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An Interval Analytic Method in Constructive Existence Theorems for Initial Value Problems

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Abstract
The method of interval analysis is employed to show that the solution, if it exists, of a first order initial value problem is majorised by an interval function whose end-functions satisfy some prescribed conditions. An interval operator is constructed and shown to be a contraction on the majorising interval function. Using this operator, the existence and uniqueness of the solution is established.

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1. INTRODUCTION

Consider the scalar initial value problem

$$u' = f(t, u(t)), \quad u(0) = u_0 \tag{1.1}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R})$ and $I = \{t : 0 \le t \le T < \infty\}$. Let us assume that there exist functions α and $\beta \in C^1(I, \mathbb{R})$ such that

$$\alpha(t) \le \beta(t),\ t \in I \ \text{ and } \ \alpha(0) \le u_0 \le \beta(0) \tag{1.2}$$

Assume further that the function f has continuous first order partial derivative with respect to its second argument and that

$$\left.\begin{array}{l} \alpha'(t) \le f(t,u(t)) + f_u(t,u(t))(\alpha(t) - u(t)) \\ \text{and} \\ \beta'(t) \ge f(t,u(t)) + f_u(t,u(t))(\beta(t) - u(t)) \end{array}\right\} \tag{1.3}$$

for any function $u \in C^1(I, \mathbb{R})$ satisfying $\alpha(t) \le u(t) \le \beta(t)$ on I.

In this paper we shall use an interval analytic method to establish existence of a solution and subsequently its uniqueness. Some authors (Chan & Vatsala, 1990; Lakshmikantham & Swansundaran, 1987) have used this method to obtain existence of solutions and sometimes solution sets of differential equations. However, the real integral operators equivalent to the problems were so constructed to ensure their

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monotonicity. The techniques employed here rely solely on the inherent monotone inclusion property of interval functions and as such neither the underlying real function nor the equivalent real integral operator need be monotone.

2. BASIC DEFINITIONS AND RESULTS IN INTERVAL ANALYSIS

We give some basic definitions and results in interval analysis that will be needed in subsequent discussions. Readers who are however not familiar with this subject are referred to (Moore, 1979; Rall, 1981; Caparani et al, 1981).

The basic objects of interval analysis used here are the closed, non-empty and bounded real intervals

$$X = [\underline{x}, \bar{x}] = \{x \mid \underline{x} \le x \le \bar{x}\} \tag{2.1}$$

and the real number x is identified with the degenerate interval

$$x = [x, x] \tag{2.2}$$

Definition 2.1: The *width* $w(X)$, *midpoint* $m(X)$ and *modulus* $|X|$ of the interval X are respectively defined as

$$w(X) = \bar{x} - \underline{x} \tag{2.3}$$

$$m(X) = \frac{1}{2}(\bar{x} + \underline{x}) \tag{2.4}$$

and

$$|X| = \max\{|\underline{x}|, |\bar{x}|\} \tag{2.5}$$

Definition 2.2: An *interval function* Y is defined as the function which assigns to each x in its interval of definition $X = [\underline{x}, \bar{x}]$ the interval denoted by

$$Y(x) = [\underline{y}(x), \bar{y}(x)] \tag{2.6}$$

where the real functions $\underline{y}$ and $\bar{y}$ are called the endfunctions of Y.

Definition 2.3: An interval function Y is said to be an *interval extension* of a real function y if it has the property of inclusion of y

$$y(X) = \{y(x) \mid x \in X\} \subseteq Y(X) \tag{2.7}$$

for each interval $X = [\underline{x}, \bar{x}]$ on which y is defined.

An interval extension Y is called a *natural interval extension* of y if it is obtained from y by replacing the real variables with the corresponding interval variables and the real arithmetic operations with the corresponding interval arithmetic operations. The interval arithmetic operations used in this work are those defined in chapter two of [2] and they preserve the inclusion property.

If y is a differentiable function with derivative y', then the interval extension Y is called an *interval mean-value extension* of y if it is given by

$$Y(X) = y(m(X)) + Y'(X)(X - m(X)) \tag{2.8}$$

where Y' is a natural interval extension of the derivative y'.

Definition 2.4: An interval function Y is said to be *inclusion monotone* if

$$X_1 \subseteq X_2 \Rightarrow Y(X_1) \subseteq Y(X_2) \tag{2.9}$$

for intervals, X_1, X_2 on which Y is defined.

Definition 2.5: Let Y be a non-degenerate interval and X another interval which may be degenerate or not. The interval Y is said to be an *interval majorant* of X if

$$X(t) \subseteq Y(t) \quad t \in I \tag{2.10}$$

for an interval I on which X and Y are defined.

Definition 2.6: The *interval integral* of an interval function Y over an interval $X = [\underline{x}, \bar{x}]$ on which it is defined is the interval

$$\int_{\underline{x}}^{\bar{x}} Y(t)dt = \int_X Y(t)dt = \left[\underline{\int}_X \underline{y}(t)dt, \overline{\int}_X \bar{y}(t)dt\right] \tag{2.11}$$

where $\underline{\int}_X$ denotes the lower Darboux integral over X and $\overline{\int}_X$ denotes the upper Darboux integral over X.

Lemma 2.1 (Rall, 1981): If X and Y are intervals, then

$$X \subseteq Y \Leftrightarrow |m(Y) - m(X)| \leq \frac{1}{2}\{w(Y) - w(X)\} \tag{2.12}$$

Theorem 2.1 (Moore, 1979): If P is an inclusion monotonic interval majorant of a real operator p and if

$$P(X_0) \subseteq X_0 \tag{2.13}$$

for an interval X_0 in the domain of P, then the sequence $\{X_k\}$ of intervals defined by

$$X_{k+1} = P(X_k), \quad k = 0, 1, 2, \ldots \tag{2.14}$$

has the following properties:

(i) $X_{k+1} \subseteq X_k, \quad k = 0, 1, 2, \ldots$

(ii) for t in the interval I of definition of X, the limit

$$X(t) = \bigcap_{k=0}^{\infty} X_k(t) \tag{2.15}$$

exists as an interval function and

$$X(t) \subseteq X_k(t), \quad k = 0, 1, 2, \ldots$$

(iii) for any solution x of the operator equation

$$x(t) = p(x)(t), \quad t \in I \tag{2.16}$$

such that $x(t) \in X_0(t), \quad t \in I$
we have $x(t) \in X_k(t) \quad \forall\, k$ and $t \in I$.

(iv) if there exists a real number c, such that $0 \leq c \leq 1$, for which

$$Z \subseteq X_0$$

$$\Rightarrow \sup_t w(P(Z)(t)) \leq c \sup_t (w(Z(t)),$$

then (2.16) has the unique solution $x(t)$ in X given by (2.15).

3. INTERVAL MAJORANT OF SOLUTION

In this section we present a result which guarantees the majorisation of a solution of the initial value problem (1.1), if it exists, by an interval function.

Theorem 3.1: Suppose that in addition to being continuous, the function f appearing in equation (1.1) has continuous first order partial derivative with respect to x and that it also satisfies conditions (1.3). Then, if u is a solution of the i.v.p. (1.1), it is majorised by the interval function Y given by

$$Y(t) = [\alpha(t), \beta(t)],$$

where α and β are the functions defined in (1.2).

Proof: If u is a solution of (1.1), we need to show that $u \in Y$ and this we shall show by contradiction.
Suppose $u(t) \notin Y(t)$ for some $t \in J \subseteq I$. Then

$$\left.\begin{array}{ll} \text{either} & u(t) < \alpha(t) \\ \text{or} & u(t) > \beta(t), \quad t \in J \end{array}\right\} \tag{3.1}$$

First we suppose $u(t) < \alpha(t)$, $t > 0$, so the interval function G defined by

$$G(t) = [u(t), \alpha(t)]$$

has a strictly positive width. Also since u is assumed to be a solution of equation (1.1), it also solves the integral equation

$$u(t) = u(0) + \int_0^t f(s, u(s))ds, \quad t \in J. \tag{3.2}$$

From (1.3) we also have

$$\alpha(t) \leq \alpha(0) + \int_0^t f(s, \alpha(s))ds, \quad t \in J. \tag{3.3}$$

From (3.2) and (3.3) we have

$$w(G(t)) \leq \alpha(0) - u(0) + \int_0^t \{f(s, \alpha(s)) - f(s, u(s))\}\, ds. \tag{3.4}$$

Since

$$f(s, \alpha(s)) - f(s, u(s)) \in F_u(s, G(s))[0, w(G(s))],$$

the integral inequality (3.4) gives

$$w(G(t)) \leq \alpha(0) - u(0) + \int_0^t |F_u(s, G(s))|\, w(G(s))ds, \quad t \in J$$

which by the application of Gronwall-Bellman's Lemma in [1] yields

$$w(G(t)) \leq (\alpha(0) - u(0)) \exp\left(\int_0^t |F_u(s, G(s))|\, ds\right),$$

showing by (1.2) and earlier assumptions that

$$0 < w(G(t)) \leq 0.$$

This contradicts the assumption that $w(G(t))$ is strictly positive. Hence the assumption that $u(t) < \alpha(t)$, $t > 0$ must have been wrong. We now consider the other assumption, that

$$u(t) > \beta(t) \quad t > 0.$$

In a similar manner we deduce that the interval function H defined by

$$H(t) = [\beta(t), u(t)]$$

has a strictly positive width.

By (1.3) we equally have

$$\beta(t) \geq \beta(0) + \int_0^t f(s, \beta(s))ds, \quad t \in J. \tag{3.5}$$

Using this and (3.2) we obtain:

$$w(H(t)) \leq u(0) - \beta(0) + \int_0^t \{f(s,u(s)) - f(s,\beta(s))\}\, ds.$$

Again by the property of the interval function F_u, we have

$$w(H(t)) \leq u(0) - \beta(0) + \int_0^t |F_u(s,H(s))|\, w(H(s))ds$$

which by Gronwall-Bellman's Lemma again yields

$$w(H(t)) \leq (u(0) - \beta(0)) \exp\left(\int_0^t |F_u(s,H(s))|\, ds\right).$$

By the assumption on $w(H)$ and (1.2) this implies that

$$0 < w(H) \leq 0$$

which is a contradiction.
Hence our assumptions must be wrong and so we have $u \in Y$ as required.

4. EXISTENCE RESULTS

In this section an interval operator is constructed. It is shown that this interval operator is a contraction. With the use of this operator, the existence of a nested sequence of interval functions is established and shown to converge to a limit containing the solution of (1.1).

Theorem 4.1: Suppose that the function f appearing in equation (1.1) is continuous and continuously differentiable with respect to its second argument. Assume further that it satisfies conditions (1.3) with the functions α and β defined in (1.2). Then the interval integral operator P defined by

$$P(U(t)) = u_0 + \int_0^t f(s, m(U(s)))ds + \frac{1}{2}\int_0^t |F_u(s,U(s))|\, w(U(s))[-1,1]ds \qquad (4.1)$$

contracts the interval function

$$Y(t) = [\alpha(t), \beta(t)], \quad t \in I. \qquad (4.2)$$

where $w(U)$ and $m(U)$ are respectively the width and midpoint of the interval function U and the interval function F_u is a natural interval extension of the partial derivative f_u of the function f chosen such that

$$|F_u(t, Y(t))| \leq 2f_u(t, m(Y(t))). \qquad (4.3)$$

Proof: To show that P contracts Y it suffices to show, by Lemma 2.1, that

$$|m(Y) - m(P(Y))| \leq \frac{1}{2}(w(Y) - w(P(Y))) \qquad (4.4)$$

Now from (4.1) and (4.2), we have, for $t \in I$

$$\begin{aligned}
-\frac{1}{2}\{w(Y(t)) - w(P(Y(t)))\} &= \frac{1}{2}\left\{\int_0^t |F_u(s, Y(s))|\, w(Y(s))ds - \{\beta(t) - \alpha(t)\}\right\} \\
&\le \cdot \int_0^t f_u(s, m(Y(s)))w(Y(s))ds - \frac{1}{2}\{\beta(t) - \alpha(t)\} \\
&\le \int_0^t \{\beta'(s) - f(s, m(Y(s)))\}\, ds - \frac{1}{2}\{\beta(t) - \alpha(t)\} \\
&\quad \text{by assumption (1.3).} \\
&\le \frac{1}{2}\{\beta(t) + \alpha(t)\} - u(0) - \int_0^t f(s, m(Y(s)))ds \\
&\quad \text{by assumption (1.2).}
\end{aligned}$$

That is

$$-\frac{1}{2}\{w(Y(t)) - w(P(Y(s)))\} \le \{m(Y(t)) - m(P(Y(t)))\} \tag{4.5}$$

We also, from (4.1) and (4.2), have

$$\begin{aligned}
\frac{1}{2}\{w(Y(t)) - w(P(Y(t)))\} &= \frac{1}{2}\left\{\beta(t) - \alpha(t) - \int_0^t |F_u(s, Y(s))|\, w(Y(s))ds\right\} \\
&\ge \frac{1}{2}\left\{\beta(t) - \alpha(t) - 2\int_0^t f_u(s, m(Y(s)))w(Y(s))ds\right\} \\
&\ge \frac{1}{2}\left\{\beta(t) - \alpha(t) + 2\int_0^t \{\alpha'(s) - f(s, m(Y(s)))\}\, ds\right\} \\
&\quad \text{by assumption (1.3).} \\
&\ge \frac{1}{2}\{\beta(t) + \alpha(t)\} - u(0) - \int_0^t f(s, m(Y(s)))ds \\
&\quad \text{by assumption (1.2).}
\end{aligned}$$

which implies that

$$\frac{1}{2}\{w(Y(t)) - w(P(Y(t)))\} \ge m(Y(t)) - m(P(Y(t))) \tag{4.6}$$

The combination of (4.5) and (4.6) yields the desired result (4.4) which by Lemma 2.1 establishes that

$$P(Y) \subseteq Y. \tag{4.7}$$

Hence the operator P contracts the interval Y.

Example: Consider the initial value problem

$$u' = u^2 - t, \quad u(0) = 1, \quad 0 \le t < 1$$

$\alpha(t) = 1 + t$, $\beta(t) = 1/(1 - t)$. These functions satisfy conditions (1.2) and (1.3) since $\alpha'(t) \le f(t, \alpha)$ *and* $\beta'(t) \ge f(t, \beta)$. So, we have $Y = [1 + t, 1/(1 - t)]$, $m(Y) = (2 - t^2)/2(1 - t)$, $f_u = 2u$, $F_u = 2U$, $f_u(t, m(Y)) = (2 - t^2)/(1 - t)$, $F_u(t, Y) = 2[1 + t, 1/(1 - t)]$, *and* $|F_u| = 2/(1 - t)$.

In this case $|Fu(t,Y)| \le 2f_u(t,m(Y)), \forall t \in [0,1)$. and $P(Y(t))$ as defined by (4.1) is given by:

$$P(Y(t)) = 1 + \int_0^t \{\frac{(2-s^2)^2}{4(1-s)^2} + \frac{s^2}{(1-s)^2}[-1,1]\}ds.$$

When the integral is evaluated it gives:

$$P(Y(t)) = 1+(t^3+3t^2)/12+[-5t/4-3t/4(1-t)-\ln(1-t)^3, 3t/4+5t/4(1-t)+\ln(1-t)].$$

It is clearly seen that $P(Y(t)) \subseteq Y(t)$ for $0 \le t < 1$.

Theorem 4.2: Let all the assumptions of Theorem 4.1 be true. Then there exists a nested sequence of interval functions $\{U_n(t) : t \in I,\ n \in \mathbb{N}\}$ with the initial interval $U_0(t) = [\alpha(t), \beta(t)]$, where α and β are the functions defined in (1.2). Furthermore the sequence is such that the limit

$$U(t) = \lim_{n\to\infty} U_n(t)$$

exists as an interval function on I and is a majorant of the function u which solves the initial value problem (1.1).

Proof: Since the function f in the i.v.p. (1.1) is continuous, the problem is equivalent to the integral equation

$$u(t) = u_0 + \int_0^t f(s, u(s))ds, \quad t \in I \tag{4.8}$$

the solution of which also satisfies the i.v.p. (1.1). We now prepare to obtain this solution by an interval analytic method:

Let U with the representation

$$U(t) = [m(U(t)) - \frac{1}{2}w(U(t)),\ m(U(t)) + \frac{1}{2}w(U(t))]$$

be an interval majorant of the solution u of equation (4.8); by Theorem 3.1, this exists. Let F_u be a natural interval extension of f_u, the partial derivative of the function f satisfying assumption (4.3). Then an interval mean value extension of f is given by

$$\begin{aligned} F(t, U(t)) &= f(t, m(U(t)) + F_u(t, U(t))(U(t) - m(U(t))) \\ &= f(t, m(U(t)) + \frac{1}{2}|F_u(t, U(t))|w(U(t))[-1,1] \end{aligned}$$

Substituting these interval functions for u and f in equation (4.8) with the interval integral and interval arithmetic operations where appropriate we obtain an interval integral operator P, which is an interval extension of the real operator (4.8), given by

$$P(U(t)) = u_0 + \int_0^t f(s, m(U(s)))ds + \frac{1}{2}\int_0^t |F_u(s, U(s))|w(U(s))[-1,1]ds \tag{4.9}$$

and we then have $u(t) \in P(U(t)),\ \ t \in I$.

We now define a sequence of interval functions $\{U_n\}$ by

$$U_{n+1}(t) = P(U_n(t)),\ \ n = 0, 1, 2, \dots \tag{4.10}$$

with

$$U_0(t) = [\alpha(t), \beta(t)].$$

All we need to establish the nestedness of this sequence by Theorem 2.1, is to show that

$$P(U_0(t)) \subseteq U_0(t).$$

This by Theorem 4.1 is true and so the sequence (4.10) is nested with each term containing the solution u of (4.8). Finally the sequence converges as well by Theorem 2.1, to the limit U, given by

$$U(t) = \bigcap_{n=1}^{\infty} U_n(t),\ \ t \in I \tag{4.11}$$

and also contains the solution u of problem (4.8). This concludes the proof.

5. UNIQUENESS OF SOLUTION

In what follows we show that the limit of the interval sequence is unique irrespective of the initial interval functions, as long as the prescribed conditions are satisfied by the end functions of such an interval. We finally give condition under which the limit interval function coincides with the real valued solution of the i.v.p. (1.1).

Theorem 5.1: The limit (4.11) of the sequence of interval functions (4.10) of Theorem 4.2 is unique as long as the endfunctions of the initial interval function satisfy conditions (1.2) and (1.3).

Proof: Suppose the limit varies with the initial interval function. Then if $\{X_n\}$ is another sequence of interval functions with initial interval function X_0 given by $X_0(t) = [\rho(t), \tau(t)]$ and limit $X(t)$ where ρ and τ satisfy conditions (1.2) and (1.3) with α and β replaced by ρ and τ respectively. Since ρ and τ satisfy conditions (1.2) and (1.3), it follows, by Theorem 3.1, that the solution u of the i.v.p (1.1) satisfies $u(t) \in X_0(t)$, $t \in I$ and therefore

$$U_0(t) \cap X_0(t) \neq \phi.$$

Let

$$Z_0(t) = X_0(t) \cap U_0(t)$$

then $u(t) \in U(t) \subseteq Z_0(t)$ and $u(t) \in X(t) \subseteq Z_0(t)$.
This implies, by the monotone inclusion property of F_u, that

$$F_u(t, U(t)) \subseteq F_u(t, Z_0(t))$$
$$\text{and} \quad F_u(t, X(t)) \subseteq F_u(t, Z_0(t)) \tag{5.1}$$

From (4.9) we have

$$U(t) = P(U(t)) = u_0 + \int_0^t f(s, m(U(s)))ds + \frac{1}{2}\int_0^t |F_u(s, U(s))| w(U(s))[-1, 1]ds \tag{5.2}$$

and since X is also the limit of the sequence $\{X_n\}$, we have, by (4.9),

$$X(t) = P(X(t)) = x_0 + \int_0^t f(s, m(X(t)))ds + \frac{1}{2}\int_0^t |F_u(s, X(s))| w(X(s))[-1, 1]ds \tag{5.3}$$

so

$$d(X(t), U(t)) = \max\{|\underline{x}(t) - \underline{u}(t)|, |\bar{x}(t) - \bar{u}(t)|\}$$

which, by the use of (5.2) and (5.3), gives

$$\begin{aligned}
= \;& \max\Big\{\Big|x_0 + \int_0^t f(s, m(X(s)))ds - \frac{1}{2}\int_0^t |F_u(s, X(s))|\, w(X(s))ds \\
& -u_0 - \int_0^t f(s, m(U(s)))ds + \frac{1}{2}\int_0^t |F_u(s, U(s))|\, w(U(s))ds\,\Big|, \\
& \Big|x_0 + \int_0^t f(s, m(X(s)))ds + \frac{1}{2}\int_0^t |F_u(s, X(s))|\, w(X(s))ds \\
& -u_0 - \int_0^t f(s, m(U(s)))ds - \frac{1}{2}\int_0^t |F_u(s, U(s))|\, w(U(s))ds\Big|\Big\} \\
= \;& \max\Big\{\Big|x_0 - u_0 + \int_0^t \{f(s, m(X(s))) - f(s, m(U(s)))\}\, ds \\
& -\frac{1}{2}\int_0^t \{|F_u(s, X(s))|\, w(X(s)) - |F_u(s, U(s))|\, w(U(s))\}\, ds\Big|, \\
& \Big|x_0 - u_0 + \int_0^t \{f(s, m(X(s))) - f(s, m(U(s)))\}\, ds \\
& +\frac{1}{2}\int_0^t \{|F_u(s, X(s))|\, w(X(s)) - |F_u(s, U(s))|\, w(U(s))\}\, ds\Big|\Big\} \\
\leq \;& |x_0 - u_0| + max\Big\{\Big|\int_0^t |F_u(s, Z_0(s))|\, \{m(X(s)) - m(U(s))\}\, ds \\
& -\frac{1}{2}\int_0^t |F_u(s, Z_0(s))|\, \{w(X(s)) - w(U(s)) - w(Z_0(s))\}\, ds\Big|, \\
& \Big|\int_0^t |F_u(s, Z_0(s))|\, \{m(X(s)) - m(U(s))\}\, ds \\
& +\frac{1}{2}\int_0^t |F_u(s, Z_0(s))|\, \{w(X(s)) - w(U(s)) + w(Z_0(s))\}\, ds\Big|\Big\} \\
\leq \;& |x_0 - u_0| + \frac{1}{2}\int_0^t |F_u(s, Z_0(s))|\, w(Z_0(s))ds
\end{aligned}$$

$$+max\left\{\left|\int_0^t |F_u(s,Z_0(s))|\left\{(m(X(s))-\frac{1}{2}w(X(s)))-(m(U(s))-\frac{1}{2}w(U(s)))\right\}ds\right|,\right.$$
$$\left.\left|\int_0^t |F_u(s,Z_0(s))|\left\{(m(X(s))+\frac{1}{2}w(X(s)))-(m(U(s))+\frac{1}{2}w(U(s)))\right\}ds\right|\right\}$$
$$\le |x_0-u_0|+\frac{1}{2}\int_0^t |F_u(s,Z_0(s))|\,w(Z_0(s))ds$$
$$+\int_0^t max\left\{\left|(m(X(s))-\frac{1}{2}w(X(s)))-(m(U(s))-\frac{1}{2}w(U(s)))\right|,\right.$$
$$\left.\left|(m(X(s))+\frac{1}{2}w(X(s)))-(m(U(s))+\frac{1}{2}w(U(s)))\right|\right\}|F_u(s,Z_0(s))|\,ds$$

That is,

$$d(X(t),U(t))\le |x_0-u_0|+\frac{1}{2}\int_0^t |F_u(s,Z_0(s))|\,w(Z_0(s))ds+\int_0^t d(X(s),U(s))|F_u(s,Z_0(s))|ds$$

and this implies

$$d(X(t),U(t))\le |x_0-u_0|\exp\left(\frac{3}{2}\int_0^t |F_u(s,Z_0(s))|ds\right)$$

which by (1.1) $\Rightarrow$ $d(X(t),U(t))\le 0$
and therefore $d(X(t),U(t))=0$.
Hence $X(t)=U(t)$, and this concludes the proof.

Theorem 5.2: Suppose the natural interval extension of the function f_u considered is such that

$$\int_0^t |F_u(s,U(s))|\,ds\le 1 \quad \text{for} \quad t\in I \tag{5.4}$$

Then the sequence of interval functions (4.9) converges to a degenerate interval function which coincides with the real valued solution u of the i.v.p. (1.1).

Proof: From Theorem 4.2, $u(t)\in U(t)=\lim_{n\to\infty} U_n(t)$. and also $U(t)=P(U(t))$.
Therefore

$$w(U(t))=w(P(U(t))).$$

From (4.8)

$$w(P(U(t)))=\int_0^t |F_u(s,U(s))|w(U(s))ds$$

and so

$$\sup_t w(U(t)) = \sup_t \int_0^t |F_u(s,U(s))|w(U(s))ds$$
$$\le \left(\sup_t w(U(t))\right)\int_0^t |F_u(s,U(s))|\,ds$$

setting

$$k=\int_0^t |F_u(s,U(s))|\,ds$$

we have

$$(1-k)\sup_t w(U(t)) \le 0.$$

which, by (5.4), implies $\sup_t w(U(t)) \le 0$ and hence $w(U(t)) = 0$.

So $U(t) = [r(t), r(t)]$, a degenerate interval, for a real function $r(t)$ defined on I, and since $u(t) \in U(t)$ we have $u(t) = r(t)$ as required.

Remarks: (a) If conditions in (1.3) are replaced with the conditions

$$f(t,x) - f(t,y) \ge -M(x-y), \quad M > 0$$

and

$$\alpha' \le f(t,\alpha); \quad \beta' \ge f(t,\beta),$$

the results of theorems 3.1, 4.1 and 4.2 would still hold. However, an interval extension for f which does not involve f_u would be needed to establish the results.

(b) If $f_x(t, x(t))$ in (1.3) is replaced with $-M$, for positive constant M, we obtain the result of theorem 2.1 in (Lakshmikantham & Swaansundaran, 1987).

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