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AN EXPLICIT FORMULA FOR THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF THE CARTESIAN PRODUCT OF THE DIHEDRAL GROUP OF ORDER 2^n WITH A CYCLIC GROUP OF ORDER 2

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Abstract

The problem of classification of fuzzy subgroups can be extended from finite *p*-groups to finite nilpotent groups. Accordingly, any finite nilpotent group can be uniquely written as a direct product of *p*-groups. In this paper, we give explicit formulae for the number of distinct fuzzy subgroups of the Cartesian product of the dihedral group of order 2^n with a cyclic group of order 2.

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Introduction

Representation, management, manipulation and processing of data and information are very imperative in this fast growing, developing and advancing world, most especially, the non-statistical aspect and the uncertainties.

As a result, fuzzy sets were introduced by Zadeh in 1965 [3]. Even though, the story of fuzzy logic started much more earlier, it was specially designed, mathematically to represent uncertainty and vagueness. It also provides formalized tools for dealing with the imprecision intrinsic to many problems.

By the way, a group is nilpotent if it has a normal series of a finite length n. That is,

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\},$$

where

$$G_i / G_{i+1} \leq Z(G/G_{i+1}).$$

By this notion, every finite p-group is nilpotent. The nilpotence property is an hereditary one. Thus, (i) any finite product of nilpotent group is nilpotent, (ii) if G is nilpotent of a class c, then every subgroup and quotient group of G is nilpotent and of class $\leq c$.

Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a *fuzzy subgroup* of G if the following two conditions are satisfied:

(i) $\lambda(ab) \geq \min\{\lambda(a), \lambda(b)\}, \quad \forall a, b \in G$; (ii) $\lambda(a^{-1}) \geq \lambda(a)$ for any $a \in G$.

And since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$.

Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$ (Tarnauceanu [1]).

Now, concerning the subgroups, the set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the *fuzzy subgroup lattice* of *G*.

We define the level subset:

$$\lambda G_\beta = \{a \in G / \lambda(a) \geq \beta\} \text{ for each } \beta \in [0, 1].$$

In what follows, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite *p*-group *G* is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of *G*. Let $h(G)$ denote the number of chains of subgroups of *G* which ends in *G*. The method of computing $h(G)$ is based on the application of the Inclusion-Exclusion Principle. If *A* is the set of chains in *G* of type $C_1 \subset C_2 \subset \cdots \subset C_r = G$, and A' represents the set of chains of A' which are contained in M_r, $r = 1, \ldots, t$, then we have:

$$|A| = 1 + |A'| = \left| \bigcup_{r=1}^{t} A_r \right|$$

$$= 1 + \sum_{r=1}^{t} |A_r| - \sum_{1 \leq r_1 \leq r_2 \leq t} h| A_{r_1} \cap A_{r_2} | + \cdots + (-1)^{t-1} \left| \bigcap_{r=1}^{t} A_r \right|.$$

Observe that, for every $1 \leq w \leq t$ and $1 \leq r_1 < r_2 < \cdots < r_w \leq t$, the set $\bigcap_{i=1}^{w} A_{r_i}$ consists of all chains of A' which are included in $\bigcap_{i=1}^{w} M_{r_i}$. We have that

$$\left| \bigcap_{i=1}^{w} A_{r_i} \right| = 2h\left(\bigcap_{i=1}^{w} M_{r_i} \right) - 1.$$

Therefore,

$$|A| = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) + C$$

and

$$C = 1 + \sum_{r=1}^{t} (-1) - \sum_{1\le r_1 < r_2 \le t} (-1) + \cdots + (-1)^{t-1}(-1)$$

$$= (1-1)^t = 0.$$

We have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \quad (*)$$

In [2], (∗) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n.

Theorem 2 [1]. *The number of distinct fuzzy subgroups of a finite p-group of order* p^n *which has a cyclic maximal subgroup is*:

(i) $h(\mathbb{Z}_{p^n}) = 2^n$, (ii) $h(D_{2^n}) = 2^{2n-1}$, (iii) $h(\varphi_{2^n}) = 2^{2n-2}$, (iv) $h(\varphi_{2^n}) = 2^{2n-2}$ *and* (v) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

An Explicit Formula for the Number of Distinct Fuzzy Subgroups of the Cartesian Product of the Dihedral Group of Order 2^n with a Cyclic Group of Order 2

The Cartesian product of the dihedral group of order 2^n with a cyclic group of order 2

We introduce here a nilpotent group of the form: $D_{2^n} \times C_2$, where C_2 is a cyclic group of order two and

$$D_{2^n} = \langle x,\ y \mid x^{2^{n-1}} = y^2 = 1,\ yxy^{-1} = x^{-1}\rangle, \quad n \geq 3.$$

Putting $n = 3,$ we have

$$D_{2^3} \times C_2 = \{1,\ x,\ x^2,\ x^3,\ y,\ xy,\ x^2y,\ x^3y\} \times \{1,\ a\}.$$

By the rule of Inclusion-Exclusion Principle, we seek the maximal subgroups involved. There exist seven distinct maximal subgroups for this group. One is abelian, generated by $\langle (1,\ a),\ (x,\ 1)\rangle$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Two are isomorphic to E_{2^3}, the elementary abelian of order eight. Other four are isomorphic to D_{2^3}, the dihedral group of order eight.

We have the seven maximal subgroups as follows.

By equation (*), let $G = D_{2^3} \times C_2$. Then we have

$$h(G) = 2\left[\sum_{r=1}^{7} h(M_r) - \sum_{1 \leq r < r_2 \leq 7} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^6 h\left(\bigcap_{r=1}^{7} M_r\right)\right],$$

$$\frac{1}{2} h(G) = h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 4h(D_{2^3}) + 2h(E_{2^3}) - 4h(\mathbb{Z}_{2^2}).$$

Therefore, $h(G) = h(D_{2^3} \times C_2) = 2 \times 216 = 432.$

The Cartesian product of the dihedral group of order 16 and a cyclic group of order 2

Recall that:

$$D_{2^4} \times C_2 = \{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6,\ x^7,\ y,\ xy,\ x^2y,$$

$$x^3y,\ x^4y,\ x^5y,\ x^6y,\ x^7y\} \times \{1,\ a\}.$$

By equation (*), setting $G = D_{2^4} \times C_2$, we have

$$\frac{1}{2}h(G) = [h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 4h(D_{2^4}) + 2h(D_{2^3} \times C_2)]$$

$$- 3[2h(\mathbb{Z}_{2^3}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 4h(D_{2^3})]$$

$$+ [28h(\mathbb{Z}_{2^2}) + 2h(\mathbb{Z}_{2^3}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 4h(D_{2^3})] - 20h(\mathbb{Z}_{2^2})$$

$$= 64 + 512 - 256 + 864 - 48 = 1136.$$

Therefore, $h(D_{2^4} \times C_2) = 2 \times 1136 = 2272.$

Following the same trend of similar pattern, we have in general that

$$\frac{1}{2}h(D_{2^n} \times C_2) = h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 4h(D_{2^n}) - 8h(D_{2^{n-1}})$$

$$- 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2h(D_{2^{n-1}} \times C_2).$$

Therefore,

$$h(D_{2^n} \times C_2) = 2[2^n + 2^{2n} + 2h(D_{2^{n-1}} \times C_2)]$$

$$= 2^{2n}(2n+1) - 2^{n+1}, \; n > 3.$$

Theorem. *Let* $G = D_{2^n} \times C_2$, *the nilpotent group formed by the Cartesian product of the dihedral group of order* 2^n *and a cyclic group of order* 2. *Then the number of distinct fuzzy subgroups of G is given by*:

$$h(G) = 2^{2n}(2n+1) - 2^{n+1}, \; \textit{for } n > 3.$$

Proof. The group $D_{2^n} \times C_2$ has one maximal subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, two maximal subgroups which are isomorphic to $D_{2^{n-1}} \times C_2$, and 2^2 which are isomorphic to D_{2^n}.

It follows from the Inclusion-Exclusion Principle using $(*)$ that:

$$\frac{1}{2}h(D_{2^n} \times C_2) = h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 4h(D_{2^n}) - 8h(D_{2^{n-1}})$$
$$- 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2h(D_{2^{n-1}} \times C_2).$$

By recurrence relation principle, we have:

$$h(D_{2^n} \times C_2) = 2^{2n}(2n+1) - 2^{n+1}, \quad n > 3.$$

By the fundamental principle of mathematical induction, set $F(n) = h(D_{2^n} \times C_2)$, assuming the truth of

$$F(k) = h(D_{2^k} \times C_2)$$
$$= 2h(\mathbb{Z}_2 \times \mathbb{Z}_{k-1}) + 8h(D_{2^k} \times C_2) - 16h(D_{2^{k-1}} \times C_2)$$
$$- 4h(\mathbb{Z}_2 \times \mathbb{Z}_{k-2}) + 4h(D_{2^{k-1}} \times C_2)$$
$$= 2^{2k}(2k+1) - 2^{k+1},$$

$$F(k+1) = h(D_{2^{k+1}} \times C_2) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) + 8h(D_{2^{k+1}} \times C_2)$$
$$- 16h(D_{2^k} \times C_2) - 4h(\mathbb{Z}_2 \times \mathbb{Z}_{k-1}) + 4h(D_{2^k} \times C_2)$$
$$= 2^2[2^{2k}(2k-3) - 2^k],$$

which is true. □

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