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Asymptotic behaviour of solutions of third order nonlinear differential equations

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Abstract. In this paper, Lyapunov direct method was employed. We present criteria for all solutions $x(t)$ its first and second derivatives of the third order nonlinear non autonomous differential equations to converge to zero as $t \to \infty$. Sufficient conditions are also established for the boundedness and uniform ultimate boundedness of solutions of the equations considered. Our results revise, improve and generalize existing results in the literature.

1 Introduction

Nonlinear differential equations of higher order have been extensively studied with high degree of generality. In particular, boundedness, uniform boundedness, ultimate boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions have in the past and also recently been discussed by remarkable authors, see for instance Reissig *et al.* [18], Rouche *et al.* [19], Yoshizawa [26] and [27] where the general results were discussed. Authors that have worked on the qualitative behaviour of solutions of third order nonlinear differential equations include Ademola *et al.* [1, 2, 3, 4, 5, 6], Chukwu [7], Ezeilo [8, 9, 10, 11, 12], Hara [13], Mehri and Shadman [14], Omeike [15, 16], Qian [17], Swick [20, 21, 22], Tejumola [23] and Tunç [24, 25]. Complete and

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incomplete Lyapunov functions were constructed and used by these authors to establish their results. The nonlinear differential equations considered are the types where the restoring nonlinear terms do not depend explicitly on the independent real variable t, except in [1, 2, 4, 13] and [14] where the restoring nonlinear terms depend or multiplied by functions of t.

Till now, according to our observation from the relevant literature, the problem of boundedness (where the bounding constant depends on the solutions in question), uniform ultimate boundedness and asymptotic behaviour of solutions of the nonlinear non autonomous third order differential equation considered, have so far remained open. In this paper therefore, using Lyapunov direct method, a complete Lyapunov function was constructed and used to obtain criteria for boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of the third order nonlinear differential equation

$$x''' + \psi(t)f(x,x',x'')x'' + \phi(t)g(x,x') + \varphi(t)h(x,x',x'') = p(t,x,x',x'') \quad (1)$$

or its equivalent system

$$x' = y, y' = z, z' = p(t,x,y,z) - \psi(t)f(x,y,z)z - \phi(t)g(x,y) + \varphi(t)h(x,y,z) \quad (2)$$

in which $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$; $f, h \in C(\mathbb{R}^3, \mathbb{R})$; $g \in C(\mathbb{R}^2, \mathbb{R})$; $\phi, \varphi, \psi \in C(\mathbb{R}^+, \mathbb{R})$; $\mathbb{R} = (-\infty, \infty)$; $\mathbb{R}^+ = [0, \infty)$; the functions $\phi, \varphi, \psi, f, g, h$ and p depend only on the arguments displaced explicitly. The derivatives $\frac{\partial}{\partial x}f(x,y,z) = f_x(x,y,z)$, $\frac{\partial}{\partial y}f(x,y,z) = f_y(x,y,z)$, $\frac{\partial}{\partial z}f(x,y,z) = f_z(x,y,z)$, $\frac{\partial}{\partial x}g(x,y) = g_x(x,y)$,
$\frac{\partial}{\partial x}h(x,y,z) = h_x(x,y,z)$, $\frac{\partial}{\partial y}h(x,y,z) = h_y(x,y,z)$, $\frac{\partial}{\partial z}h(x,y,z) = h_z(x,y,z)$, $\frac{d}{dt}\psi(t) = \psi'(t)$, $\frac{d}{dt}\phi(t) = \phi'(t)$ and $\frac{d}{dt}\varphi(t) = \varphi'(t)$ exist and are continuous for all x, y, z and t. As usual, condition for uniqueness will be assumed and x', x'', x'''as elsewhere, stand for differentiation with respect to the independent variable t. Motivation for this studies comes from the works of Hara [13], Omeike [15, 16],Tunç [24, 25] and the recent work of Ademola and Arawomo [4] where conditions for stability and uniform ultimate boundedness of solutions of (1) were proved. Our results revise and improve the results in [4] and extend the results in [13, 14, 15, 16, 24] and [25].

2 Preliminaries

Consider the system of the form

$$X'(t) = F(t, X(t)) \quad (3)$$

where $F \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^n$ is the $n-$dimensional Euclidean space.

Definition 1 *A solution* $X(t; t_0, X_0)$ *of* (3) *is bounded, if there exists a* $\beta > 0$ *such that* $\|X(t; t_0, X_0)\| < \beta$ *for all* $t \geq t_0$ *where* β *may depend on each solution.*

Definition 2 *The solutions* $X(t; t_0, X_0)$ *of* (3) *are uniformly bounded, if for any* $\alpha > 0$ *and* $t_0 \in \mathbb{R}^+$*, there exists a* $\beta(\alpha) > 0$ *such that if* $\|X_0\| < \alpha$ $\|X(t; t_0, X_0)\| < \beta$ *for all* $t \geq t_0$.

Definition 3 *The solutions of* (3) *are uniformly ultimately bounded for bound* B *if there exists a* $B > 0$ *and if corresponding to any* $\alpha > 0$ *and* $t_0 \in \mathbb{R}^+$*, there exists a* $T(\alpha) > 0$ *such that if* $\|X_0\| < \alpha$ *implies that* $\|X(t; t_0, X_0)\| < B$ *for all* $t \geq t_0 + T(\alpha)$.

Definition 4 (i) *A function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$*, continuous, strictly increasing with* $\phi(0) = 0$*, is said to be a function of class* $\mathbb{K}$ *for such function, we shall write* $\phi \in \mathbb{K}$.
(ii) *If in addition to (i)* $\phi(r) \to +\infty$ *as* $r \to \infty$, ϕ *is said to be a function of class* $\mathbb{K}^*$ *and we write* $\phi \in \mathbb{K}^*$.

The following lemmas are very important in the proofs of our results.

Lemma 1 [27] Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+, \|X\| \geq \rho$ were $\rho > 0$ may be large which satisfies the following conditions:

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$;

(ii) $V'_{(3)}(t, X) \leq 0$, for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the solutions of (3) are uniformly bounded.

Lemma 2 [27] If in addition to assumption (i) of Lemma 1, $V'_{(3)}(t, X) \leq -c(\|X\|)$, $c \in \mathbb{K}$ for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then the solutions of (3) are uniformly ultimately bounded.

Let Q be an open set in $\mathbb{R}^n$ and $Q^* \subset Q$. Consider a system of differential equation

$$X'(t) = F(t, X(t)) + G(t, X(t)) \tag{4}$$

where F, G are defined and continuous on $\mathbb{R}^+ \times Q$.

Definition 5 *A scalar function* $W(X)$ *defined for* $X \in Q$ *is said to be positive definite with respect to a set* S*, if* $W(X) = 0$ *for* $X \in S$ *and if corresponding to each* $\epsilon > 0$ *and each compact set* Q^* *in* Q *there exists a positive number* $\delta(\epsilon, Q^*)$ *such that*

$$W(X) \geq \delta(\epsilon, Q^*)$$

for $X \in Q^* - N(\epsilon, S)$. $N(\epsilon, S)$ *is the* ϵ *neighborhood of* S.

Let Ω be a closed set in Q, we have the following lemma

Lemma 3 *Suppose that there exist a nonnegative Lyapunov function* $V(t, X)$ *defined on* $\mathbb{R}^+ \times Q$ *such that*

$$V'_{(4)}(t, X) \leq -W(X)$$

where $W(X)$ *is positive definite with respect to a closed set* Ω *in the space* $\mathbb{R}^n$. *Moreover suppose that* $F(t, X)$ *of system* (4) *is bounded for all* t *when* X *belongs to an arbitrary compact set in* Q *and that* $F(t, X)$ *satisfies conditions:*

(i) $F(t, X)$ *tends to a function* $H(X)$ *for* $X \in \Omega$ *as* $t \to \infty$ *and on any compact set in* Ω *this convergence is uniform;*

(ii) *Corresponding to each* $\epsilon > 0$ *and each* $Y \in \Omega$ *there exists a* $\delta(\epsilon, Y) > 0$ *and a* $T(\epsilon, Y) > 0$ *such that if* $\|X - Y\| < \delta(\epsilon, Y)$ *and* $t \geq T(\epsilon, Y)$*, we have*

$$\|F(t, X) - F(t, Y)\| < \epsilon.$$

Then every bounded solution of (4) *approaches the largest semi-invariant set of the system*

$$X' = H(X), \ X \in \Omega \tag{5}$$

as $t \to \infty$. *In particular, if all solutions of* (4) *are bounded, every solution of* (4) *approaches the largest semi-invariant set of* (5) *contained in* Ω *as* $t \to \infty$.

3 Statement of Results

We have the following results

Theorem 1 *Further to the basic assumptions on the functions* f, g, h, ϕ, φ *and* ψ *appearing in* (2)*, suppose that* $a, a_1, b, b_1, c, \delta_0, \epsilon, \phi_0, \phi_1, \varphi_0, \varphi_1, \psi_0$ *and* ψ_1*, are positive constants such that for all* $t \geq 0$*:*

(i) $a \leq f(x, y, z) \leq a_1$ *for all* x, y, z;

(ii) $b \leq g(x, y)/y \leq b_1$ *for all* $x, y \neq 0$;

(iii) $\psi_0 \leq \psi(t) \leq \psi_1$, $\phi_0 \leq \phi(t) \leq \phi_1$, $\varphi_0 \leq \varphi(t) \leq \varphi_1$;

(iv) $h(0, 0, 0) = 0$, $\delta_0 \leq h(x, y, z)/x$ *for all* $x \neq 0, y$ *and* z;

(v) $\sup_{t \geq 0}[|\psi'(t)| + |\phi'(t)| + |\varphi'(t)|] < \epsilon$;

(vi) $g_x(x, y) \leq 0$, $yf_x(x, y, z) \leq 0$, $h_x(x, 0, 0) \leq c$ *for all* x, y *and* $ab > c$;

(vii) $h_y(x, y, 0) \geq 0$, $h_z(x, 0, z) \geq 0$, $yf_z(x, y, z) \geq 0$ *for all* x, y, z;

(viii) $\int_0^\infty |p(t, x, y, z)| dt < \infty$.

Then the solution $(x(t), y(t), z(t))$ *of* (2) *is uniformly ultimately bounded.*

Theorem 2 *In addition to the assumptions of Theorem 1,* $g(0, 0) = 0$, *then every solution* $(x(t), y(t), z(t))$ (2) *is uniformly bounded and satisfies*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0 \tag{6}$$

Theorem 3 *Suppose that* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1$ *and* ψ_0 *are positive constants such that for all* $t \geq 0$:

(i) *assumptions* (iv)-(viii) *of Theorem 1 hold;*

(ii) $a \leq f(x, y, z)$ *for all* x, y, z;

(iii) $b \leq g(x, y)/y$ *for all* x *and* $y \neq 0$;

(iv) $\phi_0 \leq \phi(t)$, $\varphi_0 \leq \varphi(t) \leq \varphi_1$, $\psi_0 \leq \psi(t)$.

Then any solution $(x(t), y(t), z(t))$ *of* (2) *with initial conditions*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \tag{7}$$

satisfies

$$|x(t)| \leq D, \quad |y(t)| \leq D, \quad |z(t)| \leq D, \tag{8}$$

for all $t \geq 0$, *where the constant* $D > 0$ *depends on* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1, \psi_0$ *as well as on* t_0, x_0, y_0, z_0 *and on the function* p *appearing in* (2).

If the function $p(t,x,y,z) \equiv p(t) \neq 0$, (2) reduces to

$$x' = y, y' = z, z' = p(t) - \psi(t)f(x,y,z)z - \phi(t)g(x,y) + \varphi(t)h(x,y,z) \quad (9)$$

where $p \in C(\mathbb{R}^+, \mathbb{R})$, with the following results:

Corollary 1 *If hypotheses* (i)-(vii) *of Theorem 1 hold true, and in addition* $\int_0^\infty |p(t)|dt < \infty$, *then the solution* $(x(t), y(t), z(t))$ *of* (9) *is uniformly ultimately bounded.*

Corollary 2 *If in addition to assumptions of Corollary 1,* $g(0,0) = 0$, *then every solution* $(x(t), y(t), z(t))$ *of* (9) *is uniformly bounded and satisfies* (6).

Corollary 3 *Suppose that* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1$ *and* ψ_0 *are positive constants such that for all* $t \geq 0$:

(i) *assumptions* (iv)-(vii) *of Theorem 1 hold;*

(ii) *assumptions* (ii)-(iv) *of Theorem 3 hold;*

(iii) $\int_0^\infty |p(t)|dt < \infty$.

Then every solution $(x(t), y(t), z(t))$ *of* (9) *with initial conditions* (7) *satisfies* (8) *for all* $t \geq 0$ *where* $D > 0$ *is a constant depending on* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1$, ψ_0 *as well as on* t_0, x_0, y_0, z_0 *and on the function* p *appearing in* (9).

Remark 1 (i) *The results in [5],[10]-[13] and [21] are special cases of Theorem 1. Also, if* $\phi(t) = \varphi(t) = \psi(t) \equiv 1$, *system* (2) *specializes to that discussed by Ademola and Arawomo [3] (the generalization of the results of Omeike [15] and Tunç [24]). Moreover, in [4] Ademola and Arawomo studied stability and uniform ultimate boundedness of solutions of* (2). *Theorem 1 revises Theorem 6 in [4]. In particular, the main tool used in this investigation weaken the hypothesis on the function* p *compared with the result in [4].*

(ii) *If* $f(x,y,z) \equiv p(t)$, $g(x,y) \equiv g(y)$, $h(x,y,z) \equiv h(x)$ *and* $p(t,x,y,z) \equiv 0$ *system* (2) *specializes to that discussed by Swick [22]. His result in Theorem 1 is a special case of Theorem 2. Moreover, if* $f(x,y,z) \equiv a$ $a > 0$ *is a constant or* $p(t)$, $g(x,y) \equiv yg(x)$ *or* $g(y)$, $p(t,x,y,z) \equiv e(t)$ *and* $\varphi(t) = \psi(t) \equiv 1$ *system* (2) *reduces to that discussed by Swick [20]. Moreover, when* $p(t,x,y,z) \equiv 0$ *in* (2) *conditions under which all solutions* $x(t)$, *its first and second derivatives converge to zero as* $t \to \infty$

had been discussed by Ademola and Arawomo [4]. Furthermore, whenever $f(x,y,z) \equiv \psi(x,y)$ *or* $\psi(x,y,z)$, $h(x,y,z) \equiv 0$ *and* $p(t,x,y,z) \equiv p(t)$ *system* (2) *specializes to that studied by Omeike [16], Qian [17] and Tunç [24]. Hence, Theorem 2 revises, improves and generalizes the results in [4, 16, 17, 20] and [24].*

(iii) *The results of Ademola* et al. *[5], Mehri and Shadman [14] and Swick [22] Theorem 5 are all special cases of Theorem 3.*

The proofs of our results depend on the function $V = V(t, x(t), y(t), z(t))$ defined as

$$V = e^{-P_*(t)}U \tag{10a}$$

where

$$P_*(t) = \int_0^t |p(\mu, x, y, z)| d\mu \tag{10b}$$

and the function $U \equiv U(t, x(t), y(t), z(t))$

$$\begin{aligned} 2U &= 2(\alpha + a\psi(t))\varphi(t)\int_0^x h(\xi, 0, 0)d\xi + 4\varphi(t)yh(x, 0, 0) \\ &+ 4\phi(t)\int_0^y g(x, \tau)d\tau + 2(\alpha + a\psi(t))\psi(t)\int_0^y \tau f(x, \tau, 0)d\tau \\ &+ 2z^2 + \beta y^2 + b\beta\phi(t)x^2 + 2a\beta\psi(t)xy + 2\beta xz + 2(\alpha + a\psi(t))yz \end{aligned} \tag{10c}$$

where α and β are positive fixed constants satisfying

$$\frac{\varphi_1 c}{\phi_0 b} < \alpha < \psi_0 a \tag{10d}$$

and

$$0 < \beta < \min\left\{b\phi_0, (ab\psi_0\phi_0 - c\varphi_1)\eta_1^{-1}, \frac{1}{2}(a\psi_0 - \alpha)\eta_2^{-1}\right\} \tag{10e}$$

where

$$\eta_1 := 1 + a\psi_1 + \delta_0^{-1}\varphi_0^{-1}\phi_0^2\left(\frac{g(x,y)}{y} - b\right)^2 \text{ and } \eta_2 := 1 + \delta_0^{-1}\varphi_0^{-1}\psi_0^2[f(x,y,z) - a]^2.$$

Remark 2 *If* $t = 0$ *in* (10b), (10a) *coincides with* (10c) *and the main tool used in [4].*

Next, we shall show that (10) and its time derivative along a solution of (2) satisfy some fundamental inequalities as presented in the following lemma.

Lemma 4 *If all the hypotheses of Theorem 1 hold true, then for the function* V *defined in* (10) *there exist positive constants* $D_1 > 0$, $D_2 > 0$ *such that*

$$D_1(x^2(t) + y^2(t) + z^2(t)) \leq V(t, x, y, z) \leq D_2(x^2(t) + y^2(t) + z^2(t)) \tag{11a}$$

and

$$V(t, x(t), y(t), z(t)) \to +\infty \text{ as } x^2(t) + y^2(t) + z^2(t) \to \infty. \tag{11b}$$

Furthermore, there exists a finite constant $D_3 > 0$ *such that along a solution of* (2)

$$V' \equiv \frac{d}{dt}V(t, x(t), y(t), z(t)) \leq -D_3(x^2(t) + y^2(t) + z^2(t)). \tag{11c}$$

Proof. Since $h(0,0,0) = 0$, (10c) can be rearranged in the form

$$\begin{aligned}
2U &= \frac{2\varphi(t)}{b\phi(t)}\int_0^x [(\alpha + a\psi(t))b\phi(t) - 2\varphi(t)h_\xi(\xi, 0, 0)]h(\xi, 0, 0)d\xi \\
&+ 4\phi(t)\int_0^y \left(\frac{g(x,\tau)}{\tau} - b\right)\tau d\tau + 2b^{-1}\phi^{-1}(t)[\varphi(t)h(x, 0, 0) + b\phi(t)y]^2 \\
&+ 2\int_0^y [(\alpha + a\psi(t))\psi(t)f(x, \tau, 0) - (\alpha^2 + a^2\psi^2(t))]\tau d\tau \\
&+ (\alpha y + z)^2 + (\beta x + a\psi(t)y + z)^2 + \beta[b\phi(t) - \beta]x^2 + \beta y^2.
\end{aligned}$$

In view of the hypotheses of Theorem 1 this equation becomes

$$\begin{aligned}
U &\geq \frac{1}{2}\left\{[(\alpha + a\psi_0)b\phi_0 - 2\varphi_1 c]b^{-1}\phi_0^{-1}\varphi_0\delta_0 + \beta(b\phi_0 - \beta)\right\}x^2 \\
&+ \frac{1}{2}\Big[\alpha(a\psi_0 - \alpha) + \beta\Big]y^2 + b^{-1}\phi_0^{-1}[\delta_0\varphi_0 x + b\phi_0 y]^2 \\
&+ \frac{1}{2}(\alpha y + z)^2 + \frac{1}{2}(\beta x + a\psi_0 y + z)^2.
\end{aligned} \tag{12}$$

From (10d) and (10e) $\alpha b\phi_0 > c\varphi_1$, $ab\phi_0\psi_0 > c\varphi_1$, $a\psi_0 > \alpha$ and $b\phi_0 > \beta$ respectively, so that the quadratic in the right hand side of the inequality (12) is positive definite, hence there exists a positive constant $\lambda_0 = \lambda_0(a, b, c, \alpha, \beta, \delta_0, \phi_0, \varphi_0, \varphi_1, \psi_0)$ such that

$$U \geq \lambda_0(x^2 + y^2 + z^2) \tag{13a}$$

for all $t \geq 0, x, y$ and z. From hypothesis (viii) of Theorem 1 and (10b) there exists a constant $P_0 > 0$ such that

$$0 \leq P_*(t) \leq P_0 \tag{13b}$$

for all $t \geq 0$.Now, using (13) in (10a) we obtain

$$V \geq \delta_1(x^2 + y^2 + z^2) \tag{14a}$$

for all $t \geq 0, x, y$ and z, where $\delta_1 := \lambda_0 \exp[-P_0] > 0$. This establishes the lower inequality in (11a). From (14a), estimate (11b) follows immediately i.e

$$V(t, x, y, z) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{14b}$$

Furthermore, $h(0, 0, 0) = 0$ implies that $h(x, 0, 0) \leq cx$ for all $x \neq 0$, using this estimate, the hypotheses of Theorem 1 and the inequalities $2|xy| \leq x^2 + y^2$, $2|xz| \leq x^2 + z^2$ and $2|yz| \leq y^2 + z^2$, (10c) yields

$$U \leq \delta_2(x^2 + y^2 + z^2) \tag{15}$$

for all $t \geq 0, x, y$ and z, where $\delta_2 := \frac{1}{2}\max\{\lambda_1, \lambda_2, \lambda_3\} > 0$, $\lambda_1 = (2 + \alpha + a\psi_1)c\varphi_1 + (1 + a\psi_1 + b\phi_1)\beta$, $\lambda_2 = (\alpha + a\psi_1)(1 + a_1\psi_1) + (1 + a\psi_1)\beta + 2(b_1\phi_1 + c\varphi_1)$ and $\lambda_3 = 2 + \alpha + \beta + a\psi_1$. Using estimates (13b) and (15) in (10a), we obtain

$$V \leq \delta_2(x^2 + y^2 + z^2) \tag{16}$$

for all $t \geq 0, x, y$ and z. Thus by (16), the upper inequality in (11a) is established.

Moreover, the derivative of V along a solution $(x(t), y(t), z(t))$ of (2), with respect to t is given by

$$V'_{(2)} = -e^{-P_*(t)}\left[U|p(t, x, y, z)| - U'_{(2)}\right], \tag{17}$$

where $P_*(t)$ and U are the functions defined in (10b) and (10c) respectively and the derivative of the function U with respect to t, along a solution of (2) is after simplifying

$$\begin{aligned}
U'_{(2)} &= \sum_{i=1}^{3} U_i - U_4x^2 - U_5y^2 - U_6z^2 - U_7 \\
&- \beta\phi(t)\left[\frac{g(x, y)}{y} - b\right]xy - \beta\psi(t)[f(x, y, z) - a]xz \\
&+ [\beta x + [\alpha + a\psi(t)]y + 2z]p(t, x, y, z),
\end{aligned} \tag{18}$$

where:

$$
\begin{aligned}
U_1 &:= \left[2\int_0^y g(x,\tau)d\tau + \frac{1}{2}b\beta x^2\right]\phi'(t) + \left[[\alpha + a\psi(t)]\int_0^x h(\xi,0,0)d\xi\right.\\
&\left. + 2yh(x,0,0) + ayz\right]\varphi'(t) + \left[a\varphi(t)\int_0^x h(\xi,0,0)d\xi + a\beta xy\right.\\
&\left. + [\alpha + 2a\psi(t)]\int_0^y \tau f(x,\tau,0)d\tau\right]\psi'(t);\\
U_2 &:= a\beta\psi(t)y^2 + 2\beta yz;\\
U_3 &:= 2\phi(t)y\int_0^y g_x(x,\tau)d\tau + [\alpha + a\psi(t)]\psi(t)y\int_0^y \tau f(x,\tau,0)d\tau;\\
U_4 &:= \beta\varphi(t)\frac{h(x,y,z)}{x}, \qquad (x \neq 0);\\
U_5 &:= [\alpha + a\psi(t)]\phi(t)\frac{g(x,y)}{y} - 2\varphi(t)h_x(x,0,0), \qquad (y \neq 0);\\
U_6 &:= 2\psi(t)f(x,y,z) - [\alpha + a\psi(t)]
\end{aligned}
$$

and

$$
\begin{aligned}
U_7 := \varphi(t)&[[\alpha + a\psi(t)]y + 2z][h(x,y,z) - h(x,0,0)]\\
&+[\alpha + a\psi(t)]\psi(t)yz[f(x,y,z) - f(x,y,0)].
\end{aligned}
$$

In view of the hypotheses of Theorem 1, we have the following estimates for U_i $(i = 1, 2, \cdots, 6)$:

$$U_1 \leq \epsilon\lambda_4(x^2 + y^2 + z^2)$$

for all $t \geq 0, x, y$ and z, where $\lambda_4 := \max\{\lambda_{41}, \lambda_{42}, \lambda_{43}\} > 0$, $\lambda_{41} := \max\{\frac{1}{2}b\beta, b_1, 1\}$, $\lambda_{42} := \frac{1}{2}\max\{(\alpha + a\psi_1 + 2)c, a + 2c, a\}$ and $\lambda_{43} := \frac{1}{2}\max\{a(\beta + c\varphi_1), a\beta + (\alpha + 2a\psi_1)a_1, 1\}$;

$$U_2 \leq \beta[(1 + a\psi_1)y^2 + z^2]$$

for all $t \geq 0, x$ and y;

$$U_3 \leq 0$$

for all $t \geq 0, x$ and y;

$$U_4 \geq \beta\delta_0\varphi_0$$

for all $t \geq 0, x \neq 0, y$ and z;

$$U_5 \geq (\alpha + a\psi_0)b\phi_0 - 2c\varphi_1$$

for all $t \geq 0, x$ and y;

$$U_6 \geq a\psi_0 - \alpha$$

for all $t \geq 0, x, y$ and z. Finally by the mean value theorem and the hypotheses of Theorem 1, we have

$$U_7 = [\alpha + a\psi(t)]\psi(t)yz^2 f_z(x, y, \theta_1 z) + [\alpha + a\psi(t)]\varphi(t)y^2 h_y(x, \theta_2 y, 0)$$

$$+2\varphi(t)z^2 h_z(x, 0, \theta_3 z) \geq 0$$

for all $t \geq 0, x, y \neq 0 \neq z$ where $0 \leq \theta_i \leq 1$ $(i = 1, 2, 3)$, but $U_7 = 0$ for $y = 0 = z$. Using estimate U_i $(i = 1, 2, \cdots, 7)$ in (18), we obtain

$$\begin{aligned}
U'_{(2)} \leq & -\frac{1}{2}\beta\delta_0\varphi_0 x^2 - [(\alpha + a\psi_0)b\phi_0 - 2c\varphi_1 - \beta(1 + a\psi_1)]y^2 \\
& - (a\psi_0 - \alpha - \beta)z^2 - \frac{1}{4}\beta\delta_0\varphi_0\left[x + 2\phi_0\varphi_0^{-1}\delta_0^{-1}\left(\frac{g(x,y)}{y} - b\right)y\right]^2 \\
& + \beta\phi_0^2\delta_0^{-1}\varphi_0^{-1}\left(\frac{g(x,y)}{y} - b\right)^2 y^2 + \beta\psi_0^2\delta_0^{-1}\varphi_0^{-1}\Big(f(x,y,z) - a\Big)^2 z^2 \\
& - \frac{1}{4}\beta\delta_0\varphi_0\Big[x + 2\psi_0\varphi_0^{-1}\delta_0^{-1}(f(x,y,z) - a)z\Big]^2 + \epsilon\lambda_4(x^2 + y^2 + z^2) \\
& + \lambda_5(|x| + |y| + |z|)|p(t,x,y,z)|,
\end{aligned} \tag{19}$$

where $\lambda_5 = \max\{\beta, \alpha + a\psi_1, 2\}$. Since, $\beta, \delta_0, \varphi_0$ are positive constants, $[x + 2\phi_0\varphi_0^{-1}\delta_0^{-1}\left(\frac{g(x,y)}{y} - b\right)y]^2 \geq 0$ and $[x + 2\psi_0\varphi_0^{-1}\delta_0^{-1}(f(x,y,z) - a)z]^2 \geq 0$ for all $t \geq 0, x, y$ and z, estimate (19) reduces to

$$\begin{aligned}
U'_{(2)} \leq & -\frac{1}{2}\beta\delta_0\varphi_0 x^2 - (\alpha b\phi_0 - c\varphi_1)y^2 - \frac{1}{2}(a\psi_0 - \alpha)z^2 \\
& - \left\{ab\phi_0\psi_0 - c\varphi_1 - \beta\left[1 + a\psi_1 + \phi_0^2\delta_0^{-1}\varphi_0^{-1}\left(\frac{g(x,y)}{y} - b\right)^2\right]\right\}y^2 \\
& - \left\{\frac{1}{2}(a\psi_0 - \alpha) - \beta\left[1 + \psi_0^2\delta_0^{-1}\varphi_0^{-1}\Big(f(x,y,z) - a\Big)^2\right]\right\}z^2 \\
& + \epsilon\lambda_4(x^2 + y^2 + z^2) + \lambda_5(|x| + |y| + |z|)|p(t,x,y,z)|.
\end{aligned}$$

Applying estimates (10d), (10e) and choosing $\epsilon < \lambda_4^{-1}\lambda_6$ where $\lambda_6 := \min\{\frac{1}{2}\beta\delta_0\varphi_0, \alpha b\phi_0 - c\varphi_1, \frac{1}{2}(a\psi_0 - \alpha)\}$, we obtain

$$U'_{(2)} \leq -\lambda_7(x^2 + y^2 + z^2) + \lambda_5(|x| + |y| + |z|)|p(t,x,y,z)|, \tag{20}$$

for all $t \geq 0, x, y$ and z, where $\lambda_7 := \lambda_6 - \epsilon\lambda_4 > 0$. Now, using estimates (13a) and (17), we find

$$V'_{(2)} \leq -e^{-P_*(t)}\Big\{[\lambda_0(x^2+y^2+z^2) - \lambda_5(|x|+|y|+|z|)]|p(t,x,y,z)| \\ + \lambda_7(x^2+y^2+z^2)\Big\} \tag{21}$$

for all $t \geq 0, x, y$ and z. Using condition (viii) of Theorem 1, noting the fact that $(|x|+|y|+|z|)^2 \leq 3(x^2+y^2+z^2)$, and choosing $(x^2+y^2+z^2)^{1/2} \geq 3^{1/2}\lambda_0^{-1}\lambda_5$, estimate (21) becomes

$$V'_{(2)} \leq -\delta_3(x^2+y^2+z^2), \tag{22}$$

for all $t \geq 0, x, y$ and z where $\delta_3 = \lambda_7 \exp[-P_*(\infty)]$. (22) establishes estimate (11c) of the lemma. This completes the proof of the lemma. □

Proof of Theorem 1. Let $(x(t), y(t), z(t))$ be any solution of (2), in view of estimates (11) the hypotheses of Lemma 2 hold true. Thus, by Lemma 2, the solution $(x(t), y(t), z(t))$ of (2) is uniformly ultimately bounded. □

Proof of Theorem 2. The proof of this theorem depends on the function V defined in (10). First, by Lemma 4, and the hypotheses of Lemma 1 are satisfied so that the solution $(x(t), y(t), z(t))$ of (2) is uniformly bounded.
Furthermore, the continuity and boundedness of the functions f, g, h, ϕ, φ and ψ imply the boundedness of the function $F(t, X)$ for all t when X belongs to any compact set in $\mathbb{R}^3$.
Next, from estimate (22), let $W(X) := \delta_3(x^2+y^2+z^2)$, clearly $W(X) \geq 0$, for all $X \in \mathbb{R}^3$. Consider the set

$$\Omega := \{X = (x, y, z) \in \mathbb{R}^3 | W(X) = 0\}. \tag{23}$$

The continuity of the function $W(X)$ implies that the set Ω is closed and $W(X)$ is positive definite with respect to Ω and

$$V'_{(2)}(t, X) \leq -W(X)$$

for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^3$. System (2) can be rewritten in the form

$$X' = F(t, X) + G(t, X)$$

where $X = (x, y, z)^T$, $F(t, X) = (y, z, -\psi(t)f(x, y, z)z - \phi(t)g(x, y) - \varphi(t)h(x, y, z))^T$ and $G(t, X) = (0, 0, p(t, x, y, z))^T$. Moreover, from the hypotheses of the theorem we have $F(t, X)$ tends to a function $F(X)$, say, for all $X \in \Omega$ as $t \to \infty$, and by (23) $W(X) = 0$ on Ω implies that $x = y = z = 0$. By system (2) and the fact that $h(0,0,0) = 0 = g(0,0)$, the largest semi invariant set of $X' = F(X)$ $X \in \Omega$ as $t \to \infty$ is the origin. Thus the hypotheses of Lemma 3 are satisfied and (6) follows. This completes the proof of the theorem. $\square$

Proof of Theorem 3. Let $(x(t), y(t), z(t))$ be any solution of (2). Under the hypotheses of Theorem 3, estimates (14a) and (21) hold. To prove (8), since $\lambda_0(x^2 + y^2 + z^2)|p(t,x,y,z)| \geq 0$, $\lambda_7(x^2 + y^2 + z^2) \geq 0$ for all $t \geq 0, x, y, z$, the fact that $|x| \leq 1 + x^2$, $|y| \leq 1 + y^2$ and $|z| \leq 1 + z^2$, estimate (21) becomes

$$V'_{(2)} \leq \lambda_5 e^{-P_*(t)}(3 + x^2 + y^2 + z^2)]|p(t,x,y,z)|$$

for all $t \geq 0, x, y$ and z. Now, from estimates (14a) and (13b) this inequality yields

$$V'_{(2)} - \delta_1^{-1}\lambda_5|p(t,x,y,z)|V \leq 3\lambda_5|p(t,x,y,z)|.$$

Solving this first order differential inequality using integrating factor $\exp[-\delta_1^{-1}\lambda_5 P_*(t)]$ and estimate (13b), we have

$$V(t,x,y,z) \leq \lambda_8 \tag{24}$$

for all $t \geq 0, x, y$ and z, where $\lambda_8 := [V(t_0, x_0, y_0, z_0) + 3\lambda_5 P_0] \exp[\delta_1^{-1}\lambda_5 P_0] > 0$ is a constant. From estimates (14a) and (24), estimate (8) follows for all $t \geq 0$, with $D \equiv \delta_1^{-1}\lambda_8$. This completes the proof of the theorem. $\square$

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