IMHOTEP

AFRICAN JOURNAL OF PURE AND APPLIED MATHEMATICS

Imhotep Mathematical Proceedings
Volume 4, Numéro 1, (2017),
pp. 34 - 43.

# On counting subgroups for a class of finite nonabelian p-groups and related problems



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**Abstract**

The main goal of this article is to review the work of Marius Tărnăuceanu, where an explicit formula for the number of subgroups of finite nonabelian $p$-groups having a cyclic maximal subgroups was given. Using examples to clarify our work and in addition we give an explicit formula to some related problems.

**Proceedings of the $6^{th}$ annual workshop on CRyptography, Algebra and Geometry (CRAG-6), 15 - 17 June 2016, University of Bamenda, Bamenda, Cameroon.**

http://imhotep-journal.org/index.php/imhotep/
**Imhotep Mathematical Proceedings**

IMHOTEP - Math. Proc. **4** (2017), 34–43
1608-9324/010034–33, DOI 13.1007/s00009-003-0000


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# On counting subgroups for a class of finite nonabelian p-groups and related problems.

Olapade Oluwafunmilayo O. and EniOluwafe M.

**Abstract.** The main goal of this article is to review the work of Marius Tărnăuceanu, where an explicit formula for the number of subgroups of finite nonabelian $p$-groups having a cyclic maximal subgroups was given. Using examples to clarify our work and in addition we give an explicit formula to some related problems.

**Mathematics Subject Classification (2010).** 20D60 (primary), 20D15, 20D30, 20D40. (secondary).

**Keywords.** Finite nonabelian p-groups, Cyclic subgroups, Number of subgroups, Recurrence relation, Cartesian products.



## I. Preliminaries

Counting subgroup of finite groups is one of the most important problems of combinatorial finite group theory. Starting with the last century, this topic has enjoyed a steady and gradual process of development. The problem was completely solved in the abelian case, by establishing an explicit expression of the number of subgroups of a finite abelian group (see[2]). Several authors have worked on this area using different methods: Gautomi Bhowmik [2] used Gaussian polynomial to evaluate divisor function of matrices, *Călugăceanu G* [3] and *J* Petrillo [6] used Goursat's lemma for groups to derive explicit formulae, Marius Tărnăuceanu [10] and EniOluwafe M. [4] used the concept of fundamental group lattice to count some types of subgroups of a finite nonabelian group; Tărnăuceanu in [11] used method based on certain attached matrix, *Lászlo Tóth* [7] and Amit Sehgal [1] use simple group-theoretic and number theoretic formulae. Unfortunately, in the nonabelian case such expression can be given only for few classes of finite groups.

In the following let $p$ be a prime, $n \geq 3$ be an integer and consider the class $\mathcal{G}$ of all finite nonabelian $p$-group of order $p$ possessing a maximal subgroup which is cyclic. A detailed description of $\mathcal{G}$ is given by Theorem 4.1, chapter 4, [8]: a group is contained in the class $\mathcal{G}$ if and only if it is isomorphic to
$M(p^n) =< x, y \mid x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{p^{n-2}+1} >$ when $p$ is odd, or to one of the next groups
- $M(2^n)$ $(n \geq 4)$,
- the dihedral group
$D_{2^n} =< x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{2^{n-1}-1} >$

**Communication presentée au 6$^{ème}$ atelier annuel sur la CRyptographie, Algèbre et Géométrie (CRAG-6), 15 - 17 Juin 2016, Université de Bamenda, Bamenda, Cameroun / Paper presented at the 6$^{th}$ annual workshop on CRyptography, Algebra and Geometry (CRAG-6), 15 - 17 June 2016, University of Bamenda, Bamenda, Cameroon.**

- the generalized quaternion group

$Q_{2^n} =< x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{2^{n-1}-1} >$
- the quasidihedral group

$S_{2^n} =< x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}-1} >$
$(n \geq 4)$
when $p = 2$. If $G$ is a group, then the set $L(G)$ consisting all subgroups of $G$ forms a complete lattice with respect to set inclusion, called the subgroup lattice of $G$. Most of our notation is standard and will usually not be repeated here. For basic definition and results on groups we refer the reader to [9] and [8]. In this paper we use examples to make the work of Marius more explicit. In his work he determine the cardinality of $L(G)$ for the groups $G$ in $\mathcal{G}$, by using the above presentation and their main properties (collected in (4.2), chapter 4, [8]).

## II. Main results

### II.1. Modular groups

First of all, *Tărăuceanu* [10] find the number of subgroups of Modular group $M(p^n)$. And state some of the property of the Modular groups:

- The commutator subgroup $D(M(p^n))$ has order $p$ and is generated by $x^q$, where $q = p^{n-2}$.
- $\Omega_1(M(p^n)) =< x^q, y >\cong \mathbb{Z}_p \times \mathbb{Z}_p$.
- $M(p^n)$ contains $p + 1$ minimal subgroups.
- The join of any two distinct minimal subgroups includes $D(M(16))$.

Let $p = 2$, then $M(2^n) =< x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}+1} >$,
We give the following examples to make the above properties more explicit:

**II.1.1. Example.** when $n = 4$,

$$\begin{aligned} M(16) &=< x, y \mid x^8 = y^2 = 1, y^{-1}xy = x^3 > \\ &= \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

$D(M(16)) =< x^q >=< x^{2^2} >=< x^4 >= \{1, x^4\}$
Clearly, $D(M(16))$ has order 2.

**II.1.2. Example.** Let $\Omega_1(M(16)) =< x^4, y >= \{1, x^4, y, x^4y\}$, so $\Omega_1(M(16))$ has order 4
$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 4.
We compute this table to see clearly the structure of $\Omega_1(M(16))$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.
Clearly from the table, $\Omega_1(M(16)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

**II.1.3. Example.** Minimal subgroups of $M(16)$ are:
$\{1, x^4\}, \{1, y\}, \{1, x^4y\}$
Clearly, $M(16)$ contains 3 minimal subgroups which is $p + 1$.

TABLE 1. Analysis of order of elements of $\Omega_1(M(16))$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

| Order of elements | 1 | 2 | 4 |
|---|---|---|---|
| $\Omega_1(M(16))$ | 1 | $x^4, y$ | $x^4y$ |
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $(0,0)$ | $(0,1)$, $(1,0)$ | $(1,1)$ |
| Total number | 1 | 2 | 1 |

**II.1.4. Example.** The join of any two distinct minimal subgroups include $D(M(p^n))$: Join $\{1, x^4\}\, and\, \{1, y\}\, gives\, \{1, x^4, y, x^4y\}$
$1, x^4 \in \{1, x^4, y, x^4y\}$
Clearly it includes $D(M(16))$.
From the above, the following results were obtained:

$$\mid L(M(p^n)) \mid = \mid L(\frac{M(p^n)}{D(M(p^n))}) \mid + p + 1. \qquad (1)$$

recall that the commutator subgroup, $D(M(p^n))$ is a minimal subgroup and that's the reason for adding $p+1$ in equation (1) above.

**II.1.5. Example.** recall: $\frac{G}{H} = \{gH \mid g \in G\}$,

$$\begin{aligned}
\frac{M(16)}{D(M(16))} &= < x, y > \\
\frac{M(16)}{D(M(16))} &= \{gD(M(16)) \mid g \in M(16)\} \\
&= \{D(M(16)), xD(M(16)), x^2D(M(16)), x^3D(M(16)), yD(M(16)), \\
&\quad xyD(M(16)), x^2yD(M(16)), x^3yD(M(16))\}
\end{aligned}$$

$\frac{M(p^n)}{D(M(p^n))}$ is an abelian group.

**II.1.6. Example.** $\frac{M(16)}{D(M(16))}$ is abelian if $xyD(M(16)) = yxD(M(16))$.

$$\begin{aligned}
yxD(M(16)) &= x^5yD(M(16)) && (yx = x^5y) \\
&= x \cdot x^4yD(M(16)) && (x \cdot x^4y = x^5y) \\
&= xyx^4D(M(16)) && (x^4y = yx^4) \\
&= xyD(M(16)) && (x^4 \in D(M(16))) \\
\therefore yxD(M(16)) &= xyD(M(16)) && \text{(that is x commutes with y)}
\end{aligned}$$

Hence $\frac{M(16)}{D(M(16))}$ is abelian.
$\frac{M(p^n)}{D(M(p^n))}$ is of order $p^{n-1}$,
that is:
$\frac{M(16)}{D(M(16))}$ is of order 8
$\frac{|M(p^n)|}{|D(M(p^n))|} = |\frac{M(16)}{D(M(16))}| = \frac{16}{2} = 8$.
This is confirmed by the number of elements in $\frac{M(16)}{D(M(16))}$ (example 2.1.5)
Next, we show that $\frac{M(p^n)}{D(M(p^n))} \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ have isomorphic lattices of subgroups. Thus, we need to determine the number of subgroups of certain order for $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ and that of $\frac{M(16)}{D(M(16))}$.

**II.1.7. Example.** Let $p = 2, n = 4$, we have:

$$\mathbb{Z}_2 \times \mathbb{Z}_4 \cong \mathbb{Z}_8$$

$$\mathbb{Z}_2 = \{1, a\}$$

$$\mathbb{Z}_4 = \{1, y, y^2, y^3\}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(1,1), (1,y), (1,y^2), (1,y^3), (x,1), (x,y), (x,y^2), (x,y^3)\}$$

$\mathbb{Z}_2 \times \mathbb{Z}_4$ is of order 8.
Likewise,

$$\frac{M(16)}{D(M(16))} = \{D(M(16)), xD(M(16)), x^2D(M(16)), x^3D(M(16)), yD(M(16)), xyD(M(16)), x^2yD(M(16)), x^3yD(M(16))\}$$

$\frac{M(16)}{D(M(16))}$ Is also of order 8.
We compute these tables to see clearly the structure of $\frac{M(16)}{D(M(16))}$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Table 2. Analysis of order of elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$

| Order of elements | 1 | 2 | 4 |
|---|---|---|---|
| $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $(1,1)$ | $(1,y), (1,y^2), (x,1)$ | $(1,y^3), (x,y), (x,y^2), (x,y^3)$ |
| Total number | 1 | 3 | 4 |

Table 3. Analysis of order of elements of $\frac{M(16)}{D(M(16))}$

| Order of elements | 1 | 2 | 4 |
|---|---|---|---|
| $\frac{M(16)}{D(M(16))}$ | $(1,1)$ | $x^2D(M(16)), yD(M(16)), x^2yD(M(16))$ | $xD(M(16)), x^3D(M(16)), xyD(M(16)), x^3yD(M(16))$ |
| Total number | 1 | 3 | 4 |

Comparing the order of $\frac{M(16)}{D(M(16))}$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$ and the order of their elements (as shown on the tables 2 and 3 above), we conclude that they are isomorphic. Therefore,

$$\frac{M(p^n)}{D(M(p^n))} \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}} \tag{2}$$

Being isomorphic, the groups $\frac{M(p^n)}{D(M(p^n))}$ and $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ have isomorphic lattices of subgroups. Thus, their is a need to determine the number of subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$. In order to do this he recall the following auxiliary result, established in [11, Theorem 3.3, pp.378].

**Lemma 1.** *For every $0 \leq \alpha \leq \alpha_1 + \alpha_2$, the number of all subgroups of order $p^{\alpha_1+\alpha_2-\alpha}$ in the finite abelian $p-group$ $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ $(\alpha_1 \leq \alpha_2)$ is:*

$$\begin{cases} \dfrac{p^{\alpha+1}-1}{p-1}, & \text{if } 0 \leq \alpha \leq \alpha_1 \\ \dfrac{p^{\alpha_1+1}-1}{p-1}, & \text{if } \alpha_1 \leq \alpha \leq \alpha_2 \\ \dfrac{p^{\alpha_1+\alpha_2-\alpha+1}-1}{p-1}, & \text{if } \alpha_2 \leq \alpha \leq \alpha_1+\alpha_2. \end{cases}$$

In particular, the total number of subgroups of $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ is:

$$\frac{1}{(p-2)^2}[(\alpha_2-\alpha_1+1)p^{\alpha_1+2}-(\alpha_2-\alpha_1-1)p^{\alpha_1+1}-(\alpha_1+\alpha_2+3)p+(\alpha_1+\alpha_2+1)]$$

For $\alpha_1 = 1$ and $\alpha_2 = n-2$, it results:

$$|L(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}})| = \frac{1}{(p-2)^2}[(n-2)p^3-(n-4)p^2-(n+2)p+n)] = (n-2)p+n. \qquad (3)$$

Now, the relation (1), (2) and (3) show that the next theorem holds.

**Theorem 2.** *The number of subgroups of the group $M(p^n)$ is given by the following equality:*

$$|L(M(p^n))| = (n-1)p+n+1.$$

**Proof.** Recall from [1] that

$$\mid L(M(p^n)) \mid = \mid L(\frac{M(p^n)}{D(M(p^n))}) \mid + p + 1$$

and from [2] that

$$\frac{M(p^n)}{D(M(p^n))} \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$$

and from [3]

$$\begin{aligned} \mid L(M(p^n)) \mid &= \mid L(\frac{M(p^n)}{D(M(p^n))}) \mid + p + 1 \\ &= \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}} + p + 1 \\ &= (n-2)p+n+p+1 \\ &= (n-2+1)p+n+1 \\ &= (n-1)p+n+1 \end{aligned}$$

Hence, $\mid L(M(p^n)) \mid = (n-1)p+n+1$

■

Next, we focus on the groups $D_{2^n}$, $Q_{2^n}$ and $SD_{2^n}$. An important property of these groups is that their centres are of order 2 (they are generated by $x^q$, where $q = 2^{n-2}$) Marius [10] gave the properties and we cite examples for clarity. That is, $Z(D_{2^n}), Z(Q_{2^n})$ and $Z(SD_{2^n})$ are of order 2 and are generated by $< x^q >$

**Example.** when $n = 4, p = 2$

$$Z(D_{2^n}) = Z(D_{16}) = \{1, x^4\}$$
$$Z(Q_{2^n}) = Z(Q_{16}) = \{1, x^4\}$$
$$Z(D_{2^n}) = Z(SD_{16}) = \{1, x^4\}$$

when $n = 5, p = 2$

$$Z(D_{2^n}) = Z(D_32) = \{1, x^8\}$$
$$Z(Q_{2^n}) = Z(Q_32) = \{1, x^8\}$$
$$Z(D_{2^n}) = Z(SD_32) = \{1, x^8\}$$

For any $G \in \{D_{2^n}, Q_{2^n}, SD_{2^n}\}$ we have:

$$\frac{G}{Z(G)} \cong D_{2^{n-1}} \qquad (4)$$

**II.2. Dihedral groups**

Let $n = 4$, $G = D_{16}, Z(D_{16}) = \{1, x^4\}$

$$\frac{G}{Z(G)} = \{gZ(G)|g \in G\}$$

$$\frac{D_{16}}{Z(D_{16})} = \{gZ(D_{16})|g \in D_{16}\}$$

$$D_{16} = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\}$$

$D_{16}$ is of order 16

$$\frac{D_{16}}{Z(D_{16})} = \{Z(D_{16}), xZ(D_{16}), x^2Z(D_{16}), x^3Z(D_{16}), yZ(D_{16}), xyZ(D_{16}), x^2yZ(D_{16}), x^3yZ(D_{16})\}$$

$\frac{D_{16}}{Z(D_{16})}$ is of order 8

$D_{2^{n-1}} = D_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ which is of order 8.

For $D_{2^n}$ this isomorphism will lead us to a recurrence relation verified by $\mid L(D_{2^n}) \mid$, but first we need to compute the number of subgroups in $D_{2^n}$ which does not contain $Z(D_{2^n})$(that is the number of subgroups of $\frac{D_{2n}}{Z(D_{2^n})}$). Clearly, the trivial subgroup of $D_{2^n}$ as well as all its minimal subgroup excepting $Z(D_{2^n})$ (that are of the form $< x^iy >, \overline{i = 0, 2^{n-1} - 1}$) satisfy this property. Since for every $i \neq j = 0, \overline{2^{n-1} - 1}$ we have $x^iyx^jy = x^{i-j}$.

**II.2.1. Example.** $x^iyx^jy = x^{i-j}$.

$$x^2yx^3y = x^2x^5yy = x^7 = x^-1\,(yx^3 = x^5y; x^-1 = x^7)$$

$$x^4yx^2y = x^4x^6yy = x^10 = x^2\,(x^8 = 1)$$

$$x^5yx^2y = x^5x^6yy = x^3\,(yx^2 = x^6y)$$

TABLE 4. Analysis of order of elements of $D_{2^{n-1}}$

| Order of elements | 1 | 2 | 4 |
|---|---|---|---|
| $D_{2^{n-1}}$ | 1 | $x^2, y, xy, x^2y, x^3y$ | $x, x^3$ |
| Total number | 1 | 5 | 2 |

TABLE 5. Analysis of order of elements of $\frac{D_16}{Z(D_16)}$

| Order of elements | 1 | 2 | 4 |
|---|---|---|---|
| $\frac{D_16}{Z(D_16)}$ | $Z(D_16)$ | $x^2Z(D_16), yZ(D_16), xyZ(D_16), x^2yZ(D_16), x^3yZ(D_16)$ | $xZ(D_16), x^3Z(D_16)$ |
| Total number | 1 | 5 | 2 |

Considering the equality of the order of elements and the order of the groups above (as we can see in table 3 and 4), we can conclude that they have the same structure and are isomorphic.

It follows again that the join of any two distinct minimal subgroups in $D_{2^n}$ includes $Z(D_{2^n})$.

Table 6. Analysis of the number of subgroups in $D_{2^n}$

| $D_{2^n}$ | $Order1$ | $Order2$ | $Order4$ | $Order8$ | $Order16$ | $Order32$ | $Order64$ | $\mid L(D_{2^{n-1}}) \mid$ | Formula |
|---|---|---|---|---|---|---|---|---|---|
| $D_8$ | 1 | 5 | 3 | 1 | – | – | 10 | – | $2^3 + 2$ |
| $D_{16}$ | 1 | 9 | 5 | 3 | 1 | – | 19 | – | $2^4 + 3$ |
| $D_{32}$ | 1 | 17 | 9 | 5 | 3 | 1 | – | – | $2^5 + 4$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | |
| $D_{2^{n-1}}$ | 1 | $2^{(n-1)-1} + 1$ | $2^{(n-1)-2} + 1$ | $2^{(n-1)-3} + 1$ | $2^{(n-1)-4} + 1$ | – | – | $\cdots$ | $2^{n-1} + (n-2)$ |

**II.2.2. Example.** Joining $\{1, y\}$ and $\{1, x^2y\}$ gives $\{1, x^2, x^4, x^6, y, x^2y, x^4y, x^6y\}$ and $\{1, x^4\} \in \{1, x^2, x^4, x^6, y, x^2y, x^4y, x^6y\}$

So, by a similar reasoning as for $M(p^n)$, we obtain that the number of subgroups of $D_{2^n}$ verifies the recurrence relation

$$|L(D_{2^n})| = |L(\frac{D_{2^n}}{Z(D_{2^n})})| + 2^{n-1} + 1$$

$$|L(D_{2^n})| = |L(D_{2^{n-1}})| + 2^{n-1} + 1. \tag{5}$$

for all $n \geq 3$. Writing (5) for $n = 3, 4, ...$ and $|L(D_{2^{n-1}})|$ is $2^{n-1}n - 2$ (from table [3]. Summing up these equalities, we find an explicit expression of $|L(D_{2^n})|$.

**Theorem 3.** *The number of subgroups of the group $D_{2^n}$ is given by the following equality:* $|L(D_{2^n})| = 2^n + n - 1$.

**Proof.** From (5) $|L(D_{2^n})| = |L(D_{2^{n-1}})| + 2^{n-1} + 1$. From table (6) $|L(D_{2^{n-1}})|$ is $2^{n-1}n - 2$ then,

$$\begin{aligned} |L(D_{2^n})| &= 2^{n-1} + (n-2) + 2^{n-1} + 1. \\ &= 2 \cdot 2^{n-1} + (n-2) + 1. \\ &= 2 \cdot 2^{n-1} + n - 1. \\ &= 2^n + n - 1. \end{aligned}$$

∎

**II.3. Quaternion groups**

Because $Q_{2^n}$ verifies also the relation (4) and $Z(Q_{2^n})$ is the unique minimal subgroup of $Q_{2^n}$, we can easily infer from Theorem 3.

**Theorem 4.** *The number of subgroups of the group $Q_{2^n}$ is given by the following equality:*

$$\begin{aligned} \mid L(Q_{2^n}) \mid &= \mid L(D_{2^{n-1}}) \mid + 1 \\ &= 2^{n-1} + (n-1) - 1 + 1 \\ &= 2^{n-1} + n - 1 \end{aligned}$$

**II.4. Quasi-dihedral groups**$(SD_{2^n})$

The method developed above can also be used to count the subgroups of the quasi-dihedral group $(SD_{2^n})n \geq 4$. For each $i \in 0, 1, \ldots, 2^{n-1} - 1$, we have $(x^iy)^2 = x^{iq}$. Hence $ord(x^iy) = 2$ when $i$ is even, while $ord(x^iy) = 4$ when $i = odd$. This shows that the minimal subgroups of $S_{2^n}$ are of the form $< x^q >$ and $< x^2jy >$, $j = \overline{0, 2^{n-2} - 1}$.

**II.4.1. Examples.** For each $i \in 0, 1, \dots, 2^{n-1} - 1$

- $(x^i y)^2 = x^{iq}$ For $n = 4$, $i = 3$, $q = 2^{n-2}$

$$\begin{aligned}(x^i y)^2 &= (x^3 y)^2 \\ &= x^3 y x^3 y \\ &= x^3 x y y (xy = yx^3) \\ &= x^4\end{aligned}$$

$$\begin{aligned}x^{iq} &= x^{3 \cdot 4} \\ &= x^12 \\ &= x^8 \cdot x^4 \\ &= x^4\end{aligned}$$

Clearly, $(x^i y)^2 = x^{iq}$

- $ord(x^i y) = 2$ when $i$ is even Let $i = 2$

$$\begin{aligned}(x^2 y)^2 &= x^2 y \cdot x^2 y \\ &= x^2 x^6 y y (x^6 y = y x^2) \\ &= x^8 y^2 \\ &= 1\end{aligned}$$

Clearly when $i$ is even $x^i y$ is of order two.

- $ord(x^i y) = 4$ when $i$ is odd Let $i = 3$

$$\begin{aligned}(x^3 y)^4 &= (x^3 y)^2 \cdot (x^3 y)^2 \qquad = x^4 \cdot x^4 \\ &= x^8 \\ &= 1\end{aligned}$$

Clearly when $i$ is odd $x^i y$ is of order four.

- Minimal subgroups are of the form $< x^q >$ and $< x^{2j} y >$,
$n = 4$, $q = 2^{n-2}$, $j = \overline{\{0, 1, \dots, 2^{n-2} - 1\}}$
For $SD_{16}$ we have:
$\{1, x^4\}$ of the form $< x^q >$.
and
$\{1, y\}, \{1, x^2 y\}\{1, x^4 y\}, \{1, x^6 y\}$ of the form $< x^{2j} y >$ which is 4 in number.
Clearly for $SD_{16}$ we have 5 minimal subgroup.
Let $n = 5$,
For $SD_{32}$ we have:
$\{1, x^8\}$ of the form $< x^q >$,
and
$\{1, y\}, \{1, x^2 y\}, \{1, x^4 y\}, \{1, x^6 y\}, \{1, x^8 y\}, \{1, x^10 y\}, \{1, x^12 y\}, \{1, x^14 y\}$ of the form $< x^{2j} y >$ which is 8 in number, that is, $2^3$
Clearly for $SD_{32}$ we have 9 minimal subgroup.
It is clear that the minimal subgroup without the centre can be written as a power of prime, and of this form: $2^{n-2}$.

The join of any two distinct minimal subgroups different from $< x^q >$ contains a nonzero power of $x$ and therefore it includes $< x^q >$.

**II.4.2. Example.** Combining $1, y$ and $1, x^2y$ we have $\{1, x^2, x^4, x^6, y, x^2y, x^4y, x^6y\}$ and $\{1, x^4\} \in \{1, x^2, x^4, x^6, y, x^2y, x^4y, x^6y\}$. Thus we conclude that the subgroups of $SD_{2^n}$ which does not contain $Z(S_{2^n})$ are:

$$< 1 >, < y >, < x^2y >, \ldots, < x^{2^{n-1}-2} > .$$

In view of the group isomorphism $\frac{SD_{2^n}}{Z(SD_{2^n})} \cong D_{2^{n-1}}$, which gives

$$|L(SD_{2^n})| = |L(D_{2^{n-1}})| + 2^{n-2} + 1, \tag{6}$$

for all $n \geq 4$. From (6) and theorem 3 we get immediately the next result.

**Theorem 5.** $\mid L(SD_{2^n}) \mid = 3 \cdot 2^{n-2} + n - 1,$

**Proof.** Recall from table 7 that

$$|L(D_{2^{n-1}})| = 2^{n-1}n - 2$$

$$\begin{aligned} \mid L(SD_{2^n}) \mid &= \mid L\frac{SD_{2^n}}{Z(SD_{2^n})} \mid + 2^{n-2} + 1 \\ &= \mid L(D_{2^{n-1}}) \mid + 2^{n-2} + 1 \\ &= 2^{n-1} + n - 2 + 2^{n-2} + 1 \\ &= 2^{n-1} + 2^{n-2} + n - 1 \\ &= 2 \cdot 2^{n-2} + 2^{n-2} + n - 1 \\ &= 3 \cdot 2^{n-2} + n - 1 \end{aligned}$$

■

Finally, for an arbitrary finite group it is not an easy task comparing the number of its subgroups and the number of its elements. But can be easily made for the 2-groups in our class $\mathcal{G}$, by using Theorems 3, 4, and 5. Obviously, it obtains:

$$\mid L(M(2^n)) \mid \leq \mid M(2^n) \mid, \text{for all } n \geq 3$$

$$\mid L(D_{2^n}) \mid > \mid D_{2^n} \mid, \text{for all } n \geq 3$$

$$\mid L(Q_{2^n}) \mid < \mid Q_{2^n} \mid, \text{for all } n \geq 3$$

$$\mid L(SD_{2^n}) \mid < \mid SD_{2^n} \mid, \text{for all } n \geq 4$$

Moreover, the following limits were calculated:

$$\lim_{n\to\infty} \frac{|L(D_{2^n})|}{|D_{2^n}|} = 1$$

$$\lim_{n\to\infty} \frac{|L(Q_{2^n})|}{|Q_{2^n}|} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{|L(SD_{2^n})|}{|SD_{2^n}|} = \frac{3}{4}.$$

For any fixed prime $p$, we also have:

$$\lim_{n\to\infty} \frac{|L(M_{p^n})|}{|M_{p^n}|} = 0$$

## III. Related Problems

Arising from this work are other related problems which we are working on. One of the problem is given below:

### III.1. Counting Subgroups of the groups of type: $D_{2^n} \times C_2$

$D_{2^n}$ is a dihedral group of order $2^n$, $n \geq 3$, and $C_2$ is a cyclic group of order 2.

Table 7. Analysis of the number of subgroups in $D_{2^n} \times C_2$

| $D_{2^n}$ | $Order1$ | $Order2$ | $Order4$ | $Order8$ | $Order16$ | $Order32$ | $Order64$ | $\mid L(D_{2^n} \times C_2) \mid$ | Formula |
|---|---|---|---|---|---|---|---|---|---|
| $D_8 \times C_2$ | 1 | 11 | 15 | 7 | 1 | – | –– | 35 | $2^5 + 3(1)$ |
| $D_{16} \times C_2$ | 1 | 19 | 27 | 15 | 7 | 1 | – | 70 | $2^6 + 3(2)$ |
| $D_{32} \times C_2$ | 1 | 35 | 51 | 27 | 15 | 7 | 1 | 137 | $2^7 + 3(3)$ |
| $D_{2^n} \times C_2$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $2^{n+2} + 3(n-2)$ |

**Theorem 6.** *For $n \geq 3$, the number of subgroups of the group $D_{2^n} \times C_2$ is given by the following equality:*

$$\mid L(D_{2^n} \times C_2) \mid = 2^{n+2} + 3(n-2)$$

*Where $\mid L(D_{2^n} \times C_2) \mid$ is the subgroup lattice of $D_{2^n} \times C_2$.*

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