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G-Theory of Group Rings for Groups of Elementary Abelian p-Groups

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Abstract

The formula for the $G-$ theory of the group ring of a finite group G given by H. Lenstra is shown to be valid for groups of elementary abelian $p-$ groups.
Keywords: Group rings , $G-$ theory.

Introduction

In [8] . H. Lenstra obtained a fundamental formula for the Grothendieck group $\mathcal{G}_0(\mathbb{Z}G)$ for G a finite abelian group and $\mathbb{Z}$ a commutative noetherian ring , in terms of Grothendieck groups of rings of fractions of algebraic integer rings. In [13], D. L. Webb established the formula

$$\mathcal{G}_n(\mathbb{Z}G) \simeq \bigoplus_{\rho \in X(G)} \mathcal{G}_n(\mathbb{Z} < \rho >), \quad n \geq 0,$$

where $\mathbb{Z} < \rho >$ denotes the ring of fractions $\mathbb{Z}(\rho)[1/|\rho|]$ obtained by inverting $|\rho|$, $\mathbb{Z}(\rho)$ denotes the quotient of the group ring $\mathbb{Z}\rho$ by the $|\rho|^{-th}$ cyclotomic polynomial $\Phi_{|\rho|}$ evaluated at a generator of ρ (the ideal factored out is independent of the choice of generator for ρ). $|.|$ denotes cardinality and $X(\pi)$ the set of cyclic quotients of π . A natural problem is that of computing $\mathcal{G}_n(\mathbb{Z}G)$ as explicitly as possible and from the formula above, when G is substituted as

$$\underbrace{\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n}_{r-times}, \quad n \geq 1, \quad r > 1, \quad n$$

a positive integer , p a prime number, it is desirable to know the number of cyclic quotients of G and the exact picture of the formula for G . Our results extend [2].

The Results and their Proofs

We established first the following lemma , which constitute the technical heart of the next theorem.

Lemma 2.1 :

Let

$$G := \underbrace{\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n}_{r-times}, \quad r > 1, \quad n \text{ a positive integer, } p$$

a prime number and H a subgroup of G. Then the number of the cyclic factor groups G/H up to isomorphism such that

$|G/H| = p^n$ is $p^{(n-1)(r-1)}(\frac{p^r-1}{p-1})$

Proof:

By the Duality Theorem for finite abelian groups , the number of subgroups H of G for which G/H is cyclic of order m is equal to the number of cyclic subgroups of G of order m .[11]

Now $G = (\mathbb{Z}/p^n)^r$.

The number of elements of order p^n in G is

$p^{nr} - p^{(n-1)r}$

and a cyclic group of order p^n contains $p^n - p^{n-1}$ such elements; so the number of cyclic subgroups is

$\frac{p^{nr}-p^{(n-1)r}}{p^n-p^{n-1}} = p^{(n-1)(r-1)}(\frac{p^r-1}{p-1})$. □

Next, consider

Theorem 2.2:

Let

$$G := \underbrace{\mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j}_{r-times}, \quad r > 1, \; j \in \{1, 2, ..., n\}, \; p$$

a prime number and H is a subgroup of G. Then the number of the cyclic factor groups G/H up to isomorphism such that $|G/H| = p^j$ for all j summed to n, is $(\frac{p^r-1}{p-1})(\frac{p^{n(r-1)}-1}{p^{r-1}-1})$.

Proof:

Using Lemma 2.1 and summing over j from 1 to n immediately gives the theorem . □

Finally , we give the proof of the following:

Proposition 2.3:

For $r > 1$, p *a prime number and* $j \in \{1, 2, ..., n\}$.

Let

$$G := \underbrace{\mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j}_{r-times}.$$

Then

$\mathcal{G}_0(\mathbb{Z}G) =$

$\mathbb{Z} \bigoplus Cl(\mathbb{Z}[\zeta_1]) \bigoplus_{j=1}^{t} (\mathbb{Z} \bigoplus Cl(\mathbb{Z}[\zeta_{p^j}, \frac{1}{p^j}])^s$

where $Cl(R)$ is the ideal class group of Dedekind ring R , t is determined from Theorem 2.2 and $s = p^{(j-1)(r-1)}(\frac{p^r-1}{p-1})$ (by Lemma 2.1)

Proof:

For $G := \underbrace{\mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j}_{r-times}$, $r > 1$,

p a prime number and $j \in \{1, 2, ..., n\}$.

Let $\{H_0, ..., H_t\}$ be the set of all subgroups of G for which G/H_j is cyclic , where t is determined from Theorem

2.2 above
Then we obtain two forms:

(I) If $\rho_0 = G/H_0$ with $|\rho_0| = 1$,
then $\mathbb{Z}(\rho_0) = \mathbb{Z}\langle\rho_0\rangle = \mathbb{Z}[\zeta_1]$ is a Dedekind ring. Where ζ_1 is the first primitive root of unity . But it is well known for any Dedekind ring R that
$\mathcal{G}_0(R) \cong \mathbb{Z} \bigoplus Cl(R)$
where $Cl(R)$ is the ideal class group of R . Thus , for this form we obtain
$\mathcal{G}_0(\mathbb{Z}\langle\rho_0\rangle) \cong \mathbb{Z} \bigoplus Cl(\mathbb{Z}[\zeta_1])$

(II) For $j > 0$, we consider
$\rho_j = G/H_j$ with $|\rho_j| = p^j$
and obtain for each $j > 0$ $\mathbb{Z}(\rho_j) \cong \mathbb{Z}[\zeta_{p^j}]$, where ζ_{p^j} denotes a primitive $p^{j\ th}$ root of unity.
Therefore , we get
$\mathbb{Z}\langle\rho_j\rangle \cong \mathbb{Z}[\zeta_{p^j}, \frac{1}{p^j}]$ a Dedekind ring.
Thus , we obtain (using Lemma 2.1 and Theorem 2.2)
$\mathcal{G}_0(\mathbb{Z}\langle\rho_j\rangle) \cong \bigoplus_{j=1}^{t}(\mathbb{Z} \bigoplus Cl(\mathbb{Z}[\zeta_{p^j}, \frac{1}{p^j}])^s$ where $s = p^{(j-1)(r-1)}(\frac{p^r-1}{p-1})$
Hence combining results from I and II , and by Lenstra's formula ,
that is,
$\mathcal{G}_0(\mathbb{Z}G) \cong \prod_{j=0}^{t} \mathcal{G}_0(\mathbb{Z}\langle\rho_j\rangle)$
we obtain
$\mathcal{G}_0(\mathbb{Z}G) =$
$\mathbb{Z} \bigoplus Cl(\mathbb{Z}[\zeta_1]) \bigoplus_{j=1}^{t}(\mathbb{Z} \bigoplus Cl(\mathbb{Z}[\zeta_{p^j}, \frac{1}{p^j}])^s$ □

Open Problems

Determine the version of the above proposition 2.3 for $\mathcal{G}_n(\mathbb{Z}G)$, $n \geq 1$. and extend to the ideas discussed in [1], [3], [4], [5], [6], [7], [9], [10], [12] and [14] respectively.

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