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A Connection between Units of Burnside Rings and the Exterior Algebra of Elementary Abelian 2-Groups

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INTRODUCTION

Let G be a finite group, $\Omega(G)$ the Burnside ring of G, that is, the Grothendieck ring obtained from the semi-ring of G-isomorphism classes of finite G-sets under addition and multiplication induced respectively by the disjoint union and the Cartesian product. The goal of this paper is to give the connection between the structure of the group $\Omega(G)^*$ of units of $\Omega(G)$ and the associated Exterior Algebra, where

$$G := \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n\text{-times}}$$

is an elementary abelian 2-group of order 2^n.

In Section 1 we discuss the condition (UB) and show how an element of $\Omega(G)^*$ can be identified. In Section 2, we show that the map

$$\omega_i\colon G^i \to \Omega(G)^*_{i+1}/\Omega(G)^*_i$$

is multilinear and that

$$\omega_i(g_1, \dots, g_i) = 0$$

if

$$\#\{g_1, \dots, g_i\} < i.$$

Finally, we show that ω_i induces an isomorphism between $\bigwedge^i(G)$ and $\Omega(G)^*_{i+1}/\Omega(G)^*_i$.

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1. CONDITION (UB)

1.1. Let

$$G := \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n\text{-times}}$$

be an elementary abelian 2-group of order 2^n and let $\mathrm{Sub}(G)$ denote its subgroup lattice. It is well known that the group $\Omega(G)^*$ of units in the Burnside ring $\Omega(G)$ of G is canonically isomorphic to the group of maps

$$e\colon \mathrm{Sub}(G) \to \{\pm 1\}$$

satisfying the following condition:

(UB) For all $U, U_1, U_2, U_3, V \in \mathrm{Sub}(G)$ with $(V : U) = 4$ such that

$$\{U_1, U_2, U_3\} := \{W \in \mathrm{Sub}(G) \mid U \subseteq W \subseteq V\},$$

we have

$$e(U) \cdot e(U_1) \cdot e(U_2) \cdot e(U_3) = 1.$$

1.2. THEOREM. *For every $H \leq G$, the map*

$$e_H\colon \mathrm{Sub}(G) \to \{\pm 1\}$$

$$U \mapsto 1 - 2\delta_{G, H\cdot U} := \begin{cases} 1, & \text{if } H \cdot U \neq G \\ -1, & \text{if } H \cdot U = G \end{cases}$$

satisfies (UB) *and hence represents an element in* $\Omega(G)^*$.

Proof. Assume that U, V, U_1, U_2, U_3 are as in (UB). We distinguish the following cases:

Case 1. If

$$e_H(U) = e_H(U_1) = e_H(U_2) = e_H(U_3) = 1$$

there is nothing to prove.

Case 2. If

$$e_H(U) = -1, \qquad \text{that is, } H \cdot U = G,$$

then

$$G \supseteq H \cdot U_i \supseteq H \cdot U = G,$$

so $H \cdot U_i = G$ for $i = 1, 2, 3$ and therefore $e_H(U_i) = -1$ for $i = 1, 2, 3$. So also in this case

$$e_H(U) \cdot e_H(U_1) \cdot e_H(U_2) \cdot e_H(U_3) = (-1)^4 = 1.$$

Case 3. If

$$e_H(U) = 1, \qquad e_H(U_i) = -1$$

for at least one $i \in \{1,2,3\}$, say $i = 1$, then we argue as follows. We have

$$U \cdot H \neq G$$

and

$$U_1 H = G.$$

We have to show that we can neither have

$$U_2 \cdot H = U_3 \cdot H = G$$

nor

$$U_2 \cdot H \neq G, \qquad U_3 \cdot H \neq G.$$

To this end we prove first the following.

1.3. LEMMA. *If G is a group, H a normal subgroup, and W_1, W_2 are subgroups of G with $W_1 \subseteq W_2$, then*

$$\frac{(W_2 : W_1)}{(H \cdot W_2 : H \cdot W_1)}.$$

Proof. Given that $W_1 \subseteq W_2$ and $H \trianglelefteq G$, then we have

$$HW_1 \leq HW_2 \leq G, \qquad W_1 \cap H \trianglelefteq W_1, \qquad \text{and} \qquad W_2 \cap H \trianglelefteq W_2.$$

Consider

$$W_1 \cap H.$$

Let α be an arbitrary element of $W_1 \cap H$. Then $\alpha \in H$ and $\alpha \in W_1$. But $W_1 \subseteq W_2$; this implies $\alpha \in W_2$. Now $\alpha \in W_2$ and $\alpha \in H$; this implies $\alpha \in W_2 \cap H$.

So we have

$$\alpha \in W_1 \cap H \Rightarrow \alpha \in W_2 \cap H.$$

Hence,

$$W_1 \cap H \subseteq W_2 \cap H$$

and as both intersection are subgroups

$$W_1 \cap H \leq W_2 \cap H.$$

In particular,

$$\frac{|H \cap W_2|}{|H \cap W_1|}$$

is a positive integer. We now consider

$$\frac{|H \cdot W_2|}{|H \cdot W_1|} := \frac{(|H| \cdot |W_2|)/|H \cap W_2|}{(|H| \cdot |W_1|)/|H \cap W_1|} := \frac{|W_2| \cdot |H \cap W_1|}{|W_1| \cdot |H \cap W_2|}$$

which implies that

$$(W_2 : W_1) = (H \cap W_2 : H \cap W_1)(H \cdot W_2 : HW_1).$$

This implies that $(HW_2 : HW_1)$ divides $(W_2 : W_1)$, since $(H \cap W_2 : H \cap W_1)$ is a positive integer.

Hence

$$\frac{(W_2 : W_1)}{(H \cdot W_2 : H \cdot W_1)}.$$

It also follows from this result that if

$$(W_2 : W_1) = 2, \qquad \text{then } (H \cdot W_2 : H \cdot W_1) \leq 2.$$

Next, we show that with $G, H, U, U_1, U_2, U_3, V$ as above.

1.4. LEMMA. *If* $(G : U \cdot H) = 2$, *then one has*

$$U_j \cdot H = G \Leftrightarrow U_j \not\subseteq U \cdot H \qquad \text{for } j = 1, 2, 3.$$

Proof. Assume first that $U_j \cdot H = G$. Then since $U \cdot H \neq G$, we obtain that $U \cdot H \subseteq U_j \cdot H$, and this implies that $U_j \cdot H \not\subseteq U \cdot H$, so we have that $U_j \not\subseteq U \cdot H$, since $H \subseteq U \cdot H$. Thus, $U_j \cdot H = G \Rightarrow U_j \not\subseteq U \cdot H$.

Conversely, suppose that $U_j \not\subseteq U \cdot H$. Then $U_j \cdot H \not\subseteq U \cdot H$. It also follows that $U \cdot H \subseteq U_j \cdot H$, since $U \subseteq U_j$ and therefore $U \cdot H \subseteq U_j \cdot H$ but $U \cdot H \neq U_j \cdot H$. Now, as $U \cdot H \neq G$, we get that $U \cdot H \subseteq G$. We know also that $U_j \cdot H \subseteq G$. So it follows that

$$U \cdot H \subseteq U_j \cdot H \subseteq G.$$

Hence

$$2 = (G : U \cdot H) = (G : U_j \cdot H) \cdot (U_j \cdot H : U \cdot H)$$

together with $(U_j \cdot H : U \cdot H) \neq 1$ and therefore (by Lemma 1.3)

$$(U_j \cdot H : U \cdot H) = 2.$$

This implies

$$2 = (G : U_j \cdot H) \cdot 2$$

or

$$(G : U_j \cdot H) = 1.$$

So we must have that

$$G = U_j \cdot H.$$

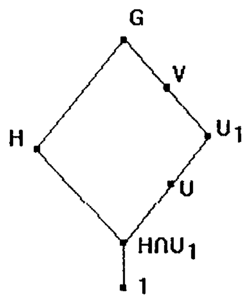

FIG. 1. Step 1.

Thus,

$$U_j \nsubseteq U \cdot H \Rightarrow U_j \cdot H = G,$$

and so

$$U_j \cdot H = G \Leftrightarrow U_1 \nsubseteq U \cdot H.$$

Now we continue with the proof of Case 3.

Consider the stepwise diagrams shown in Figs. 1–3 with the motive of getting a final result for Case 3.

Step 1. Consider $H \cdot U_1 = G$ (see Fig. 1).

Step 2. See Fig. 2.

(iii) To show that $U \cdot H \cap V \neq U_1$, assume $U \cdot H \neq G$. We must show that

(ii) $H \cap U_1 \subseteq U$ by first showing that

(i) $H \cdot U \cap U_1 = U$.

Proof of (i). In the first place, it is evident that

$$U \subseteq U \cdot H \cap U_1,$$

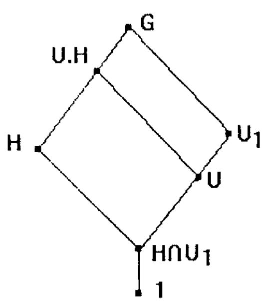

FIG. 2. Step 2.

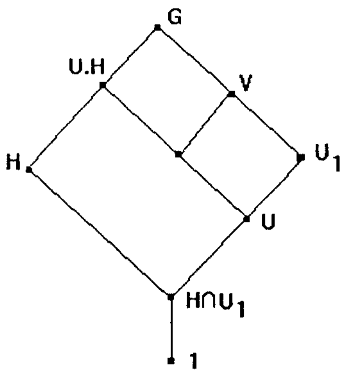

FIG. 3. Step 3.

since

$$U \subseteq U \cdot H, \qquad U \subseteq U_1.$$

To prove

$$U \cdot H \cap U_1 \subseteq U,$$

consider

$$U \cdot H \cap U_1.$$

By definition of intersection,

$$U \cdot H \cap U_1 \subseteq U_1.$$

So,

$$U \subseteq U \cdot H \cap U_1 \subseteq U_1.$$

But since $(U_1 : U) = 2$, it then follows that either

$$U = U \cdot H \cap U_1$$

or

$$U \cdot H \cap U_1 = U_1.$$

But then, by our assumption that

$$U \cdot H \neq G; \qquad U_1 \cdot H = G,$$

which also implies

$$U_1 \nsubseteq U \cdot H \qquad \text{(see Lemma 1.4)}$$

we obtain that

$$U \cdot H \cap U_1 \neq U_1$$

because $U \cdot H \cap U_1 = U_1$ would imply that $U_1 \subseteq U \cdot H$ which in turn gives a contradiction to our assumption. Hence

$$U \cdot H \cap U_1 = U.$$

Step 3 (see Fig. 3). Since we obtain from Step 2 that

$$U = U \cdot H \cap U_1 \neq U_1,$$

this implies $U_1 \not\subseteq U \cdot H$ and therefore

$$U \cdot H \cap V \neq U_1.$$

Observe that

$$U \subseteq U \cdot H \cap V \subseteq V.$$

Step 4. See Fig. 4.

Step 5. See Fig. 5.

Without loss of generality, say, $U \cdot H \cap V = U_2$ and $U \cdot H \cap V \neq U_3$. We must show first that

$$U \cdot H \cap V \in \{U_2, U_3\}.$$

Proof. As

$$U \cdot H \leq U \cdot H \cdot V = G$$

and

$$V \trianglelefteq U \cdot H \cdot V,$$

we obtain

$$U \cdot H \cap V \trianglelefteq U \cdot H,$$

$$|U \cdot H \cdot V| = |G| = 2^n.$$

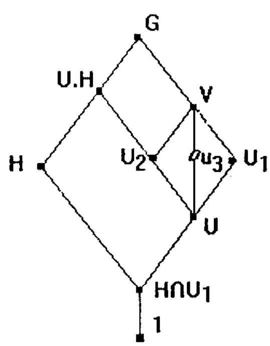

FIG. 4. Step 4.

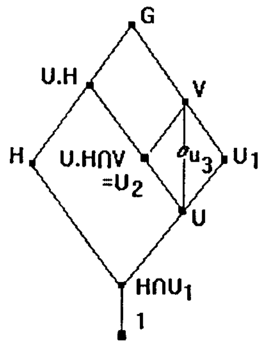

FIG. 5. Step 5.

We also have that

$$|U \cdot H| < 2^n,$$

since $U \cdot H \neq G$, and $U_i \cdot H = G$ for any i together implies

$$1 < (G : U \cdot H) \leq 2.$$

This implies

$$(G : U \cdot H) = 2$$

and hence that

$$|U \cdot H| = 2^{n-1}.$$

Next, we consider the equation

$$|U \cdot H \cdot V| = \frac{|U \cdot H| \cdot |V|}{|(U \cdot H) \cap V|}$$

which implies that

$$2^n = \frac{2^{n-1} \cdot |V|}{|(U \cdot H) \cap V|} = 2^{n-1}(V : (U \cdot H) \cap V).$$

This implies

$$(V : (U \cdot H) \cap V) = \frac{2^n}{2^{n-1}} = 2,$$

and since $U \subseteq U \cdot H$ and $U \subseteq V$ implies $U \subseteq U \cdot H \cap V$, because

$$(V : (U \cdot H) \cap V) = 2 \neq (V : V), (V : U)$$

we obtain

$$U \cdot H \cap V \neq V, U$$

and also by Step 3,

$$U \cdot H \cap V \neq U_1.$$

Hence

$$U \cdot H \cap V \in \{U_2, U_3\}.$$

We shall finally prove Step 5.

Since

$$U \cdot H \cap V \in \{U_2, U_3\}$$

by Step 4, we may then assume without loss of generality that

$$U \cdot H \cap V = U_2; \qquad U \cdot H \cap V \neq U_3.$$

This implies

$$U_2 \subseteq U \cdot H; \qquad U_3 \not\subseteq U \cdot H, \text{ since } U_3 \subseteq V.$$

This implies

$$U_2 \cdot H \subseteq UH \neq G; \qquad U_3 \not\subset U \cdot H.$$

Hence,

$$U_2 H \neq G; \qquad U_3 \cdot H = G \text{ (by Lemma 1.4)}.$$

Therefore the proof of Case 3 is complete.

So we conclude by Case 1, Case 2, and Case 3 that the map

$$e_H : \mathrm{Sub}(G) \to \{\pm 1\}$$

satisfies condition (UB).

2. MULTILINEARITY CONDITION

2.1. Now for each $i = 0, 1, 2, \ldots,$ we define

$$\Omega(G)_i^* := \left\{e \in \Omega(G)^* \mid e(U) = 1 \text{ for all } U \leq G \text{ with } |U| \leq 2^{n-i}\right\}$$

and observe that

$$e_H \in \Omega(G)_i^* \qquad \text{if } |H| \leq 2^{i-1},\ H \leq G.$$

2.2. THEOREM. *Define the map*

$$\omega_l : G^i \to \Omega(G)_{i+1}^* / \Omega(G)_i^*$$

by

$$: (g_1, \ldots, g_i) \mapsto e_{\langle g_1, \ldots, g_i \rangle} \Omega(G)_i^*.$$

Then

2.2.1. ω_i *is multilinear, and*

2.2.2. $\omega_i(g_1, \ldots, g_i) = 0$ *if* $\#\{g_1, \ldots, g_i\} < i$.

Before we prove Statement 2.2.1, we shall first state and prove the following useful lemmata:

2.3. LEMMA. *Let G be a group of order 2^n and K, $H \trianglelefteq G$, such that $|K| := 2^i$ and $|H| := 2^{n-i}$. Then $K \cdot H = G \Leftrightarrow K \cap H = 1$.*

Proof. As $K \leq G$ and $H \trianglelefteq G$ imply

$$K \cdot H = \langle K, H \rangle \leq G,$$

we consider the equation

$$\begin{aligned} |K \cdot H| &= \frac{|K| \cdot |H|}{|K \cap H|} \\ &= \frac{2^i \cdot 2^{n-i}}{|K \cap H|} \\ &= \frac{2^{i+n-i}}{|K \cap H|} \\ &= \frac{2^n}{|K \cap H|}. \end{aligned}$$

Hence

$$|K \cdot H| = |G| = 2^n \Leftrightarrow |K \cap H| = 1.$$

Therefore

$$G = K \cdot H \Leftrightarrow K \cap H = 1.$$

2.4. LEMMA. *Let*

$$G := \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n\text{-times}}, \qquad A \subseteq G.$$

Then

$$|\langle A \rangle| \leq 2^{\#A}.$$

Proof. Assume $\#A := i$. Label the elements in A as, say, $a_1, \ldots, a_i$, so that

$$A := \{a_1, \ldots, a_i\}.$$

Then

$$\langle A \rangle = \{a_1^{\epsilon_1}, \ldots, a_i^{\epsilon_i} \mid \epsilon_1, \ldots, \epsilon_i \in \{0, 1\}\}.$$

To see this, let H be the set on the righthand side above. Since $\langle A \rangle$ is closed under multiplication and the forming of inverses, $H \subseteq \langle A \rangle$. But also, by definition, $\langle A \rangle$ is the unique smallest subgroup of G containing A in the sense that, for all $U \leq G$, whenever $A \subseteq U \leq G$, then $\langle A \rangle \leq U$. Obviously, since all the elements of A are used up in the construction of an element of H, $A \subseteq H$. Now, let $h_1, h_2 \in H$. Then $h_1 := a_1^{\epsilon_1} \dots a_i^{\epsilon_i}$ for all choices of $\epsilon_1, \dots, \epsilon_i \in \{0,1\}$, and $h_2 := a_1^{\eta_1} \dots a_i^{\eta_i}$ for every choice of $\eta_1, \dots, \eta_i \in \{0,1\}$. Next we consider

$$\begin{aligned} h_1 \cdot h_2^{-1} &:= \left(a_1^{\epsilon_1} \cdot a_2^{\epsilon_2} \dots a_{i-1}^{\epsilon_i - 1} \cdot a_i^{\epsilon_i}\right)\left(a_1^{\eta_1} \cdot a_2^{\eta_2} \dots a_{i-1}^{\eta_{i-1}} \cdot a_i^{\eta_i}\right)^{-1}, \\ &= a_1^{\epsilon_1} \cdot a_2^{\epsilon_2} \dots a_{i-1}^{\epsilon_{i-1}} \cdot a_i^{\epsilon_i} \cdot a_i^{-\eta_i} \cdot a_{i-1}^{-\eta_{i-1}} \dots a_2^{-\eta_2} \cdot a_1^{-\eta_1} \\ &= a_1^{\epsilon_1} \cdot a_1^{-\eta_1} \cdot a_2^{\epsilon_2} \dots a_{i-1}^{\epsilon_{i-1}} \cdot a_i^{\epsilon_i} \cdot a_i^{-\eta_i} \cdot a_{i-1}^{-\eta_{i-1}} \dots a_2^{-\eta_2} \\ &= a_1^{\epsilon_1} \cdot a_1^{-\eta_1} \cdot a_2^{\epsilon_2} \cdot a_2^{-\eta_2} \dots a_{i-1}^{\epsilon_{i-1}} \cdot a_i^{\epsilon_i} \cdot a_i^{-\eta_i} \cdot a_{i-1}^{-\eta_{i-1}} \dots . \end{aligned}$$

Continuing in this way, we get

$$\begin{aligned} h_1 \cdot h_2^{-1} &= a_1^{\epsilon_1 - \eta_1} \cdot a_2^{\epsilon_2 - \eta_2} \dots a_{i-1}^{\epsilon_{i-1} - \eta_{i-1}} \cdot a_i^{\epsilon_i - \eta_i} \\ &= a_1^{\alpha_1} \cdot a_2^{\alpha_2} \dots a_{i-1}^{\alpha_{i-1}} \cdot a_i^{\alpha_i}, \end{aligned}$$

where in view of the special structure of G, α_i is determined by ϵ_i and η_i according to the scheme

$\epsilon \mid \eta$	0	1
0	0	1
1	1	0

Hence $h_1 \cdot h_2^{-1} \in H$, and this implies that $H \leq G$. So we have

$$\langle A \rangle \leq H, \qquad \text{and therefore } \langle A \rangle = H.$$

Hence, we obtain $|\langle A \rangle| \leq 2^{\# A}$, since

$$2^i = \#\{(\epsilon_1, \dots, \epsilon_i) \mid \epsilon_i \in \{0,1\}\}.$$

Proof of Statement 2.2.1. Let r be such that $1 \leq r \leq i$. For every r and $g_r, h_r \in G$. Consider

$$\begin{aligned} &\omega_i(g_1, \dots, g_{r-1}, g_r * h_r, g_{r+1}, \dots, g_i) \\ &\qquad = \omega_i(g_1, \dots, g_r, \dots, g_i) \cdot \omega_i(g_1, \dots, h_r, \dots, g_i). \end{aligned}$$

To see this, we must prove that

$$e_{\langle g_1, \dots, g_r * h_r, \dots, g_i \rangle}(H) e_{\langle g_1, \dots, g_r, \dots, g_i \rangle}(H) e_{\langle g_1, \dots, h_r, \dots, g_i \rangle}(H) = 1$$

for all $H \leq G$ with $|H| \leq 2^{n-i}$.

Without loss of generality we may assume that $r = i$, put $a := g_i$, $b = h_i$, $c := g_i * h_i$ so that $a \cdot b \cdot c = 1$. Then we can define

$$A := \langle g_1, \dots, g_{i-1}, a \rangle$$
$$B := \langle g_1, \dots, g_{i-1}, b \rangle$$
$$C := \langle g_1, \dots, g_{i-1}, c \rangle.$$

Note that since

$$|A| \leq 2^i,$$
$$|B| \leq 2^i,$$
$$|C| \leq 2^i,$$

we have by the above result that

$$e_A \in \Omega(G)^*_{i+1},$$
$$e_B \in \Omega(G)^*_{i+1},$$

and

$$e_C \in \Omega(G)^*_{i+1},$$

respectively, that is, the following case is obvious. For any $H \leq G$ with $|H| < 2^{n-i}$, we get that

$$e_A(H) = e_B(H) = e_C(H) = 1.$$

So, we consider the only non-trivial case $|H| = 2^{n-i}$.

Next we shall discuss under this case some of the useful consequences derived for members in the set

$$\{A, B, C\}$$

and with respect to distinguished cases as

(i) Assume $|A| = |B| = |C| = 2^i$.
(ii) Assume $|A| < 2^i$, $|B| = |C| = 2^i$.
(iii) Assume $|A| < 2^i$, $|B| < 2^i$, $|C| = 2^i$.
(iv) Assume $|A| < 2^i$, $|B| < 2^i$, $|C| < 2^i$.

First, we discuss case (iv) as follows: As

$$|A| \leq 2^{i-1}, \qquad |B| \leq 2^{i-1}, \qquad |C| \leq 2^{i-1},$$

we obtain

$$|H \cdot A| \leq 2^{n-i} \cdot 2^{i-1} = 2^{n-1} < |G|,$$
$$|H \cdot B| \leq 2^{n-i} \cdot 2^{i-1} = 2^{n-1} < |G|,$$
$$|H \cdot C| \leq 2^{n-i} \cdot 2^{i-1} = 2^{n-1} < |G|,$$

and it follows by definition that

$$e_A(H) = e_B(H) = e_C(H) = 1.$$

Second, we discuss cases (ii) and (iii) by proving the following lemma:

2.5. LEMMA. *The following are equivalent*

(i) $A = \langle g_1, \dots, g_{i-1} \rangle$

(ii) $a \in \langle g_1, \dots, g_{i-1} \rangle$

(iii) $B = C$.

Proof. (i) $\Rightarrow$ (ii), i.e., $A = \langle g_1, \dots, g_{i-1} \rangle \Rightarrow a \in \langle g_1, \dots, g_{i-1} \rangle$. Assume

$$A = \langle g_1, \dots, g_{i-1} \rangle.$$

Since

$$a \in A := \langle g_1, \dots, g_{i-1}, a \rangle \qquad \text{and} \qquad A := \langle g_1, \dots, g_{i-1} \rangle$$

it follows that $a \in \langle g_1, \dots, g_{i-1} \rangle$.

(ii) $\Rightarrow$ (iii), i.e.,

$$a \in \langle g_1, \dots, g_{i-1} \rangle \Rightarrow B = C.$$

Assume $a \in \langle g_1, \dots, g_{i-1} \rangle$. Then we have

$$a = g_1^{\epsilon_1}, \dots, g_{i-1}^{\epsilon_{i-1}} \qquad \text{for some choices } \epsilon_1, \dots, \epsilon_{i-1} \in \{0, 1\}.$$

In view of $g_1, \dots, g_{i-1} \in C$ by definition, it is enough to observe that

$$\begin{aligned} b = ac &= g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot c \\ &\in \langle g_1, \dots, g_{i-1}, c \rangle := C, \end{aligned}$$

hence, $B \subseteq C$.

Similarly, on the other hand, in view of $g_1, \dots, g_{i-1} \in B$ we observe that

$$\begin{aligned} c = ab &= g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot b \\ &\in \langle g_1, \dots, g_{i-1}, b \rangle := B. \end{aligned}$$

Hence, $C \subseteq B$; therefore, $B = C$.

(iii) $\Rightarrow$ (ii), i.e., $B = C \Rightarrow a \in \langle g_1, \dots, g_{i-1} \rangle$. Assume $B = C$. Then there exist $\epsilon_1, \dots, \epsilon_{i-1}, \epsilon$ and $\eta_1, \dots, \eta_{i-1}, \eta$ with $b = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot c^{\epsilon}$ and $c = g_1^{\eta_1} \dots g_{i-1}^{\eta_{i-1}} \cdot b^{\eta}$. Now if $\epsilon = 1$, then

$$a = bc = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot c \cdot c = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \in \langle g_1, \dots, g_{i-1} \rangle.$$

Similarly, if $\eta = 1$, then

$$a = bc = cb = g_1^{\eta_1} \dots g_{i-1}^{\eta_{i-1}} \cdot b \cdot b = g_1^{\eta_1} \dots g_{i-1}^{\eta_{i-1}} \in \langle g_1, \dots, g_{i-1} \rangle$$

and if $\epsilon = \eta = 0$ then

$$b \in \langle g_1, \ldots, g_{i-1} \rangle, \qquad c \in \langle g_1, \ldots, g_{i-1} \rangle.$$

This implies

$$a = bc \in \langle g_1, \ldots, g_{i-1} \rangle$$

hence,

$$a \in \langle g_1, \ldots, g_{i-1} \rangle.$$

(ii) $\Rightarrow$ (i), i.e., $a \in \langle g_1, \ldots, g_{i-1} \rangle \Rightarrow A = \langle g_1, \ldots, g_{i-1} \rangle$. Assume

$$a \in \langle g_1, \ldots, g_{i-1} \rangle.$$

Then we have

$$\langle g_1, \ldots, g_{i-1} \rangle \subseteq A,$$

since

$$a \in A := \langle g_1, \ldots, g_{i-1}, a \rangle$$

also

$$A \subseteq \langle g_1, \ldots, g_{i-1} \rangle,$$

since by assumption

$$a \in \langle g_1, \ldots, g_{i-1} \rangle,$$

hence

$$A = \langle g_1, \ldots, g_{i-1} \rangle,$$

and the proof of the lemma is complete.

Continuation of the Proof of Statement 2.2.1. By Lemma 2.4, we get that

$$|\langle g_1, \ldots, g_{i-1} \rangle| = 2^{i-1}.$$

Hence, in this case,

$$|A| < 2^i \Leftrightarrow |A| = 2^{i-1} \Leftrightarrow A = \langle g_1, \ldots, g_{i-1} \rangle.$$

In view of the above considerations we conclude that Case (ii) is possible and our formula

$$e_A(H) \cdot e_B(H) \cdot e_C(H) = 1$$

is almost trivially satisfied, since

$$|H \cdot A| \leq 2^{n-i} \cdot 2^{i-1} = 2^{n-1} < |G|, \qquad \text{implies } e_A(H) = 1,$$

and either $H \cdot B = H \cdot C = G$ and then we have

$$|H \cdot B| = 2^{n-i} \cdot 2^{i} - 2^{n} = |G|, \qquad \text{implies } e_B(H) = -1,$$

and

$$|H \cdot C| = 2^{n-i} \cdot 2^{i} = 2^{n} = |G|, \qquad \text{implies } e_C(H) = -1,$$

or

$$H \cdot B = H \cdot C \neq G.$$

Then we obtain

$$|H \cdot B| = 2^{n-i} \cdot 2^{i-1} = 2^{n-1} < |G|, \qquad \text{implies } e_B(H) = 1,$$

$$|H \cdot C| \leq 2^{n-i} \cdot 2^{i-1} = 2^{n-1} < |G|, \qquad \text{implies } e_C(H) = 1.$$

But case (iii) is not possible. So we are left to discuss case (i) as follows. Assume

$$|A| = |B| = |C| = 2^{i}$$

and consider

$$A = \langle g_1, \ldots, g_{i-1} \rangle \langle a \rangle,$$

$$B = \langle g_1, \ldots, g_{i-1} \rangle \cdot \langle b \rangle,$$

$$C = \langle g_1, \ldots, g_{i-1} \rangle \cdot \langle c \rangle,$$

$$\frac{|\langle a \rangle|}{|\langle g_1, \ldots, g_{i-1} \rangle \cap \langle a \rangle|} = (A : \langle g_1, \ldots, g_{i-1} \rangle) = 2.$$

Since

$$|\langle g_1, \ldots, g_{i-1} \rangle| = 2^{i-1},$$

this implies

$$|\langle a \rangle| = 2, |\langle g_1, \ldots, g_{i-1} \rangle \cap \langle a \rangle| = 1.$$

Similarly, we obtain

$$(B : \langle g_1, \ldots, g_{i-1} \rangle) = 2,$$

$$(C : \langle g_1, \ldots, g_{i-1} \rangle) = 2.$$

Also, since it is clear that $g_1, \ldots, g_{i-1} \in A$ and $g_1, \ldots, g_{i-1} \in B$ implies $g_1, \ldots, g_{i-1} \in A \cap B$, this implies

$$\langle g_1, \ldots, g_{i-1} \rangle \subseteq A \cap B \subseteq A, B$$

and $A \cap B \neq A$ or B because $A \neq B$ by Lemma 2.5.

Similarly, we obtain

$$\langle g_1, \ldots, g_{i-1} \rangle \subseteq B \cap C \subseteq B, C, \qquad B \neq C$$
$$\langle g_1, \ldots, g_{i-1} \rangle \subseteq A \cap C \subseteq A, C, \qquad A \neq C$$

so we must have

$$(A : A \cap B) = 2, \qquad (B : A \cap B) = 2,$$
$$(C : B \cap C) = 2, \qquad (B : B \cap C) = 2,$$
$$(C : A \cap C) = 2, \qquad (A : A \cap C) = 2.$$

Next we consider

$$(A : A \cap B) \cdot (A \cap B : \langle g_1, \ldots, g_{i-1} \rangle) = (A : \langle g_1, \ldots, g_{i-1} \rangle).$$

Then we have

$$2 \cdot (A \cap B : \langle g_1, \ldots, g_{i-1} \rangle) = 2.$$

This implies

$$(A \cap B : \langle g_1, \ldots, g_{i-1} \rangle) = 1$$

hence,

$$A \cap B = \langle g_1, \ldots, g_{i-1} \rangle.$$

Similarly, we obtain

$$B \cap C = \langle g_1, \ldots, g_{i-1} \rangle,$$
$$A \cap C = \langle g_1, \ldots, g_{i-1} \rangle.$$

It also follows that

$$|A \cap B| = |B \cap C| = |A \cap C| = 2^{i-1}.$$

Also, since

$$g_1, \ldots, g_{i-1}, a, b \in \langle g_1, \ldots, g_{i-1}, a \rangle \cdot \langle g_1, \ldots, g_{i-1}, b \rangle,$$
$$g_1, \ldots, g_{i-1}, b, c \in \langle g_1, \ldots, g_{i-1}, b \rangle \cdot \langle g_1, \ldots, g_{i-1}, c \rangle,$$
$$g_1, \ldots, g_{i-1}, a, c \in \langle g_1, \ldots, g_{i-1}, a \rangle \cdot \langle g_1, \ldots, g_{i-1}, c \rangle,$$

and $a \cdot b \cdot c = 1$, we get that

$$D \subseteq A \cdot B, \qquad \text{because } c = a \cdot b,$$
$$D \subseteq B \cdot C, \qquad \text{because } a = b \cdot c,$$
$$D \subseteq A \cdot C, \qquad \text{because } b = a \cdot c,$$

where

$$D := \langle g_1, \ldots, g_{i-1}, a, b \rangle,$$

and since

$$A \subseteq D, B \subseteq D \text{ implies } A \cdot B \subseteq D,$$
$$B \subseteq D, C \subseteq D \text{ implies } B \cdot C \subseteq D,$$
$$A \subseteq D, C \subseteq D \text{ implies } A \cdot C \subseteq D,$$

it follows that

$$D = A \cdot B = B \cdot C = A \cdot C.$$

Now we compute

$$|D| = \frac{|A|\,|B|}{|A \cap B|} = \frac{|B|\,|C|}{|B \cap C|} = \frac{|A|\,|C|}{|A \cap C|} = 2^{i+1}.$$

2.6. We are now set to give the proof of non-trivial case: That is, we must prove that if $H \leq G$, $|H| = 2^{n-i}$ and $H \cdot A = G$ then either $H \cdot B = G$ and $H \cdot C \neq G$ or vice-versa.

Note that $H \cdot A = G$ implies $H \cdot D = G$.

Proof. Since

$$|H \cdot D| = \frac{|H|\,|D|}{|H \cap D|},$$

we have

$$2^n = \frac{2^{n-i} \cdot 2^{i+1}}{|H \cap D|},$$

and hence

$$|H \cap D| = \frac{2^{n-i} \cdot 2^{i+1}}{2^n} = 2^{n-i+i+1-n} = 2^1 = 2.$$

Similarly, we obtain

$$|H \cap \langle A \cdot B \rangle| = |H \cap \langle B \cdot C \rangle| = |H \cap \langle A \cdot C \rangle| = 2,$$

since

$$D = A \cdot B = B \cdot C = A \cdot C,$$

Now as $|H \cap \langle A \cdot C \rangle| = 2$, there exists precisely one element, say $u \in G$, such that $u \neq 1$ and $|\langle u \rangle| = 2$ with $\langle u \rangle \subseteq H$ and $\langle u \rangle \subseteq \langle A \cdot C \rangle$, since $H \cap \langle A \cdot C \rangle = \langle u \rangle$. Similarly, we get $H \cap \langle A \cdot B \rangle = H \cap \langle B \cdot C \rangle = H \cap D = \langle u \rangle$. But then by our hypothesis $H \cdot A = G$ implies $H \cap A = 1$, and it follows that $\langle u \rangle \nsubseteq A$, as $\langle u \rangle \subseteq H$.

Now we know the following:

$H \cdot A = G$ and $H \cap A = 1$, and this implies $u \notin A$,
$H \cdot B = G \Leftrightarrow H \cap B = 1$,
$H \cdot C = G \Leftrightarrow H \cap C = 1$,
$H \cap D = \langle u \rangle \supseteq H \cap B$, $H \cap C \supseteq 1$, and this implies

$$H \cap B = 1 \Leftrightarrow u \notin B \qquad \text{and} \qquad H \cap C \neq 1 \Leftrightarrow u \in C$$

or, in other words,

$$H \cap B \neq 1 \Leftrightarrow u \in B \qquad \text{and} \qquad H \cap C = 1 \Leftrightarrow u \notin C.$$

We must show either

$$u \notin B \qquad \text{and} \qquad u \in C$$

or

$$u \in B \qquad \text{and} \qquad u \notin C$$

To see this, we have to show that neither

$$u \in B \qquad \text{and} \qquad u \in C$$

nor

$$u \notin B \qquad \text{and} \qquad u \notin C$$

can hold.

Hence, assume first that on the contrary $u \in B$ and $u \in C$. Then we consider $B \cap C$, and use the fact that

$$(A : \langle g_1, \dots, g_{i-1} \rangle) \leq 2, \qquad B \cap C = \langle g_1, \dots, g_{i-1} \rangle.$$

We obtain

$$B \cap C \subseteq A,$$

and it follows that $u \in A$, a contradiction. So we can't have

$$u \in B \qquad \text{and} \qquad u \in C.$$

Assume second that

$$u \notin B \qquad \text{and} \qquad u \notin C.$$

Then we have

$$u \in B \cdot C = D = \langle g_1, \dots, g_{i-1}, a, b \rangle.$$

This implies

$$u = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot a^{\epsilon_i} \cdot b^{\epsilon_{i+1}}$$

for some choices of $\epsilon_1, \dots, \epsilon_{i+1} \in \{0, 1\}$.

Now, by hypotheses

$u \notin A$ implies $u \neq g_1^{\epsilon'_1} \dots g_{i-1}^{\epsilon'_{i-1}} \cdot a^{\epsilon'_i}$ for every choice of $\epsilon'_1, \dots, \epsilon'_i \in \{0, 1\}$,

$u \notin B$ implies $u \neq g_1^{\epsilon''_1} \dots g_{i-1}^{\epsilon''_{i-1}} \cdot b^{\epsilon''_i}$ for every choice of $\epsilon''_1, \dots, \epsilon''_i \in \{0, 1\}$, and

$u \notin C$ implies $u \neq g_1^{\epsilon'''_1} \dots g_{i-1}^{\epsilon'''_{i-1}} \cdot c^{\epsilon'''_i}$ for every choice of $\epsilon'''_1, \dots, \epsilon'''_i \in \{0, 1\}$.

Now if $\epsilon_{i+1} = 0$, then we shall have

$$u = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot a^{\epsilon_i} \in A := \langle g_1, \dots, g_{i-1}, a \rangle,$$

hence

$$\epsilon_{i+1} \neq 0 \Rightarrow \epsilon_{i+1} = 1.$$

If $\epsilon_i = 0$, then we get that

$$u = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot b^{\epsilon_{i+1}} \in B := \langle g_1, \dots, g_{i-1}, b \rangle,$$

hence

$$\epsilon_i \neq 0 \Rightarrow \epsilon_i = 1,$$

and so we have

$$\epsilon_i = 1, \qquad \epsilon_{i+1} = 1.$$

Hence we obtain

$$u = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot (ab)^1 = g_1^{\epsilon_1} \dots g_{i-1}^{\epsilon_{i-1}} \cdot c^1 \qquad \text{since } abc = 1$$
$$\in C := \langle g_1, \dots, g_{i-1}, c \rangle, \text{ a contradiction.}$$

So we cannot have

$$u \notin B \qquad \text{and} \qquad u \notin C.$$

Therefore, the proof of the non-trivial case is complete.

Hence

$$e_{\langle g_1, \dots, g_r * h_r, \dots, g_i \rangle}(H) \cdot e_{\langle g_1, \dots, g_i \rangle}(H) \cdot e_{\langle g_1, \dots, h_r, \dots, g_i \rangle}(H) = 1$$

for all $H \leq G$ with $|H| \leq 2^{n-i}$, and by definition, we obtain

$$\omega_i(g_1, \dots, g_r * h_r, \dots, g_i) \\ = \omega_i(g_1, \dots, g_i) \cdot \omega_i(g_1, \dots, h_r, \dots, g_i).$$

Thus, ω_i is a multilinear map.

Now, consider again the map

$$\omega_i : G^i \to \Omega(G)^*_{i+1}/\Omega(G)^*_i \\ : (g_1, \dots, g_i) \mapsto e_{\langle g_1, \dots, g_i \rangle}\Omega(G)^*_i.$$

If we impose on this map the condition that $\#\{g_1, \dots, g_i\} < i$ then we obtain as follows. As $\#\{g_1, \dots, g_i\} \leq i - 1$, then we obtain by Lemma 2.4 that

$$|\langle g_1, \dots, g_i \rangle| \leq 2^{i-1}$$

and this implies

$$e_{\langle g_1, \dots, g_i \rangle} \in \Omega(G)^*_i$$

or

$$e_{\langle g_1, \dots, g_i \rangle}\Omega(G)^*_i = \Omega(G)^*_i.$$

So by definition, we get

$$\omega_i(g_1, \dots, g_i) = 0.$$

Hence, $\omega_i(g_1, \dots, g_i) = 0$ whenever $\#\{g_1, \dots, g_i\} < i$, and the proof of Theorem 2.2 is complete.

3. ISOMORPHISM BETWEEN $\wedge^i(G)$ AND $\Omega(G)^*_{i+1}/\Omega(G)^*_i$

3.1. ω_i induces a canonical map

$$\hat{\omega}_i : \bigwedge^i (G) \to \Omega(G)^*_{i+1}/\Omega(G)^*_i$$

which maps

$$g_1 \wedge \cdots \wedge g_i \in \bigwedge^i (G) \text{ onto} \\ e_{\langle g_1, \dots, g_i \rangle}\Omega(G)^*_i.$$

3.2. CLAIM. $\hat{\omega}_i$ is an isomorphism.

Proof. First, we note that $\hat{\omega}_i$ is a well defined linear map, because of the universal properties of $\bigwedge^i(G)$ and the particular properties established above of the map ω_i.

Second,

$$\bigwedge^i(G) \qquad \text{and} \qquad \Omega(G)^*_{i+1}/\Omega(G)^*_i$$

are both of the same order, because

$$(\Omega(G)^*_{i+1} : \Omega(G)^*_i) = 2^{\binom{n}{i}}$$

and by standard results, we know that as $\dim(G/\mathbb{F}_2) = n$, we have $\dim \bigwedge^i(G) = \binom{n}{i}$, and this implies

$$\left| \bigwedge^i(G) \right| = 2^{\binom{n}{i}}.$$

Third, we establish injectivity of $\hat{\omega}_i$ in the following way. Now assume that $\lambda \in \bigwedge^i(G)$ satisfies

$$\hat{\omega}_i(\lambda) = 0, \qquad \text{that is,}$$

$$\gamma_T(\hat{\omega}_i(\lambda)) = 1 \qquad \text{for all } T \in \binom{\{1, \dots, n\}}{n-i},$$

$$\text{where } \gamma_T : \Omega(G)^*_{i+1} \to \{\pm 1\}.$$

This implies that for every

$$T \in \binom{\{1, 2, \dots, n\}}{n-i}$$

we have

$$\hat{\omega}_i(\lambda)(\langle e_i \mid i \in T \rangle) = 1,$$

$$\text{where } e_i := (1, \dots, 1, \underset{\substack{\uparrow \\ i\text{th position}}}{-1}, 1, \dots, 1) \in G$$

for all $i = 1, \dots, n$.

As $G = \langle e_1, \dots, e_n \rangle$ and $\lambda \in \bigwedge^i(G)$ there are unique coefficients $C_{k_1, \dots, k_i} \in \mathbb{F}_2$ $(1 \le k_1 < \cdots < k_i \le n)$ such that

$$\lambda = \sum_{1 \le k_1 < k_2 < \cdots < k_i \le n} C_{k_1 \dots k_i} e_{k_1} \wedge \cdots \wedge e_{k_i}.$$

Hence, for any

$$T \in \binom{\{1, \ldots, n\}}{n-i},$$

we have

$$\gamma_T(\hat{\omega}_i(\lambda)) = \hat{\omega}_i(\lambda)(\langle e_i \,|\, i \in T \rangle)$$
$$= \prod_{1 \le k_1 < k_2 < \cdots < k_i \le n} e_{\langle e_{k_1}, \ldots, e_{k_i} \rangle}(\langle e_i \,|\, i \in T \rangle)^{C_{k_1 \ldots k_i}} = 1,$$

where

$$e_{\langle e_{k_1}, \ldots, e_{k_i} \rangle}(\langle e_i \,|\, i \in T \rangle) = (-1)^{\delta^T_{\{k_1, \ldots, k_i\}}},$$

we define with

$$\delta^T_{\{k_1, \ldots, k_i\}} := \begin{cases} 0, & \text{if } \{k_1, \ldots, k_i\} \cap T \neq \varnothing \\ 1, & \text{if } \{k_1, \ldots, k_i\} \cap T = \varnothing. \end{cases}$$

This means that

$$\delta^T_{\{k_1, \ldots, k_i\}} = 1 \qquad \text{if and only if } T := \{1, \ldots, n\} \backslash \{k_1, \ldots, k_i\}.$$

Applying this definition on individual factors of the above products relation, we obtain for a fixed $\{k_1^0, \ldots, k_i^0\}$ with $T := \{1, \ldots, n\} \backslash \{k_1^0, \ldots, k_i^0\}$ that

$$\delta^T_{\{k_1^0, \ldots, k_i^0\}} = 1 \qquad \text{and} \qquad \delta^T_{\{k_1, \ldots, k_i\}} = 0$$

for

$$\{k_1, \ldots, k_i\} \neq \{k_1^0, \ldots, k_i^0\}.$$

This implies

$$e_{\langle e_{k_1^0}, \ldots, e_{k_i^0} \rangle}(\langle e_i \,|\, i \in T \rangle) = -1, \qquad e_{\langle e_{k_1}, \ldots, e_{k_i} \rangle}(\langle e_i \,|\, i \in T \rangle) = 1$$

and substituting this in the above products relation, we get that

$$1 = \gamma_T(\hat{\omega}_i(\lambda)) = \prod_{1 \le k_1 < \cdots < k_i \le n} \left((-1)^{\delta^T_{\{k_1, \ldots, k_i\}}} \right)^{C_{k_1 \ldots k_i}}$$
$$= (-1)^{C_{k_1^0 \ldots k_i^0}},$$

and therefore,

$$C_{k_1^0 \ldots k_i^0} = 0.$$

Hence, for every $\{k_1^0, \ldots, k_i^0\}$ we must have

$$C_{k_1^0 \ldots k_i^0} = 0,$$

which implies that

$$\lambda = \sum_{1 \le k_1 < k_2 < \cdots < k_i \le n} C_{k_1 \ldots k_i} e_{k_1} \wedge \cdots \wedge e_{k_i} = 0,$$

hence, $\hat{\omega}_i$ is injective.

So it is clear from the above considerations that $\hat{\omega}_i$ is surjective and that the vectors

$$\left[e_{\langle e_{k_1}, \ldots, e_{k_i} \rangle} \right], \{k_1, \ldots, k_i\} \subseteq \{1, 2, \ldots, n\}$$

generate

$$\Omega(G)^*_{i+1} / \Omega(G)^*_i.$$

Therefore, $\hat{\omega}_i$ is an isomorphism.

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