U if

(5.1.1) $$T(\xi + \eta) = T(\xi)T(\eta)$$

for all $\xi, \eta \in R_n^+$, i.e. $T(\xi + \eta)f = T(\xi)[T(\eta)f]$ for all $f \in U$ and $\xi, \eta \in R_n^+$. We shall, in the present chapter, obtain a generalisation to n-parameter semigroups of operators, of Theorems 1.1 and 1.2 of [13]. The results of Chapter 4 are of course analogues also of Theorems 1.1 and 1.2 of [13].

For each $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in R_n^+ ,

$T(\xi) = T(\xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_n e_n) = T(\xi_1 e_1)\, T(\xi_2 e_2) \ldots T(\xi_n e_n)$.

Setting $T(\xi_k e_k) = T_k(\xi_k)$, $k = 1, 2, \ldots, n$, we see that $\{T(\xi) : \xi \in R_n^+\}$ is a direct product of n one-parameter semigroups $\mathcal{J}_k = \{T_k(\xi_k) : \xi_k \geq 0\}$, and for each $\xi \in R_n^+$, we have

$$(5.1.2) \qquad T(\xi) = T_1(\xi_1) T_2(\xi_2) \ldots T_n(\xi_n)$$

The operators $T_k(\xi_k)$ commute with each other, since for instance, $\xi_1 e_1 + \xi_2 e_2 = \xi_2 e_2 + \xi_1 e_1$ implies that

$$T_1(\xi_1) T_2(\xi_2) = T(\xi_1 e_1 + \xi_2 e_2) = T(\xi_2 e_2 + \xi_1 e_1) = T_2(\xi_2) T_1(\xi_1).$$

The boundedness of $T(\xi)$, $\xi \in R_n^+$ implies the boundedness of every $T_k(\xi_k)$, since for each $k = 1, 2, \ldots, n$, $T_k(\xi_k) = T(\xi^{(k)})$, where $\xi^{(k)}$ is the element $(0, 0, \ldots, 0, \xi_k, 0, \ldots, 0)$ of R_n^+ which has ξ_k in the k^{th}-place and zero elsewhere. Moreover, the linearity of $T(\xi)$ implies the linearity of each $T_k(\xi_k)$. For let $T(\xi)$ be linear, and suppose that exactly one of the $T_k(\xi_k)$'s , say $T_1(\xi_1)$, is not linear. [The choice of $T_1(\xi_1)$ is indeed arbitrary; because the operators $T_k(\xi_k)$ commute, we can always rearrange the expression $T_1(\xi_1)\, T_2(\xi_2) \ldots T_n(\xi_n)$ so as to have the operator which is assumed non-linear to occupy the position of $T_1(\xi_1)$]. Thus for $f_1, f_2 \in U$ and complex numbers α_1, α_2, we have

$$T(\xi)(\alpha_1 f_1 + \alpha_2 f_2) = T_1(\xi_1)[T_2(\xi_2) \ldots T_n(\xi_n)(\alpha_1 f_1 + \alpha_2 f_2)]$$

$$= T_1(\xi_1)[\alpha_1 T_2(\xi_2) \ldots T_n(\xi_n) f_1 + \alpha_2 T_2(\xi_2) \ldots T_n(\xi_n) f_2]$$

$$\neq \alpha_1 T_1(\xi_1) \ldots T_n(\xi_n) f_1 + \alpha_2 T_1(\xi_1) \ldots T_n(\xi_n) f_2 \quad (T_1(\xi_1) \text{ is non-linear })$$

$$= \alpha_1 T(\xi) f_1 + \alpha_2 T(\xi) f_2$$

which contradicts the linearity of $T(\xi)$. Thus $\mathcal{J} = \{T(\xi) : \xi \in R_n^+\}$ in $B(U)$ is a direct product of n one-parameter semigroups $\mathcal{J}_k = \{T_k(\xi_k) : \xi_k \geq 0\}$ of operators in $B(U)$.

To each $\xi \in R_n^+$ corresponds an infinitesimal operator $A_0(\xi)$ of $\{T(\xi) : \xi \in R_n^+\}$ defined by, ([10], [3]),

(5.1.3) $$A_0(\xi) f = \lim_{t \to 0^+} \frac{1}{t}[T(t\xi) f - f]$$

wherever this limit exists. If we denote by A_k the infinitesimal operator of $\{T_k(\xi_k) : \xi_k \geq 0\}$, then ([10], p. 336), for each $\xi \in R_n^+$, we have $A_0(\xi) = \sum_{k=1}^{n} \xi_k A_k$. Thus the set $\{A_0(\xi) : \xi \in R_n^+\}$ of infinitesimal operators of $\mathcal{J}$ is itself an additive abelian semigroup of operators.

The following theorems are the main results of this chapter.

5.1.1 Theorem: Let $\mathcal{J} = \{T(\xi) : \xi \in R_n^+\}$ be a semigroup of bounded linear operators on U. If, for each $\xi \in R_n^+$, the operator $T(\xi)$ commutes with translations, then $\mathcal{J}$ determines a collection $\{\phi_\xi : \xi \in R_n^+\}$ of (U, U)-multipliers such that

(i) for each $\xi \in R_n^+$, $\phi_\xi \hat{f} = [T(\xi)f]^\wedge$ for all $f \in U$, and

(ii) $\phi_{\xi+\eta}(\sigma) = \phi_\xi(\sigma)\phi_\eta(\sigma)$, for all $\xi, \eta \in R_n^+$, $\sigma \in \hat{G}$.

If, in addition, $\mathcal{J}$ is weakly measurable, then there exists a subset $\hat{G}_0$ of $\hat{G}$ and a mapping $\nu : \sigma \to \nu(\sigma)$ of $\hat{G}_0$ into K_n, such that

$$\phi_\xi(\sigma) = \begin{cases} e^{\xi.\nu(\sigma)} & \text{if } \sigma \in \hat{G}_0 \\ 0 & \text{if } \sigma \notin \hat{G}_0 \end{cases}$$

Proof: Since the bounded linear operators $T(\xi)$, $\xi \in R_n^+$, commute with translations, Theorem 0.1.1 of [12] implies that there exists, for each $\xi \in R_n^+$, a (U, U)- multiplier ϕ_ξ such that $[T(\xi)f]^\wedge = \phi_\xi \hat{f}$ for all $f \in U$. Statement (ii) of the theorem follows from the semigroup property of $\mathcal{J}$, since for all $\xi, \eta \in R_n^+$ and $f \in U$, we have

$$\phi_{\xi+\eta} \hat{f} = [T(\xi+\eta)f]^\wedge = [T(\xi)(T(\eta)f)]^\wedge = \phi_\xi[T(\eta)f]^\wedge = \phi_\xi \phi_\eta \hat{f} .$$

Now, $\mathcal{J}$ is a direct product of n one-parameter semigroups $\mathcal{J}_k = \{T_k(\xi_k) : \xi_k \geq 0\}$ of bounded linear operators on U. The fact that $T(\xi)$, $\xi \in R_n^+$, commutes with translations implies that, for each k, $T_k(\xi_k)$ also commutes with translations. For if we suppose (as was done earlier to prove linearity of each $T_k(\xi_k)$) that $T_1(\xi_1)$ does not commute with translations, then, for any $f \in U$ and $a \in G$, we have

$$T(\xi)f_a = T_1(\xi_1)\, T_2(\xi_2) \ldots T_n(\xi_n)f_a$$

$$= T_1(\xi_1) \ldots T_{n-1}(\xi_{n-1})[T_n(\xi_n)f_a]$$

$$= T_1(\xi_1) \ldots T_{n-1}(\xi_{n-1})[T_n(\xi_n)f]_a$$

$$= T_1(\xi_1)[T_2(\xi_2) \ldots T_n(\xi_n)f]_a$$

$$\neq [T_1(\xi_1)T_2(\xi_2) \ldots T_n(\xi_n)f]_a$$

$$= [T(\xi)f]_a$$

which contradicts $T(\xi)$ commuting with translations. Thus each $T_k(\xi_k)$, $\xi_k \geq 0$, is a bounded linear operator on U which commutes with translations. Theorem 0.1.1 of [12] then implies that there exists for each k, a (U, U)- multiplier ϕ_{ξ_k} such that $\phi_{\xi_k}\hat{f} = [T_k(\xi_k)f]^\wedge$ for all $f \in U$. Hence, for $f \in U$ and $\xi \in R_n^+$, we have

$$[T(\xi)f]^\wedge = [T_1(\xi_1) \ldots T_n(\xi_n)f]^\wedge = \phi_{\xi_1}[T_2(\xi_2)\ldots T_n(\xi_n)f]^\wedge$$

$= \phi_{\xi_1}\phi_{\xi_2} \ldots \phi_{\xi_n}\hat{f}$. But $[T(\xi)f]^\wedge = \phi_\xi \hat{f}$ for some (U,U)-multiplier ϕ_ξ. It follows that

$$\phi_\xi = \phi_{\xi_1}\phi_{\xi_2} \ldots . \phi_{\xi_n} \tag{5.1.4}$$

for each $\xi \in R_n^+$. Now, the semigroup property of $T_k(\xi_k)$, for fixed k, implies that $\phi_{\xi_k+\eta_k}(\sigma) = \phi_{\xi_k}(\sigma)\,\phi_{\eta_k}(\sigma)$ for all $\sigma \in \hat{G}$ and $\xi_k, \eta_k > 0$.

Moreover, the weak measurability of $T(\xi)$ implies that, for fixed $\sigma \in \hat{G}$, $\phi_{\xi_k}(\sigma) = \psi_\sigma[T(\xi_k e_k)\chi_\sigma]$ is Lebesgue measurable. It follows by lemma 4.1.3 that, for each $\sigma \epsilon \hat{G}$, either $\phi_{\xi_k}(\sigma)$ is identically zero or $\phi_{\xi_k}(\sigma) = e^{\xi_k \nu_k(\sigma)}$ for some complex number $\nu_k(\sigma)$. Let $\hat{G}_k = \{\sigma \epsilon \hat{G} : \phi_{\xi_k}(\sigma) \neq 0\}$, $k = 1, 2, \ldots, n$, and let $\hat{G}_0 = \bigcap_{k=1}^{n} \hat{G}_k$. Since for fixed $\sigma \in \hat{G}$,

$\phi_\xi(\sigma) = \phi_{\xi_1}(\sigma)\, \phi_{\xi_2}(\sigma) \ldots \phi_{\xi_n}(\sigma)$, it follows that if $\sigma \notin \hat{G}_0$, then $\phi_\xi(\sigma)$ is identically zero, and if $\sigma \in \hat{G}_0$, then

$$\phi_\xi(\sigma) = e^{\xi_1 \nu_1(\sigma)} e^{\xi_2 \nu_2(\sigma)} \ldots e^{\xi_n \nu_n(\sigma)}$$

$$= e^{\xi_1 \nu_1(\sigma) + \xi_2 \nu_2(\sigma) + \ldots + \xi_n \nu_n(\sigma)} = e^{\xi . \nu(\sigma)},$$ where

$\nu(\sigma) = (\nu_1(\sigma), \nu_2(\sigma), \ldots, \nu_n(\sigma)) \in K_n$. This completes the proof of the theorem.

5.1.2 <u>Theorem</u>: For a fixed subset $\hat{G}_0$ of $\hat{G}$, let $\nu : \sigma \to \nu(\sigma)$ be a mapping of $\hat{G}_0$ into K_n. Assume that the function ϕ_ξ, $\xi \in R_n^+$, defined on $\hat{G}$ by

$$\phi_\xi(\sigma) = \begin{cases} e^{\xi . \nu(\sigma)} & \text{if } \sigma \in \hat{G}_0 \\ 0 & \text{if } \sigma \notin \hat{G}_0 \end{cases}$$

is a (U, U)- multiplier. Define a mapping $T(\xi)$ of U into itself by

$$[T(\xi)f]^\wedge = \phi_\xi \hat{f} \qquad (f \in U)$$

Then $\{T(\xi) : \xi \in R_n^+\}$ is a strongly continuous semigroup of bounded linear operators on U, the elements of which commute with translations.

For arbitrary, but fixed, $\xi \in R_n^+$ let $A_0(\xi)$ be an infinitesimal operator of $\{T(\xi) : \xi \in R_n^+\}$, and let $D(A_0(\xi))$ denote the domain of $A_0(\xi)$. Then, for each $f \in D(A_0(\xi))$ and $\sigma \notin \hat{G}_0$, we have $\hat{f}(\sigma) = 0$. Moreover, $[A_0(\xi)f]^\wedge(\sigma) = \xi.\nu(\sigma)\hat{f}(\sigma)$ for all $f \in D(A_0(\xi))$, $\sigma \in \hat{G}_0$.

<u>Proof</u>: That $T(\xi)$ is a bounded linear operator, for each $\xi \in R_n^+$, follows from (35.2) of [8]. The semigroup property and the fact that the operators commute with translations are immediate from the definition of $T(\xi)$. The proof of strong continuity of $T(\xi)$ is as in Theorem 4.2.2. First, we suppose that t is a trigonometric polynomial on G, say $t = \alpha_1\chi_{\sigma_1} + \alpha_2\chi_{\sigma_2} + \ldots + \alpha_k\chi_{\sigma_k}$. The orthogonality relations of the elements of $\hat{G}$ imply that

$$T(\xi)t = \alpha_1 e^{\xi.\nu(\sigma_1)}\chi_{\sigma_1} + \ldots + \alpha_k e^{\xi.\nu(\sigma_k)}\chi_{\sigma_k}.$$ We then have

$$\begin{aligned}||T(\xi)t - T(\xi_0)t|| = ||(\alpha_1 e^{\xi.\nu(\sigma_1)}\chi_{\sigma_1} - \alpha_1 e^{\xi_0.\nu(\sigma_1)}\chi_{\sigma_1}) \\ + \ldots + (\alpha_k e^{\xi.\nu(\sigma_k)}\chi_{\sigma_k} - \alpha_k e^{\xi_0.\nu(\sigma_k)}\chi_{\sigma_k})|| \\ \leq |\alpha_1|\,|e^{\xi.\nu(\sigma_1)} - e^{\xi_0.\nu(\sigma_1)}| + \ldots + |\alpha_k|\,|e^{\xi.\nu(\sigma_k)} - e^{\xi_0.\nu(\sigma_k)}| \\ \to 0\end{aligned}$$

as $\xi \to \xi_0$ (convergence in K_n is defined componentwise, i.e. $\xi, \eta \in K_n$, $\xi \to \eta$ iff $\xi_i \to \eta_i$, for all $i = 1, 2, \ldots, n$).

Suppose now that f is arbitrary in U, and let $\varepsilon > 0$ be given. Then ([14], 1.5.2), there exists a trigonometric polynomial t on G such that $\|f - t\| < \varepsilon$. Since $\|T(\xi)f - T(\xi)t\| \leq \|T(\xi)\|\|f-t\|$ for each $\xi \in R_n^+$, it follows that $T(\xi)f$ is strongly measurable. Hence $T(\xi)$ is strongly continuous (for these last assertions, see proof of Theorem 4.2.2).

Let ξ be arbitrary, but fixed, in R_n^+, and let $f \in D(A_0(\xi))$. Then there exists $g = A_0(\xi) f \in U$ such that $g = \lim\limits_{t \to 0^+} \frac{1}{t} [T(t\xi)f - f]$, in norm. For each $\sigma \in \hat{G}$, $\hat{g}(\sigma) = \lim\limits_{t \to 0^+} \frac{1}{t}[(T(t\xi)f)^\wedge(\sigma) - \hat{f}(\sigma)]$.

If $\sigma \notin \hat{G}_0$, then $\hat{g}(\sigma) = -\lim\limits_{t \to 0^+} \frac{1}{t} \hat{f}(\sigma)$, which implies $\hat{f}(\sigma) = 0$.

If $\sigma \in \hat{G}_0$, then $\hat{g}(\sigma) = \lim\limits_{t \to 0^+} \frac{1}{t}(e^{t\xi.\nu(\sigma)} - 1)\hat{f}(\sigma)$

$$\text{i.e. } [A_0(\xi)f]^\wedge(\sigma) = \lim_{t \to 0^+} \frac{1}{t}\left(1 + t\xi.\nu(\sigma) + \frac{(t\xi.\nu(\sigma))^2}{2!} + \dots - 1\right) \hat{f}(\sigma)$$

$$= \lim_{t \to 0^+} \left(\xi.\nu(\sigma) + \frac{t(\xi.\nu(\sigma))^2}{2!} + \dots \right) \hat{f}(\sigma)$$

$$= \xi.\nu(\sigma) \hat{f}(\sigma).$$

CHAPTER 6

AN APPROXIMATION THEOREM FOR SEMIGROUPS OF OPERATORS

Let G be a compact abelian group with character group $\hat{G}$, and let U be an arbitrary, but fixed, member of the set $\{C(G), L_p(G) : 1 \leq p < \infty\}$, as in Chapter 4. Let $\nu : \sigma \to \nu(\sigma)$ be a mapping of $\hat{G}$ into K such that, for each $\xi > 0$, $e^{\xi\nu}$ is a (U, U) - multiplier. Define, for each $\xi > 0$, an operator $T(\xi)$ on U, by

$$(6.1.1) \qquad [T(\xi)f]^\wedge(\sigma) = e^{\xi\nu(\sigma)} \hat{f}(\sigma)$$

for all $f \in U$, $\sigma \in \hat{G}$. We investigate, in this chapter, the degree of approximation of the identity operator by the operator $T(\xi)$ for small values of the parameter ξ, i.e. the order of magnitude of $\|T(\xi)f - f\|$, as a function of ξ. Our main result generalises to compact abelian groups Hille and Phillips' result ([10], Theorem 20.6.1), proved for the circle group. Such results concerning approximation of the identity are of interest for applications to the theory of summability and singular integrals ([2], [9]).

6.1.1 <u>Definition</u>: Let $\mathcal{J} = \{T(\xi) : \xi \geq 0\}$ be a strongly continuous semigroup of bounded linear operators on U. $\mathcal{J}$ is said to be of class $(1, C_1)$, ([10], p .322), if

(i) $\displaystyle\int_0^1 \|T(\xi)\| d\xi < \infty$

and

(ii) $$\lim_{\eta \to 0^+} \frac{1}{\eta}\int_0^\eta T(\xi)f \, d\xi = f$$

in norm, for each $f \in U$.

For the basic classes of semigroups of operators on a Banach space, see 10.6 of [10].

6.1.2 <u>Definition</u>: Let J be a subset of $\hat{G}$. The linear extension of J, denoted by $\mathcal{L}_J$, is the set of all finite linear combinations of elements of J. $\bar{\mathcal{L}}_J$, the closure of $\mathcal{L}_J$ in the norm of U, is called the closed linear extension of J.

Since the closed linear extension of J is the smallest subspace of U containing all the characters χ_σ, $\sigma \in J$, we see that $\bar{\mathcal{L}}_J$ is identifiable with the set of trigonometric polynomials on J([8],(27.8)). Moreover, if $f \in U$ is such that $\hat{f}(\sigma) = 0$ for all $\sigma \notin J$, then ([8], p. 98) there exists a sequence $\{t_n\}$ in $\bar{\mathcal{L}}_J$, such that $\|f - t_n\| \to 0$. Since $\bar{\mathcal{L}}_J$ is closed, this implies that $f \in \bar{\mathcal{L}}_J$.

6.1.3 <u>Theorem</u>: Let G be a compact abelian group and let U be an arbitrary, but fixed, member of the set $\{C(G), L_p(G): 1 \leq p < \infty\}$. Suppose the operator $T(\xi)$, $\xi > 0$, on U, defined by

(6.1.1) satisfies

(i) $$\int_0^1 \|T(\xi)\| d\xi < \infty$$

and

(ii) $$\lim_{\eta \to 0^+} \frac{1}{\eta}\int_0^\eta T(\xi)f \, d\xi = f \qquad \text{for each } f \in U.$$

Then, (a) Set $J = \{\sigma \in \hat{G} : \nu(\sigma) = 0\}$. We have

$$(6.1.2) \qquad \liminf_{\xi \to 0^+} \frac{1}{\xi} \|T(\xi)f - f\| = 0$$

iff f belongs to the closed linear extension of J;

(b) Let A be the infinitesimal generator of $\{T(\xi) : \xi > 0\}$. For each $f \in D(A)$, the domain of A, we have

$$(6.1.3) \qquad T(\xi)f - f = \xi(Af + o(1))$$

for all $\xi > 0$, and $D(A) = \{f \in U : \nu\hat{f} \in \hat{U}\}$.

<u>Proof</u>: (a) By Theorem 1.2 of [13], $\mathcal{J} = \{T(\xi) : \xi > 0\}$ is a strongly continuous semigroup of operators on U. The assumptions (i) and (ii) of the theorem imply that $\mathcal{J}$ is of class $(1, C_1)$. Suppose $f \in U$ satisfies (6.1.2) ; by Theorem 10.7.2 of [10], $T(\xi)f = f$ for all $\xi > 0$. Conversely, if $f \in U$ is such that $T(\xi)f = f$ for all $\xi > 0$, then it is clear that f satisfies (6.1.2). Thus f satisfies $\liminf_{\xi \to 0^+} \frac{1}{\xi} \|T(\xi)f - f\| = 0$ iff $T(\xi)f = f$ for all $\xi > 0$.

Next, we show that $T(\xi)f = f$ for all $\xi > 0$ iff for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. Suppose $T(\xi)f = f$ for all $\xi > 0$. Then for each $\sigma \in \hat{G}$, we have $[T(\xi)f]^\wedge(\sigma) = \hat{f}(\sigma)$. Since $\hat{f}(\sigma)$ is independent of ξ, it follows that $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ, for each $\sigma \in \hat{G}$. Conversely, suppose that for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. Then $e^{\xi\nu(\sigma)}\hat{f}(\sigma) = e^{2\xi\nu(\sigma)}\hat{f}(\sigma)$, which implies $\hat{f}(\sigma) = e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ for each $\sigma \in \hat{G}$.

We then have $[T(\xi)f]^\wedge(\sigma) = \hat{f}(\sigma)$ for each $\sigma \in \hat{G}$, which implies $T(\xi)f = f$ for all $\xi > 0$. The proof of (a) of the theorem will be complete if we show that an $f \in U$ belongs to the closed linear extension of J iff for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. So, suppose that f is in the closed linear extension of J; then there is a sequence $\{f_n\}$ in $\mathcal{L}_J$ such that $\|f_n - f\| \to 0$. Let $\sigma \in \hat{G}$. If $\sigma \in J$, then $\nu(\sigma) = 0$ and hence $e^{\xi\nu(\sigma)}\hat{f}(\sigma) = \hat{f}(\sigma)$ is independent of ξ. If $\sigma \notin J$, then $\hat{f}_n(\sigma) = 0$ for each n. Since then, $|\hat{f}(\sigma)| \leq |\hat{f}(\sigma) - \hat{f}_n(\sigma)| + |\hat{f}_n(\sigma)| = |(f-f_n)^\wedge(\sigma)| \leq \|f-f_n\|$, and $\|f-f_n\| \to 0$, it follows that $\hat{f}(\sigma) = 0$. Hence $e^{\xi\nu(\sigma)}\hat{f}(\sigma) = 0$ is independent of ξ. Conversely, suppose $f \in U$ is such that, for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. If $\sigma \notin J$, then $\nu(\sigma) \neq 0$, so we must have $\hat{f}(\sigma) = 0$. By the remarks following Definition 6.1.2, f must be in the closed linear extension of J.

(b) Let A_0 be the infinitesimal operator of $\mathcal{J} = \{T(\xi) : \xi > 0\}$. Since $\mathcal{J}$ is of class $(1, C_1)$, Theorem 10.7.2 of [10] implies that for each $f \in D(A_0)$, we have $T(\xi)f - f = \xi\,(A_0 f + O(1))$, for all $\xi > 0$. But since $\mathcal{J}$ is of class $(1, C_1)$, A_0 is closed ([10], Theorem 10.5.3); hence $A_0 = A$, the infinitesimal generator of $\mathcal{J}$. It follows that if $f \in D(A)$, then $T(\xi)f - f = \xi(Af + O(1))$ for all $\xi > 0$.

The fact that $\mathcal{J}$ is of class $(1, C_1)$ implies that $\mathcal{J}$ is of class (A) ([10], Theorem 10.6.1). It now follows from Theorem 1.2 of [13] that $D(A) = \{ f \in U : \nu\hat{f} \in \hat{U} \}$.

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