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UNIFORM STABILITY AND BOUNDEDNESS OF SOLUTIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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ABSTRACT. In this paper, a complete Lyapunov functional was constructed and used to obtain criteria (when $p = 0$) for uniform asymptotic stability of the zero solution of the nonlinear delay differential equation (1.1). When $p \neq 0$, sufficient conditions are also established for uniform boundedness and uniform ultimate boundedness of solutions of this equation. Our results improve and extend some well known results in the literature.

1. INTRODUCTION

Many works have been done by several authors on the properties of solutions of ordinary differential equations of the second, third, fourth, fifth and higher order in which the unknown functions and its derivatives are all evaluated at the same instant, t, see for instance Reissig *et. al.*, [10], a survey book and Ademola *et. al.*, [1], Ezeilo [6], Omeike [8], Tejumola [12], Tunç [17] and the references cited therein to mention few. A more general type of differential equation often called a functional differential equation is one in which the unknown function occurs with various different arguments. This means an equation expressing some derivatives of x at time t in term of x (and its lower order derivative if any) at time t and at earlier instants.

With respect to our observation in the relevant literature, interesting results have been obtained on the properties of solutions of these classes of equations see for examples Burton [4, 5], Hale [7] and Yoshizawa [18] which contain general results on the subject matter. Notable authors that have contributed immensely to the qualitative properties of solutions (in particular stability and boundedness of solutions) of nonlinear third order delay differential equations include Afuwape and Omeike [2], Omeike [9], Sadek [11], Tunç [13, 14, 15, 16, 17] and Zhu [19]. These authors dealt with the problems by constructing Lyapunov functionals and obtain criteria for stability and boundedness of solutions.

The purpose of this paper is to obtain criteria for uniform stability, uniform boundedness and uniform ultimate boundedness of solutions for a more

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general third order nonlinear delay differential equation

$$\dddot{x}+f(x,\dot{x},\ddot{x})\ddot{x}+g(x(t-r(t)),\dot{x}(t-r(t)))+h(x(t-r(t))) = p(t,x,\dot{x},\ddot{x}) \tag{1.1}$$

or its equivalent system given by

$$\begin{aligned} &\dot{x} = y,\ \dot{y} = z,\ \dot{z} = p(t,x,y,z) - f(x,y,z)z - g(x,y) - h(x) \\ &+ \int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds + \int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds \\ &+ \int_{t-r(t)}^{t} h'(x(s))y(s)ds \end{aligned} \tag{1.2}$$

where $0 \leq r(t) \leq \gamma$, $\gamma > 0$ is a constant which will be determined later, the functions f, g, h and p are continuous in their respective arguments and the derivatives $f_x(x,y,z)$ $f_z(x,y,z)$, $g_x(x,y)$, $g_y(x,y)$, $h'(x)$ exist and are continuous for all x, y, z with $h(0) = g(0,0) = g(x,0) = 0$ for all x. The dots, as elsewhere, stands for differentiation with respect to the independent variable t. Also, as usual, condition for uniqueness of solutions of (1.2) will be assumed. The results obtained in this investigation improve, generalize and complement existing results on the third order nonlinear delay differential equation in the literature.

2. Preliminaries

Consider the general autonomous delay differential system

$$\dot{X} = F(X_t),\ X_t(\theta) = X(t+\theta),\ -r \leq \theta \leq 0.\ t \geq 0 \tag{2.1}$$

where $F : C_H \to \mathbb{R}^n$ is a continuous mapping , $F(0) = 0$, we suppose that F takes closed bounded set of $\mathbb{R}^n$. Here $(C, \|\cdot\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$, C_H is an open ball of radius H in C; $C_H := \{\phi \in (C[-r,0], \mathbb{R}^n) : \|\phi\| < H\}$. It has been shown by Burton [5], that if $\phi \in C_H$, and $t \geq 0$, then there is at least one continuous solution $X(t, t_0, \phi)$ satisfying (2.1) for $t > t_0$ on the interval $[t_0, t_0 + \alpha)$, such that $X_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remain in B, then $\alpha = \infty$.

Definition 1. [5]. A continuous function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with $W(0) = 0$, $W(s) > 0$ if $s \neq 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i is an integer).

Definition 2. [5]. The zero solution of (2.1) is asymptotically stable if it is stable and if for each $t_0 \geq 0$ there is an $\eta > 0$ such that $\|\phi\| \leq \eta$ implies that

$$X(t, t_0, \phi) \to 0 \text{ as } t \to \infty.$$

Definition 3. [4]. An element $\psi \in C_H$ is in the $\omega-$limit set of ϕ, say $\Omega(\phi)$, if $X(t, 0, \phi)$ is defined on $\mathbb{R}^+$ and there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to +\infty$, with $\|X_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$, where $X_{t_n}(\phi) = X(t_n + \theta, 0, \phi)$ for $-r \leq \theta < 0$.

Definition 4. [19]. A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution $X(t, 0, \phi)$ of (2.1) is defined on $\mathbb{R}^+$ and $X_t(\phi) \in Q$ for $t \in \mathbb{R}^+$.

Lemma 2.1. [5, 7, 19]. *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2.1) with $x_0(\phi) = \phi$ is defined on $\mathbb{R}^+$ and $\|X_t(\phi)\| \leq H_1 < H$ for $t \in \mathbb{R}^+$, then $\Omega(\phi)$ is nonempty, compact, invariant set, and*

$$dist(X_t(\phi), \Omega(\phi)) \to 0, \text{ as } t \to \infty.$$

Lemma 2.2. [4, 19].*) Let $V(\phi) : C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(0) = 0$ and such that*

(i) $W(|\phi(0)|) \leq V(\phi) \leq W_2(\|\phi\|)$ *where $W_1(r)$ and $W_2(r)$ are wedges;*
(ii) $\dot{V}_{(2.1)}(\phi) \leq 0$ *for $\phi \in C_H$.*

Then the zero solution of (2.1) is uniformly stable.
If we define $Z = \{\phi \in C_H : \dot{V}_{(2.1)}(\phi) = 0\}$, then $X_t = 0$ of (2.1) is asymptotically stable, provided that the largest invariant set in Z is $M = \{\boldsymbol{0}\}$.

Next, consider the system

$$\dot{X} = F(t, X_t),\ X_t = X(t + \theta),\ -r \leq \theta \leq 0,\ t \geq 0, \tag{2.2}$$

Where $F : [0, \infty) \times C \to \mathbb{R}$ is continuous and takes bounded sets into bounded sets. The following lemma is a well-known result obtained by Burton [4].

Lemma 2.3. [4]. *Let $V : \mathbb{R}^+ \times C \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) $W_0(|X_t|) \leq V(t, X_t) \leq W_1(|X_t|) + W_2\left(\int\limits_{t-r(t)}^{t} W_3(X_t(s))ds\right)$ *and*

(ii) $\dot{V}_{(2.2)}(t, X_t) \leq -W_4(|X_t|) + N$, *for some $N > 0$ where W_i $(i = 0, 1, 2, 3, 4)$ are wedges.*
Then X_t of (2.2) is uniformly bounded and uniformly ultimately bounded for bound B.

3. Main Results

In the case when $p(t,x,y,z) \equiv 0$. (1.2) becomes

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = -f(x,y,z)z - g(x,y) - h(x) + \int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds + \int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds + \int_{t-r(t)}^{t} h'(x(s))y(s)ds \tag{3.1}$$

where f, g, h and r are the functions defined in Section 1. Let (x_t, y_t, z_t) be any solution of (3.1), the main tool in the proofs of our results is the continuously differentiable functional $V \equiv V(x_t, y_t, z_t)$, defined by

$$\begin{aligned} 2V &= 2(\alpha + a)\int_0^x h(\xi)d\xi + 4\int_0^y g(x,\tau)d\tau + 4yh(x) + b\beta x^2 \\ &+ 2(\alpha+a)\int_0^y \tau f(x,\tau,0)d\tau + \beta y^2 + 2z^2 + 2a\beta xy + 2\beta xz \\ &+ 2(\alpha+a)yz + \int_{-r(t)}^{0}\int_{t+s}^{t}\Big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big]d\theta ds \end{aligned} \tag{3.2a}$$

where α, β are positive fixed constants satisfying

$$b^{-1}c < \alpha < a; \tag{3.2b}$$

$$0 < \beta < \min\Big\{(ab-c)a^{-1}, (ab-c)A_1^{-1}, \frac{1}{2}(a-\alpha)A_2^{-1}\Big\}, \tag{3.2c}$$

where $A_1 := 1 + a + \delta^{-1}\Big(\frac{g(x,y)}{y} - b\Big)^2$ and $A_2 := 1 + \delta^{-1}\Big(f(x,y,z) - a\Big)^2$; λ_1 and λ_2 are also positive constants which will be determined later.We have the following result.

Theorem 3.1. *Further to the fundamental assumptions on f, g, h and r appearing in* (3.1), *suppose that $a, a_1, b, c, \delta, \gamma$ and M are positive constants and that:*

(i) $a \le f(x,y,z) \le a_1$, $yf_x(x,y,0) \le 0$, $yf_z(x,y,z) \ge 0$ *for all* x, y, z;

(ii) $g(0,0) = 0 = g(x,0)$, $b \le \frac{g(x,y)}{y}$ *for all* $x, y \ne 0$, $|g_x(x,y)| \le L$ *for some* $L \ge 0$ *and* $|g_y(x,y)| \le M$ *for all* x, y;

(iii) $h(0) = 0$, $\frac{h(x)}{x} \ge \delta$ $(x \ne 0)$, $h'(x) \le c$ *for all* x *and* $ab > c$;

(iv) $0 \le r(t) \le \gamma$, $r'(t) \le \beta_0$ $0 < \beta_0 < 1$.

Then the trivial solution of (3.1) *is uniformly asymptotically stable, provided that*

$$\gamma < \min\left\{\delta A_3^{-1}, 2(1-\beta_0)(\alpha b - c)A_4^{-1}, (a-\alpha)(1-\beta_0)A_5^{-1}\right\} \tag{3.3}$$

where $A_3 := c + L + M$, $A_4 := (\alpha + a)(1-\beta_0) + (c+L)(2+\alpha+\beta+a)$ *and* $A_5 := 2(1-\beta_0)(c+L+M) + M(2+\alpha+\beta+a)$.

Remark. *Whenever* $f(x,\dot{x},\ddot{x}) \equiv a$ *(*$a > 0$ *is a constant) and* $g(x(t - r(t)), \dot{x}(t-r(t))) \equiv g(\dot{x}(t-r(t)))$*, Eq.* (1.1) *reduces to that discussed by Sadek* [11] *and Tunç* [14]*. Also, if* $f(x,\dot{x},\ddot{x}) \equiv f(\dot{x})$ (1.1) *specializes to that discussed by Afuwape and Omeike* [2]*. Besides, an incomplete Lyapunov functionals were constructed and used to obtain stability results in* [2, 11] *and* [14] *compared with a complete Lyapunov functional used in this investigation. Hence our result includes and generalizes theirs.*

Proof of Theorem 3.1. Let (x_t, y_t, z_t) be any solution of (3.1), the time derivative of the functional defined in (3.2) along a solution of (3.1) is simplified to give

$$\begin{aligned}
\dot{V}_{(3.1)} &= (\alpha + a)y\int_0^y \tau f_x(x,\tau,0)d\tau + a\beta y^2 + [\beta x + (\alpha+a)y + 2z] \\
&\times \int_{t-r(t)}^t \Big(g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s) + h'(x(s))y(s)\Big)ds \\
&+ r(t)\Big(\lambda_1 y^2 + \lambda_2 z^2\Big) + 2\beta yz + 2y\int_0^y g_x(x,\tau)d\tau - \beta\Big(\frac{g(x,y)}{y} - b\Big)xy \\
&- \beta\Big(f(x,y,z) - a\Big)xz - \beta\frac{h(x)}{x}x^2 - \Big[(\alpha+a)\frac{g(x,y)}{y} - 2h'(x)\Big]y^2 \\
&- \Big[2f(x,y,z) - (\alpha+a)\Big]z^2 - (\alpha+a)yz\Big[f(x,y,z) - f(x,y,0)\Big] \\
&- (1 - r'(t))\int_{t-r(t)}^t \Big(\lambda_1 y^2 + \lambda_2 z^2\Big)ds.
\end{aligned}$$

On applying the hypotheses of the theorem and the fact that $2pq \leq p^2 + q^2$, we obtain

$$\begin{aligned}
\dot{V}_{(3.1)} &\leq \frac{1}{2}(c + L + M)(\beta x^2 + (\alpha + a)y^2 + 2z^2)r(t) + \lambda_1 y^2 r(t) \\
&+ \lambda_2 z^2 r(t) - \frac{1}{2}\beta\delta x^2 - (\alpha b - c)y^2 - \frac{1}{2}(a - \alpha)z^2 \\
&- \left\{ab - c - \beta\left[1 + a + \delta^{-1}\left(\frac{g(x,y)}{y} - b\right)^2\right]\right\}y^2 \\
&- \left\{\frac{1}{2}(a - \alpha) - \beta\left[1 + \delta^{-1}\left(f(x,y,z) - a\right)^2\right]\right\}z^2 \\
&- \frac{\beta}{4\delta}\left[\delta x + 2\left(\frac{g(x,y)}{y} - b\right)y\right]^2 - \frac{\beta}{4\delta}\left[\delta x + 2\left(f(x,y,z) - a\right)z\right]^2 \\
&- \left[\lambda_1(1 - r'(t)) - \frac{1}{2}(c + L)(2 + \alpha + \beta + a)\right]\int_{t-r(t)}^{t} y^2(s)ds \\
&- \left[\lambda_2(1 - r'(t)) - \frac{M}{2}(2 + \alpha + \beta + a)\right]\int_{t-r(t)}^{t} z^2(s)ds
\end{aligned}$$

Now, in view of estimates (3.2b), (3.2c), the fact that $0 \leq r(t) \leq \gamma$ and $r'(t) \leq \beta_0$ $0 < \beta_0 < 1$, this inequality becomes

$$\begin{aligned}
\dot{V}_{(3.1)} &\leq -\frac{\beta}{2}\left[\delta - (c + L + M)\gamma\right]x^2 \\
&- \left\{ab - c - \left[\frac{1}{2}(\alpha + a)(c + L + M) + \lambda_1\right]\gamma\right\}y^2 \\
&- \left\{\frac{1}{2}(a - \alpha) - (c + L + M + \lambda_2)\gamma\right\}z^2 \\
&- \left[\lambda_1(1 - \beta_0) - \frac{1}{2}(c + L)(2 + \alpha + \beta + a)\right]\int_{t-r(t)}^{t} y^2(s)ds \\
&- \left[\lambda_2(1 - \beta_0) - \frac{M}{2}(2 + \alpha + \beta + a)\right]\int_{t-r(t)}^{t} z^2(s)ds.
\end{aligned}$$

Choosing $\lambda_1 := 2^{-1}(1 - \beta_0)^{-1}(c + L)(2 + \alpha + \beta + a) > 0$ and $\lambda_2 := 2^{-1}(1 - \beta_0)^{-1}M(2 + \alpha + \beta + a) > 0$, it follows that

$$\begin{aligned}
\dot{V}_{(3.1)} &\leq -\frac{\beta}{2}\left[\delta - (c + L + M)\gamma\right]x^2 \\
&- \left\{ab - c - \left[\frac{1}{2}(\alpha + a)(c + L + M) + \frac{(c + L)(2 + \alpha + \beta + a)}{2(1 - \beta_0)}\right]\gamma\right\}y^2 \\
&\left\{\frac{1}{2}(a - \alpha) - \left[c + L + M + \frac{M(2 + \alpha + \beta + a)}{2(1 - \beta_0)}\right]\gamma\right\}z^2.
\end{aligned}$$

Applying estimate (3.3), there exists a constant $\delta_0 > 0$ such that

$$\dot{V}_{(3.1)} \leq -\delta_0(x^2 + y^2 + z^2) \tag{3.4}$$

for all x, y, z where $\delta_0 = \min\Big\{\frac{\beta}{2}\Big[\delta - (c + L + M)\gamma\Big], ab - c - \Big[\frac{1}{2}(\alpha + a)(c + L + M) + \frac{(c + L)(2 + \alpha + \beta + a)}{2(1 - \beta_0)}\Big]\gamma, \frac{1}{2}(a - \alpha) - \Big[c + L + M + \frac{M(2 + \alpha + \beta + a)}{2(1 - \beta_0)}\Big]\gamma\Big\}$.
By (3.4) $\dot{V}_{(3.1)} = 0$ and the system (3.1), we can easily obtain $x = y = z = 0$. Hence, condition (ii) of Lemma 2.2 is satisfied.
Next, since $h(0) = 0$, the functional V defined in (3.2) can be recast in the form

$$\begin{aligned} V &= \frac{1}{b}\int_0^x \Big[(\alpha + a)b - 2h'(\xi)\Big]h(\xi)d\xi + 2\int_0^y \Big(\frac{g(x,\tau)}{\tau} - b\Big)\tau d\tau \\ &+ \frac{1}{b}\Big(h(x) + by\Big)^2 + \int_0^y \Big[(\alpha + a)f(x,\tau,0) - (\alpha^2 + a^2)\Big]\tau d\tau \\ &+ \frac{1}{2}(\alpha y + z)^2 + \frac{1}{2}(\beta x + ay + z)^2 + \frac{1}{2}\beta(b - \beta)x^2 + \frac{1}{2}\beta y^2 \\ &+ \frac{1}{2}\int_{-r(t)}^0 \int_{t+s}^t \Big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big]d\theta ds. \end{aligned}$$

Now, by hypotheses of the theorem and the fact that the double integrals

$$\int_{-r(t)}^0 \int_{t+s}^t \Big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big]d\theta ds$$

are non negative, it follows that

$$\begin{aligned} V &\geq \frac{1}{2b}\Big[\delta(\alpha b - c + ab - c) + b\beta(b - \beta)\Big]x^2 + \frac{1}{2}\Big[\alpha(a - \alpha) + \beta\Big]y^2 \\ &+ \frac{1}{2}(\alpha y + z)^2 + \frac{1}{2}(\beta x + ay + z)^2. \end{aligned} \tag{3.5}$$

By estimates (3.2b) and (3.2c), the quadratic in the right hand side of (3.5) is positive definite, hence there exists a positive constant $\delta_1 = \delta_1(a, b, c, \alpha, \beta, \delta)$ such that

$$V \geq \delta_1(x^2 + y^2 + z^2) \tag{3.6}$$

for all x, y and z. Estimate (3.6) establishes the lower inequality in condition (i) of Lemma 2.2.
Furthermore, since $g(x, 0) = 0$ for all x implies that $g(x, y) \leq My$ for all $x, y \neq 0$ and $h(0) = 0$ implies $h(x) \leq cx$ $(x \neq 0)$. These inequalities and the

fact that $2pq \leq p^2 + q^2$, the functional V defined in (3.2) yields

$$V \leq \eta_1(x^2 + y^2 + z^2) + \int_{-r(t)}^{0} \int_{t+s}^{t} \eta_2 \Big[x^2(\theta) + y^2(\theta) + z^2(\theta)\Big] d\theta ds \tag{3.7}$$

where $\eta_1 := \frac{1}{2}\max\{(1+a+b)\beta+(2+\alpha+a)c, (1+a_1)(\alpha+a)+2(c+M)+(1+a)\beta, 2+\alpha+\beta\}$ and $\eta_2 := \frac{1}{2}\max\{1, \lambda_1, \lambda_2\}$. From estimate (3.7) the upper inequality in codition (i) of Lemma 2.2 follows. By estimates (3.4), (3.6) and (3.7) the hypotheses of Lemma 2.2 are satisfied. Hence, by Lemma 2.2 the trivial solution of (3.1) is uniformly stable and uniformly asymptotically stable. □

Next, if $p(t, x, y, z) \neq 0$, we have the following results.

Theorem 3.2. *If the assumptions of Theorem 3.1 hold true and*

$$|p(t, x, y, z)| \leq p_1(t) + p_2(t)(|x| + |y| + |z|) \tag{3.8a}$$

where $p_1(t)$ and $p_2(t)$ are continuous functions satisfying

$$p_1(t) \leq P_0 \tag{3.8b}$$

$0 < P_0 < \infty$ *and there exists* $\epsilon > 0$ *such that*

$$0 \leq p_2(t) \leq \epsilon. \tag{3.8c}$$

Then the solutions of the system (1.2) are uniformly bounded and uniformly ultimately bounded, provided that the inequality in (3.3) holds true.

If $p(t, x, y, z) \equiv p(t) \neq 0$, Eq. (1.2) becomes

$$\begin{aligned} &\dot{x} = y,\ \ \dot{y} = z,\ \ \dot{z} = p(t) - f(x, y, z)z - g(x, y) - h(x) \\ &+ \int_{t-r(t)}^{t} g_x(x(s), y(s))y(s)ds + \int_{t-r(t)}^{t} g_y(x(s), y(s))z(s)ds \\ &+ \int_{t-r(t)}^{t} h'(x(s))y(s)ds \end{aligned} \tag{3.9}$$

Corollary 1. If the assumptions of Theorem 3.1 are satisfied and $|p(t)| \leq P_1$, $0 < P_1 < \infty$ then the solutions of (3.9) are uniformly bounded and uniformly ultimately bounded, provided that the inequality in (3.3) holds.

Remark. *If $f(x, \dot{x}, \ddot{x}) \equiv a$, $a > 0$ is a constant, $g(x(t-r(t)), \dot{x}(t-r(t))) \equiv g(\dot{x}(t - r(t)))$ and $p(t, x, \dot{x}, \ddot{x}) \equiv p(t)$, (1.1) specializes to that discussed by Sadek [11], our assumptions and conclusion coincide with his. Also, if $p_2(t) \equiv 0$, (1.1) reduces to the one discussed by Tunç [13]. Finally, whenever*

$f(x,\dot{x},\ddot{x}) = h(\dot{x})$ and $p_1(t) \equiv m$, $m > 0$ is a constant, then the hypotheses and conclusion of Theorem 3.2 coincide with that of Afuwape and Omeike in [2] *except $f(x,y,z) \leq a_1$ a necessary and sufficient condition for uniform boundedness and uniform ultimate boundedness of solutions of* (1.1) *which could not be found in* [2]. *Hence, our result revises and generalizes the situation given in* [2, 11] *and* [14].

Proof of Theorem 3.2. Let (x_t, y_t, z_t) be any solution of (1.2), the derivative of the functional V defined in (3.2) along a solution of (1.2) is

$$\dot{V}_{(1.2)} = \dot{V}_{(3.1)} + (\beta x + (\alpha + a)y + 2z)p(t,x,y,z).$$

Now, from estimate (3.4) and the fact that $q \leq |q|$, this equation becomes

$$\dot{V}_{(1.2)} \leq -\delta_0(x^2 + y^2 + z^2) + \eta_3(|x| + |y| + |z|)|p(t,x,y,z)|$$

where $\eta_3 = \max\{\beta, \alpha + a, 2\}$. In view of (3.8) and choosing $\epsilon < 3^{-1}\eta_3^{-1}\delta_0$ there exists $\eta_4 = \delta_0 - 3\eta_3\epsilon > 0$ such that

$$\dot{V}_{(1.2)} \leq -\frac{\eta_4}{2}(x^2 + y^2 + z^2) + \frac{3}{2}\eta_3^2 P_0^2 \eta_4^{-1}, \tag{3.10}$$

since

$$\frac{\eta_4}{2}\left[\left(|x| - \eta_3 P_0 \eta_4^{-1}\right)^2 + \left(|y| - \eta_3 P_0 \eta_4^{-1}\right)^2 + \left(|z| - \eta_3 P_0 \eta_4^{-1}\right)^2\right] \geq 0$$

for all x, y and z. From estimate (3.10), hypothesis (ii) of Lemma 2.3 is satisfied. Also from estimates (3.6) and (3.7), condition (i) of Lemma 2.3 follows. This completes the proof of the theorem. □

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