STABILITY RESULTS FOR THE SOLUTIONS OF A CERTAIN THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

Ademola, Timothy A.
Department of Mathematics and Statistics,
Bowen University, P. M. B. 284, Iwo, Osun State, Nigeria
e-mail: ademola672000@yahoo.com

Ogundiran, Michael O.
College of Natural and Applied Sciences, Department of Physical Sciences,
Bells University of Technology, P.M.B 1015, Ota, Ogun State, Nigeria
e-mail: adeolu74113@yahoo.com

Arawomo, Peter O.
Department of Mathematics, University of Ibadan
Ibadan, Nigeria
e-mail: arawomopet@yahoo.com

Adesina, Olufemi Adeyinka[1]
Department of Mathematics,
Obafemi Awolowo University, Ile-Ife, Nigeria
[1]Corresponding author's e-mail: oadesina@oauife.edu.ng

Abstract

The paper is concerned with the uniform asymptotic stability for all solutions of a third order nonlinear differential equation (1.1). Sufficient conditions under which all solutions $x(t)$, its first and second derivatives tend to zero as $t \to \infty$ are given.

Keywords and Phrases. Differential equations of third order; Lyapunov functions; uniform asymptotic stability.

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1 Introduction

Stability analysis of nonlinear systems is an important area of current research and many concept of stability analysis have in the past and also recently been studied, see for instance Reissig *et. al.,* [11], Rouche *et. al.,* [12] and Yoshizawa [16]. The study of the qualitative behaviour of solutions to third order nonlinear differential equations has been discussed by many authors in a series of papers, see for instance Afuwape [1], Afuwape and Adesina [2], Andres [3], Bereketoğlu, and Györi [4], Ezeilo [5, 6], Ezeilo and Tejumola [7], Hara [8], Ogundare [9], Qian [10], Tejumola [13] and Tunc [14, 15]. These works were done with the aid of Lyapunov functions except in [2], where frequency domain approach was used. With respect to our observation in the relevant literature, works on the uniform asymptotic stability for third order nonlinear differential equation (1.1) using a complete Lyapunov function are scarce.

The purpose of this paper is to study the uniform asymptotic stability of the third order nonlinear autonomous ordinary differential equation

$$\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = 0 \tag{1.1}$$

or its equivalent system of differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -f(z) - g(y) - h(x), \end{aligned} \tag{1.2}$$

where $f, g, h \in C(\mathbb{R},\ \mathbb{R})$ and $\mathbb{R} = (-\infty,\ \infty)$. In what follows, we assume that the functions f, g and h depend only on the arguments displayed explicitly. The dots, as usual, indicate differentiation with respect to t. Furthermore we shall require that the derivative $h'(x) = \dfrac{dh(x)}{dx}$ exists and continuous and the uniqueness of solutions to (1.1) or (1.2) will be assumed. We shall use Lyapunov's second (or direct) method as our basic tool to achieve the desired results.

The results obtained in this work complement existing results on third order nonlinear differential equations in the literature.

2 Main Results

We have the following results:

Theorem 2.1 *Suppose that a, b, c, δ_0 are positive constants and that*

(i) $h(0) = 0, \quad \delta_0 \leq h(x)/x, \quad$ *for all* $x \neq 0$;

(ii) $h'(x) \leq c$ *for all* x;

(iii) $b \leq g(y)/y \leq b_1, \quad$ *for all* $y \neq 0$;

(iv) $a \leq f(z)/z, \quad$ *for all* $z \neq 0$;

Then the zero solution of the system (1.2) is uniformly asymptotically stable.

Remark 2.2 *Hypotheses (ii), (iii) and (iv) of the theorem imply the existence of arbitrary positive constant α, satisfying*

$$0 < \alpha < b - \frac{c}{a} \tag{2.1}$$

Remark 2.3 *If (1.1) is a constant coefficient differential equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$, then conditions (i)-(iv) of Theorem 2.1 reduce to the Routh-Hurwitz conditions $a > 0, ab > c$ and $c > 0$. To show this, we set $f(\ddot{x}) = a\ddot{x}, \ g(\dot{x}) = b\dot{x}$ and $h(x) = cx$.*

For the proof of Theorem 2.1 our main tool is the continuous differentiable function $V = V(x, y, z)$ defined by

$$\begin{aligned} 2V = 2a\int_0^x h(\xi)d\xi + 2\int_0^y g(\tau)d\tau + 2yh(x) + \alpha b x^2 \\ +(\alpha + a^2)y^2 + z^2 + 2\alpha a x y + 2\alpha x z + 2ayz \end{aligned} \tag{2.2}$$

where α is defined in (2.1).The following lemmas are used for proving that the function V defined in (2.2) is a Lyapunov function for the system (1.2).

Lemma 2.4 *Suppose that all the conditions of the Theorem 2.1 hold. Then there are positive constants $D_i = D_i(a, b, c, \delta_0, b_1), \ (i = 0, 1)$ such that for all $(x, y, z) \in \mathbb{R}^3$*

$$D_0(x^2 + y^2 + z^2) \leq V \leq D_1(x^2 + y^2 + z^2) \tag{2.3}$$

Proof. We observe that the function V defined in (2.2) can be rewritten as follows:

$$2V = V_1 + V_2$$

where

$$V_1 = 2a\int_0^x h(\xi)d\xi + 2\int_0^y g(\tau)d\tau + 2yh(x)$$

and

$$V_2 = \alpha b x^2 + (\alpha + a^2)y^2 + z^2 + 2\alpha axy + 2\alpha xz + 2ayz.$$

In view of hypotheses (ii) and (iii), and the fact that $h(0) = 0$, we obtain

$$V_1 \geq (ab - c)b^{-1}\delta_0 x^2, \tag{2.4}$$

for all x. V_2 can be rewritten as

$$V_2 = XQX^T$$

where $X = \begin{pmatrix} x & y & z \end{pmatrix}$, $Q = \begin{pmatrix} \alpha b & \alpha a & \alpha \\ \alpha a & \alpha + a^2 & a \\ \alpha & a & 1 \end{pmatrix}$ and $\det Q = \alpha^2(b - \alpha) > \alpha^2$, since $b - \alpha > 0$ from (2.1). Hence for all $(x, y, z) \in \mathbb{R}^3$, we obtain

$$V_2 \geq \alpha^2(x^2 + y^2 + z^2). \tag{2.5}$$

On combining the estimates (2.4) and (2.5), we obtain the first inequality in (2.3) for all $(x, y, z) \in \mathbb{R}^3$. To establish the second inequality, let us observe that hypothesis (ii) of the theorem implies that $h(x) \leq cx$ for all x, since $h(0) = 0$. It follows from hypotheses (ii) and (iii) that

$$V_1 \leq c(a+1)x^2 + (b_1 + c)y^2 \tag{2.6}$$

$$V_2 \leq \alpha(a + b + 1)x^2 + (a+1)(\alpha + a)y^2 + (\alpha + a + 1)z^2. \tag{2.7}$$

On gathering the estimates (2.6) and (2.7), we obtain the upper inequality in (2.3) which proves the lemma. $\square$

Lemma 2.5 *Under the hypotheses of the Theorem* 2.1, *there exist positive constants* $D_i = D_i(a, b, c, \delta_0)$, $(i = 2, 3, 4)$, *such that if* $(x(t), y(t), z(t))$ *is any solution of the system (1.2) then*

$$\dot{V} \equiv \frac{d}{dt}V(x, y, z) \leq -(D_2x^2 + D_3y^2 + D_4z^2). \tag{2.8}$$

Proof. Along any solution $(x(t), y(t), z(t))$ of the system (1.2), it follows from the system (1.2) and the equation (2.2) that

$$\begin{aligned} \dot{V}_{(1.2)} &= -\alpha x h(x) - (ayg(y) - y^2 h'(x)) - \alpha(g(y) - by)x \\ &\quad - (\alpha x + ay + z)(f(z) - az) + \alpha Y Q_1 Y^T, \end{aligned}$$

where $Y = \begin{pmatrix} y & z \end{pmatrix}$, $Q_1 = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $\det Q_1 = -1$. On applying conditions (i)-(iv), we have

$$\dot{V}_{(1.2)} \leq -\frac{1}{2}\alpha\delta_0 x^2 - \frac{7}{8}(\alpha + ab - c\alpha)y^2 - \frac{1}{2}\alpha z^2 - W_j, \quad (j = i, 2, 3) \tag{2.9}$$

where

$$W_1 = \alpha\left(\frac{1}{4}\delta_0 x^2 + (g(y) - by)x + \frac{1}{16\alpha}(\alpha + ab - c)y^2\right), \tag{2.10}$$

$$W_2 = \alpha\left(\frac{1}{4}\delta_0 x^2 + (f(z) - az)x + \frac{1}{4}z^2\right), \tag{2.11}$$

$$W_3 = a\left(\frac{1}{16a}(\alpha + ab - c)y^2 + (f(z) - az)y + \frac{1}{4a}\alpha z^2\right). \tag{2.12}$$

On using the following inequalities

$$(g(y) - by)^2 < \frac{\delta_0(\alpha + ab - c)}{16\alpha}y^2, \tag{2.13}$$

$$(f(z) - az)^2 < \frac{\delta_0}{4}z^2, \tag{2.14}$$

$$(f(z) - az)^2 < \frac{(\alpha + ab - c)}{16a^2}z^2, \tag{2.15}$$

we have respectively,

$$W_1 \geq \frac{\alpha}{4}\left(2\sqrt{\delta_0}|x| - \sqrt{\frac{\alpha + ab - c}{\alpha}}|y|\right)^2 \geq 0 \text{ for all } x, y \tag{2.16}$$

$$W_2 \geq \frac{\alpha}{4}\left(\sqrt{\delta_0}|x| - \sqrt{\frac{\alpha}{a}}|z|\right)^2 \geq 0 \text{ for all } x, z, \tag{2.17}$$

$$W_3 \geq \frac{a}{4}\left(\sqrt{\frac{\alpha + ab - c}{a}}|y| - 2\sqrt{\frac{\alpha}{a}}|z|\right)^2 \geq 0 \text{ for all } y, z. \tag{2.18}$$

On gathering the estimates (2.16)-(2.18), the inequality (2.9) becomes

$$\dot{V}_{(1.2)} \leq -\frac{1}{2}\alpha\delta_0 x^2 - \frac{7}{8}(\alpha + ab - c)y^2 - \frac{1}{2}\alpha z^2$$

which proves the lemma. □

Proof of Theorem 2.1. To prove the Theorem 2.1, we shall use the usual limit point argument as is contained in [16] to show that when Lemma 2.4 and Lemma 2.5 hold, then $V(t) \equiv V(x(t), y(t), z(t)) \to 0$

as t$\to \infty$. In view of the fact that from Lemma 2.5,

$$\begin{aligned} V(x,y,z) &= 0 \quad \text{if and if only if } \; x^2+y^2+z^2=0, \\ V(x,y,z) &> 0 \quad \text{if and if only if } \; x^2+y^2+z^2 \neq 0, \\ V(x,y,z) &\to \infty \quad \text{if and if only if } \; x^2+y^2+z^2 \to \infty. \end{aligned}$$

Now assume that $\vartheta = (x, y, z)$ is any solution of the system (1.2), and consider the function $V(t) \equiv V(x(t), y(t), z(t))$ which corresponds to this solution. By Lemma 2.5 (inequality (2.8))

$$\dot{V} \leq -(D_2x^2 + D_3y^2 + D_4z^2) \leq V(0) \quad \forall t \geq 0.$$

Furthermore, $V(t)$ is non-negative and non-increasing, and thus tends to a non-negative limit, $V(\infty)$ as $t \to \infty$. The Theorem 2.1 will be proved if we can show that $V(\infty) = 0$. Therefore, suppose $V(\infty) > 0$ and consider the set

$$S = \{(x, y, z)|V(x, y, z) \leq V(0)\}.$$

It follows from the properties of the function V that S is bounded and hence the set $\vartheta \subset S$ is also bounded. Also, the non-empty set of all limits points of ϑ consists of whole trajectories of the system (1.2) lying on the surface $V(x, y, z) = V(\infty)$. However, if P is a limit point of ϑ, then there exists a half trajectory, say ϑ_P, issuing from P and lying on the surface $V(x, y, z) = V(\infty)$. Since for every point (x, y, z) on ϑ_P, $V(x, y, z) \geq V(\infty)$, we deduce at once that $\dot{V} = 0$ on ϑ_P. It follows from Lemma 2.5 (inequality (2.8)) that $x = y = z = 0$.

Thus the point $(0, 0, 0)$ lies on the surface $V(x, y, z) = V(\infty)$ which contradicts $V(x, y, z) \geq V(\infty)$. Hence $V(\infty) = 0$. This completes the proof of the Theorem 2.1. □

Theorem 2.6 *Suppose that* $g(0) = 0 = h(0)$ *and that*

(i) conditions (i)-(iv) of Theorem 2.1 hold;

(ii) $H(x) \to \infty$ *as* $|x| \to \infty$.

Then every solution $(x(t), y(t), z(t))$ *of (1.2) satisfies*

$$x(t) \to 0,\ \ y(t) \to 0,\ \ z(t) \to 0 \tag{2.19}$$

as $t \to \infty$.

For the proof of Theorem 2.6 we shall require the following preliminary lemma. Let

$$H(x) = \int_0^x h(\xi)d\xi \text{ and } G(y) = \int_0^y g(\tau)d\tau$$

for all x, y then we have the following results:

Lemma 2.7 *Subject to the conditions* $g(0) = 0 = h(0)$, $\ h(x)/x > 0$ $(x \neq 0)$ $\delta(g(y)/y) - h'(x) > 0$ $\ (y \neq 0)$, *where* $\delta > 0$*is a constant, the functions* h, G, H *satisfy the inequality*

$$4\delta H(x)G(y) \geq y^2h^2(x).$$

Proof. See [5]. □

Proof. The proof of Theorem 2.6 depends on some fundamental properties of a continuously differentiable function $V = V(x, y, z)$ defined by:

$$2V = 2aH(x) + 2G(y) + 2yh(x) + a^2y^2 + z^2 + 2ayz. \tag{2.20}$$

This function and its time derivatives satisfy some fundamental inequalities. First, we shall show, under the hypotheses of Theorem 2.6, that the function V defined in (2.20) is positive definite. It is clear from the equation (2.20) that $V(0, 0, 0) = 0$, and in view of hypotheses (i)-(iv) of the Theorem 2.1, the equation (2.20) becomes

$$V \geq \frac{1}{2}\Big[(ab - c)b^{-1}\delta_0x^2 + (ay + z)^2\Big].$$

The quadratic form on the right hand side of the above inequality is positive definite, hence there exists a positive constant δ_1 such that for all $(x, y, z) \in \mathbb{R}^3$

$$V \geq \delta_1(x^2 + y^2 + z^2). \tag{2.21}$$

Next, we shall show that any solution $(x(t), y(t), z(t))$ of the system (1.2) is bounded. To see this, we shall show that the inequality

$$V(x(t), y(t), z(t)) \leq K < \infty, \ t > 0, \tag{2.22}$$

where $K > 0$ is a constant, necessarily implies (under the present condition) the boundedness of $x(t), y(t)$ and $z(t)$ for all $t \geq 0$. Now, the system (2.20) can be rearranged as

$$2V = 2aH(x) + 2G(y) + 2yh(x) + (ay + z)^2,$$

and if the inequality (2.22) holds, then

$$|z(t) + ay(t)| < K_0 \equiv (2K)^{1/2}, \quad \Phi_0(x(t), y(t)) \leq K \ \ t \geq 0. \tag{2.23}$$

This follows since $\Phi_0(x(t), y(t)) \equiv aH(x) + G(y) + yh(x) \geq 0$ from the inequality (2.4). On following the procedure in [5], we can conclude that the solutions $x(t), y(t)$ and $z(t)$ are bounded.

In order to prove (2.19), let $(x(t), y(t), z(t))$ be any solution of the system (1.2) and consider the function defined in the equation (2.20), then

$$\begin{aligned} \dot{V}_{(1.2)} &\equiv \frac{d}{dt} V(x(t), y(t), z(t)) \\ &= -\frac{1}{2}(ayg(y) - y^2 h'(x)) - W - z(f(z) - az) \\ &\leq -\frac{1}{2}(ab - c)y^2 - W, \end{aligned}$$

where

$$W = a\left(\frac{ab - c}{2a} y^2 + (f(z) - az)y\right). \tag{2.24}$$

From the equation (2.24) and on using the inequality

$$[(f(z) - az)y]^2 < \left(\sqrt{\frac{ab - c}{2a}} |y||z| - \frac{1}{4} z^2\right)^2,$$

we have that

$$W \geq a\left(\sqrt{\frac{ab - c}{2a}} |y| - \frac{1}{2}|z|\right)^2 \geq 0 \tag{2.25}$$

for all y and z. Hence, from (2.25), we obtain

$$\dot{V}_{(1.2)} \leq -\frac{1}{2}(ab - c)y^2 \leq 0$$

for all y. The conclusion of remaining part of the proof follows the steps in the proof of the Theorem 2.1 and hence it is ommited. $\square$

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