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On the Extension Problem and the Nil Groups of Rings of Finite Global Dimension

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Abstract

A vanishing result is obtained in respect of nil groups of rings of finite global dimension. Also a connection is established with the extension problem.

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1 Introduction

Let C be an admissible subcategory of an abelian category. We are interested in the category $Nil(\mathrm{C})$ whose objects are the pairs (M, ν), where $M \in \mathrm{C}$ and $\nu \in End_{\mathrm{C}}(M)$ is nilpotent. The nil groups have geometrical significance as they occur as obstructions to geometrical problems ([4],[7]). The Nil group vanishes for any abelian category (Proposition 6.1 on page 653 of [2]), and it is an interesting problem to determine under what conditions will the Nil group vanish for a nonabelian category [11]. Some known vanishing results in respect of nil groups for rings are those for the group ring $\mathbb{Z}[G]$ where G is a finite group of square - free order [5], regular rings ([7],[11]), quasi-regular rings [7], the cyclic group C_n of finite order $n \geq 2$ and the finite group of finite type $G \cong F \rtimes \mathbb{Z}$ where F is a finite subgroup of G [8]. This paper gives a solution of the above stated problem in respect of the nonabelian category of rings. We obtain that

$$NK_i(R) = 0 \ \forall \ i$$

for a ring R of finite global dimension.

Let M', M'' be R - modules. The question is asked as regards the R - modules M such that M' is a submodule of M and M'' be its quotient. Equivalently, this question can be posed as follows: which are the R - modules M such that the sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be exact? The classification of such R - modules M constitute what is known as the extension problem [10]. In this paper, we establish a result that relates the extension problem to rings of finite global dimension. Thus giving us a supply of rings of finite global dimension.

2 Extensions, Global Dimensions and Nil Groups

Definition 2.1 *An extension of an R - module M' by M'' is a short exact sequence $E : 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$.*

Definition 2.2 *(i) Let R be a ring and E an R - module. E is said to have homological dimension (denoted $hd_R E$) $\leq n$ if $\exists$ an exact sequence*

$$0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow P_0 \longrightarrow E \longrightarrow 0$$

where each P_i is projective (i.e. a sum of free R - modules).

(ii) The global dimension of R denoted by $gl.dimR$ is defined as

$gl.dimR =sup\ hd_R E \quad \forall \quad R$ - modules E

(iii) *The finitistic global dimension of* R *denoted by* $f.gl.dimR$ *is defined as* $f.gl.dimR = sup\ hd_R E \quad \forall \quad R$ *- modules* $E \ni hd_R E < \infty$.

(iv) *Let* G *be a group,* $R[G]$ *its corresponding*

group ring and $\epsilon : R[t] \longrightarrow R$ *the augmentation map. The* i *- th nil group of* R *is defined as*

$$NK_i(R) = ker(K_i(R[t]) \xrightarrow{\epsilon_*} K_i(R))$$

(v) *Let* G *be a finite group,* $\alpha : \mathbb{Z} \longrightarrow Aut(G)$ *a homomorphism and* $V = G \rtimes_\alpha \mathbb{Z}$ *the semidirect product. The nil groups of* $\mathbb{Z}[V]$ *which takes into account the automorphism* α*, are called twisted nil groups and are denoted by* $NK_i^\alpha(\mathbb{Z}[G])$ *i.e.*

$$NK_i^\alpha(R) = ker(K_i(R_\alpha[t]) \xrightarrow{\epsilon_*} K_i(R))$$

(vi) *Let* τ *be the category of triples* $\boldsymbol{R} = (R; B_0, B_1)$*, where* $B_i, i = 0, 1$ *are two bimodules. A morphism in* τ *is a triple*

$$(\phi, f_0, f_1) : (R; B_0, B_1) \longrightarrow (S, C_0, C_1)$$

where $\phi : R \longrightarrow S$ *is a ring homomorphism and* $f_i : B_i \otimes_R S \longrightarrow C_i,\ i = 0, 1,$ *are* $R - S$ *- bimodule homomorphisms. Waldhausen nil groups are functors from the category* τ *to abelian groups. For an object* $\boldsymbol{R}$ *in* τ*, we first define an exact category* $Nil(\boldsymbol{R})$ *with objects quadruples* $(P, Q, ; p, q)$*, where* P *and* Q *are finitely generated projective right* R *- modules and*

$$p : P \longrightarrow Q \otimes B_0,\ \ q : Q \longrightarrow P \otimes B_1$$

is a pair of R *- module homomorphisms such that the compositions*

$$P \xrightarrow{p} Q \otimes B_0 \xrightarrow{q\otimes 1} P \otimes B_1 \otimes B_0 \cdots$$

$$Q \xrightarrow{q} P \otimes B_1 \xrightarrow{p\otimes 1} P \otimes B_0 \otimes B_1 \cdots$$

are zero after finitely many steps. Morphisms are homomorphisms on the modules that are compatible with the maps. There is a forgetful functor $\phi : Nil(\boldsymbol{R}) \longrightarrow \mathcal{P}_\mathcal{R} \times \mathcal{P}_\mathcal{S}$*, where* $\mathcal{P}_\mathcal{R}$ *is finitely generated projective right* R *- modules. Then the Waldhausen Nil - groups ([7], [9] and [11])* $\widetilde{Nil}_i^W \boldsymbol{R}$ *are defined as*

$$\widetilde{Nil}_i^W \boldsymbol{R} = ker(K_i(Nil(\boldsymbol{R})) \xrightarrow{\phi_i} K_i(\mathcal{P}_\mathcal{R} \times \mathcal{P}_\mathcal{R})), for\rangle \in \mathbb{Z}.$$

Remark 2.3 *There is a natural isomorphism between the NK - groups and the Waldhausen Nil - groups*

$$NK_i(R) \cong \widetilde{Nil}^W_{i-1}(R), \textit{for } i \leq 1.$$

Thus the vanishing results of Waldhausen Nil - groups can be applied to the NK - groups. ([3])

The group of equivalence classes of extensions of M' by M'' under the Baer sum is denoted by $Ex(M'', M')$ and it is isomorphic to $Ext^1(M'', M')$ [10].

3 Nil Groups of Rings of Finite Global Dimension

It is known that the relationship between the extension bifunctor and the Baer sum is illustrated in the solution to the extension problem ([10]). Then by results from ([12]), we relate rings of finite global dimension to the extension problem. Thus giving a condition for the supply of rings of finite global dimension.

Theorem 3.1 *Let R be a ring of finite global dimension. Then $NK_i(R) = 0 \ \forall \ i$.*

Proof: Let R be a ring of finite global dimension. Then $hd_R E \leq n < \infty$ for any R - module E and exact sequence

$$0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \cdots \longrightarrow P_0 \longrightarrow E \longrightarrow 0$$

where each P_i is projective. Let $\underline{p}$ be a prime ideal of R and S a multiplicatively closed subset of R not containing 0 given by $S = R - \underline{p}$. The ring of fractions $R_{\underline{p}} = S^{-1}R$ can give an indication on whether R is regular ([1]). Since R is of finite global dimension, it follows that $R_{\underline{p}}$ also has finite global dimension and is regular. Thus R is regular. It is known that every R - module has a projective resolution and since R is of finite global dimension, it means that every R - module is of homological dimension $\leq n (n \in \mathbb{N})$. Now the category of finitely generated projective modules is a full subcategory of the category of R - modules. Therefore $K_i(Nil(\mathrm{R})) \longrightarrow K_i(Nil(Mod_R))$ is an isomorphism. Using Corollary 6.3 on p. 654 of [2] and the fact that the groups NK_i are isomorphic to the Waldhausen's groups $\widetilde{Nil}^W_{i-1}$ for regular rings (see Proof of Proposition 3.8 in [3]), it follows that $NK_i(R) = 0 \ \forall \ i$.

Theorem 3.2 *Let A be a ring such that the extensions of any simple module S by S over A splits and consists of exactly only one element. Then A is of finite global dimension.*

Proof: Suppose that A be a ring such that the extensions of any simple A - module S by S over A splits and consists of only one element. Then by Theorem 3.14 on page 15 of [10] we have that $Ext^n_A(S, S) = 0 \quad \forall \quad n \geq 1$ and all simple A - modules S and consequently $Ext^n_A(S, S) = 0$ for all simple A - modules S and all $n >> 0$. Then by Theorem 2 of [12], A is of finite global dimension.

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