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ON HYERS-ULAM STABILITY OF NONLINEAR SECOND ORDER ORDINARY AND FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we consider the Hyers-Ulam stability of some nonlinear second order ordinary and functional differential equations. As mathematical technique, we use some nonlinear extension of the Grönwall integral inequality.

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1. Introduction

Hyers-Ulam stability which started with Ulam [19] at a wide range talk given before the Mathematics-Club of the University of Wincosin in 1940 on many important unsolved problems had attracted the interest of many researchers. Since then several authors have investigated the stability of linear differential equations, some of these authors include: Hyers [11], Alsina and Ger [1], Miura

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et al [8,9], Takahasi et al [17,18].

However, only few authors have considered the Hyers-Ulam stabilty of nonlinear differential equations. These include: Rus[15,16], Ravi et al[14], Jinghao[6], Qarawani[11,12], Qusuay et al[10] and Motaza and Omid[7]. Motivation for this study came from the work of Qusuay et al[10] where Hyers-Ulam stability of nonlinear differential equations of second order was considered using Grönwall lemma.

In this paper, we consider the Hyers-Ulam stability of the nonlinear differential equation

$$u''(t) + f(t, u(t), u'(t)) = 0 \tag{1.1}$$

where $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$, $\mathbf{I} = [t_0, b)(b < \infty)$ and $f \in C(\mathbf{I} \times {\mathbf{R}_+}^2, \mathbf{R}_+)$

2. Preliminaries

In this section, we give some definitions, theorems, and lemma which will be useful in subsequent discussion.

Lemma 2.1. (see Bihari [2,3]) *Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \le t \le b, (\le \infty)$ and $K > 0$, $M \ge 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \ge 0$, then the inequality*

$$u(t) \le K + M \int_a^t f(s)\omega(u(s))ds, \quad a \le t < b \tag{2.1}$$

implies the inequality

$$u(t) \le \Omega^{-1}\left(\Omega(k) + M \int_a^t f(s)ds\right), \quad a \le t \le b' \le b \tag{2.2}$$

where

$$\Omega(u) = \int_{u_0}^u \frac{dt}{\omega(t)}, \qquad 0 < u_0 < u \tag{2.3}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω for t in the subinterval $[a, b']$ of $[a, b]$ such that

$$\Omega(k) + M \int_a^t f(s)ds \in Dom(\Omega^{-1}).$$

Definition 2.2. A function ω is said to belong to a class S if it satisfies the following conditions:

i $\omega(u) > 0$ is nondecreasing and $\omega \in C(\mathbf{R}_+, \mathbf{R}_+)$ for $u > 0$

ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$

iii there exists a function ϕ, continuous on $[0, \infty)$ with $\omega(u) \leq \omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha > 0$, $u > 0$.

Theorem 2.3. (see [4]) *Suppose that*

i $u(t), r(t), g(t) \in C(\mathbf{I}, \mathbf{R}_+)$

ii $\omega(u)$ *is a nonnegative, monotonic nondecreasing, continuous, submultiplicative function for* $u > 0$

if

$$u(t) \leq K + \int_{t_0}^{t} r(s)u(s)ds + \int_{t_0}^{t} g(s)\omega(s)ds, \quad t \in \mathbf{I} \tag{2.4}$$

for $K > 0$*, a constant, then*

$$u(t) \exp\left(-\int_{t_0}^{t} r(s)ds\right) \leq \Omega^{-1}\left(\Omega(K) + \int_{t_0}^{t} g(s)\omega\left(\exp\int_{t_0}^{s} r(\delta)d\delta\right) ds\right) \quad t \in \mathbf{I} \tag{2.5}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{ds}{\omega(s)}, \quad 0 < u_0 \leq u \tag{2.6}$$

and Ω^{-1} *is the inverse of* Ω *and* t *is in the subinterval* $(0, b) \in \mathbf{I}$ *so that*

$$\Omega(K) + \int_{t_0}^{t} g(s)\omega\left(\exp\int_{t_0}^{s} r(\delta)d\delta\right) ds \in Dom(\Omega^{-1}). \tag{2.7}$$

Definition 2.4. Equation (1.1) is said to be Hyers-Ulam stable if there exists constants $K > 0$, $\epsilon > 0$ and the solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of

$$|u''(t) + f(t, u(t), u'(t))| \leq \epsilon \tag{2.8}$$

satisfies

$$|u(t) - u_0(t)| \leq K\epsilon.$$

where $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is a solution of equation (1.1) and K is the Hyers-Ulam constant.

3. Main Result

Our main results are presented in the following theorems.

In the first theorem we consider the Hyes-Ulam stability of a second order differential equation which is nonlinear in only $u(t)$

$$u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) = 0 \tag{3.1}$$

where $a, b, g, \in C(\mathbf{I}, \mathbf{R}_+)$. and $f \in C(\mathbf{R}_+, \mathbf{R}_+)$ with $f \in S$.

Theorem 3.1. *Suppose that $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies the differential inequality:*

$$|u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t))| \leq \epsilon \qquad (3.2).$$

Then, if

i $\displaystyle\int_{t_0}^{t} \frac{1}{b(s)} ds \leq p$ *for* $p > 0$, *and all* $t \in \mathbf{R}_+$

ii $\displaystyle\int_{t_0}^{t} \left(\frac{a(s)}{b(s)} - 1\right) |u'(s)| ds \leq m.$ *for* $m \geq 0$

iii $|u'(t)| \leq \lambda$ *for* $\lambda \geq 0$.

iv $\displaystyle \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{b^2(s)} ds = M < \infty$ *for* $M > 0$

$\displaystyle \lim_{t\to\infty} \int_{t_0}^{t} \frac{g(s)}{b(s)} ds = T < \infty$ *for* $T > 0$

are satisfied, equation (3.1) is Hyers-Ulam stable with the Hyers-Ulam constant K defined as

$$K = (|E(t_0)| + \lambda + p + m)\, M\Omega^{-1} (\Omega(1) + T\omega(\exp M)) \tag{3.3}$$

Proof. Inequality (3.2), implies that

$$-\epsilon \leq u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) \leq \epsilon \tag{3.4}$$

it follows that

$$u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) \leq \epsilon \tag{3.5}$$

Define

$$E(t) = \frac{u'(t)}{b(t)} + u(t), \quad u(t) \neq 0, \quad b(0) \neq 0 \tag{3.6}$$

clearly

$$E(t) = E(t_0) + \int_{t_0}^{t} \frac{d}{ds}\left(\frac{u'(s)}{b(s)} + u(s)\right) ds \tag{3.7}$$

Where

$$E(t_0) = \frac{u'(t_0)}{b(t_0)} + u(t_0)$$

$$E(t) = E(t_0) + \int_{t_0}^{t}\left(u'(s) + \frac{u''(s)}{b(s)} - \frac{db(s)}{ds}\frac{u'(s)}{b^2(s)}\right) ds \tag{3.8}$$

Since $b(t)$ is an increasing function, $\frac{db(t)}{dt} \geq 0$.

It follows from(3.8) that

$$E(t) \leq E(t_0) + \int_{t_0}^{t}\left(u'(s) + \frac{u''(s)}{b(s)}\right) ds \tag{3.9}$$

Substituting for $u''(t)$ in (3.9) using (3.5), we have

$$E(t) \leq E(t_0) + \int_{t_0}^{t}\left(u'(s) - \frac{1}{b(s)}\left(a(s)u'(s) + \frac{1}{b(s)}u(s) + g(s)f(u(s)) - \epsilon\right)\right) ds$$

$$E(t) \leq E(t_0) - \int_{t_0}^{t} \Bigg(-u'(s) + \frac{a(s)}{b(s)}u'(s) + \frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)}\Bigg) ds$$

$$E(t) \leq E(t_0) - \int_{t_0}^{t}\Bigg(\left(\frac{a(s)}{b(s)} - 1\right)u'(s) + \frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)}\Bigg) ds \tag{3.10}$$

Replacing $E(t)$ in (3.10) with (3.6), we have

$$u(t) \leq E(t_0) - \frac{u'(t)}{b(t)} - \int_{t_0}^{t}\left(\left(\frac{a(s)}{b(s)} - 1\right)u'(s)\right.$$

$$+\frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)}\Bigg) ds.$$

Taking the absolute value of both sides, we get

$$|u(t)| \leq |E(t_0)| + \frac{|u'(t)|}{b(t)}$$
$$+ \int_{t_0}^{t} \left(\left(\frac{a(s)}{b(s)} - 1 \right) |u'(s)| + \frac{1}{b^2(s)}|u(s)| + \frac{g(s)}{b(s)} f(|u(s|)) + \frac{\epsilon}{b(s)} \right) ds$$

Using conditions (i-iii) with $\frac{1}{b(t)} \leq 1$, we get

$$|u(t)| \leq |E(t_0)| + \lambda + \epsilon p + m + \int_{t_0}^{t} \frac{1}{b^2(s)}|u(s)|ds + \int_{t_0}^{t} \frac{g(s)}{b(s)} f(|u(s)|)ds \quad (3.11)$$

Setting

$$\frac{1}{b^2(t)} = \alpha(t), \quad \frac{g(t)}{b(t)} = \gamma(t)$$

and using them in (3.11), we have

$$|u(t)| \leq |E(t_0)| + \lambda + \epsilon p + m + \int_{t_0}^{t} \alpha(s)|u(s)|ds + \int_{t_0}^{t} \gamma(s) f(|u(s)|)ds \quad (3.12)$$

with

$$f(|u(t)|) = \omega(|u(t)|) \text{ and } \epsilon \geq 1$$

(3.12) becomes

$$|u(t)| \leq C + \int_{t_0}^{t} \alpha(s)|u(s)|ds + \int_{t_0}^{t} \gamma(s)\omega(|u(s)|)ds \quad (3.13)$$

where

$$C = \epsilon \left(|E(t_0)| + \lambda + p + m \right)$$

thus, we have

$$\frac{|u(t)|}{C} \leq 1 + \int_{t_0}^{t} \alpha(s)\frac{|u(s)|}{C}ds + \int_{t_0}^{t} \gamma(s)\omega(\frac{|u(s)|}{C})ds \quad (3.14)$$

the application of theorem 2.3 then yields

$$|u(t)| \leq C\Omega^{-1} \left(\Omega(1) \right.$$

$$+ \int_{t_0}^{t} \gamma(s)\omega\left(\exp\int_{t_0}^{s}\alpha(\delta)d\delta\right)ds\Big)\left(\exp\int_{t_0}^{t}\alpha(s)ds\right)$$

using the condition (iv), we obtain

$$|u(t)| \leq CM\Omega^{-1}\left(\Omega(1) + T\omega(\exp M)\right) \tag{3.15}$$

Substituting for C in (3.15) gives

$$|u(t) - u_0(t)| \leq |u(t)| \leq \epsilon\left(|E(t_0)| + \lambda + p + m\right)M\Omega^{-1}\left(\Omega(1) + T\omega(\exp M)\right)$$

Therefore, equation (3.1) is Hyers-Ulam stable with the Hyers-Ulam constant

$$K = \left(|E(t_0)| + \lambda + p + m\right)M\Omega^{-1}\left(\Omega(1) + T\omega(\exp M)\right).$$

Example. To investigate Hyers-Ulam stability of the second order nonlinear differential equation

$$u''(t) + t^2u'(t) + t^{-2}u(t) + t^{-6}u^2(t) = 0$$

all the conditions (i-iv) of Theorem 3.1 are satisfied so the equations is Hyers-Ulam stable.

Next we consider the Hyers-Ulam stability of a second order differential equation which is nonlinear in both $u(t)$. and $u'(t)$

$$u''(t) + \phi(t)g(u(t))h(u'(t)) = 0 \tag{3.16}$$

together with initial condition

$$u(t_0) = u'(t_0) = 0$$

where $h, \phi \in C(\mathbf{I}, \mathbf{R}_+)$, $g \in C(\mathbf{R}_+, \mathbf{R}_+)$ and $h(u') > 0$.

Theorem 3.2. *Let $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfy the differential inequality*

$$|u''(t) + \phi(t)g(u(t))h(u'(t))| \leq \epsilon \tag{3.17}$$

for all $t \in \mathbf{I}$ and for some $\epsilon > 0$, then there exists a solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of equation(3.17) such that $|u(t) - u_0(t)| \leq K\epsilon$, for

$$K = \frac{1}{\delta\lambda}P\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda}M\right) \tag{3.18}$$

with $G \in S$, provided the following conditions are satisfied:

$i \;\; G(u(t)) = \int_{u(t_0)}^{u(t)} g(s)ds < \infty$

$ii \;\; R(u(t)) = \int_{u(t_0)}^{u(t)} \frac{s}{h(s)}ds < \infty$

$iii \;\; \lim_{t\to\infty} \int_{t_0}^{t} |\phi'(s)|ds = M < \infty$

$iv \;\; \alpha(t) \geq \delta, \quad where \;\; \delta > 0$

$v \;\; \int_{t_0}^{t} \frac{|u'(s)|}{|h(u'(s))|}ds \leq P$

Proof. From (3.17), it follows that

$$-\epsilon \leq u''(t) + \phi(t)g(u(t))h(u'(t)) \leq \epsilon \text{ for all } \;\; t \geq t_0 \tag{3.19}$$

$$u''(t) + \phi(t)g(u(t))h(u'(t)) \leq \epsilon \text{ for all } \;\; t \geq t_0$$
$$\frac{u''(t)u'(t)}{h(u'(t))} + \phi(t)g(u(t))u'(t) \leq \frac{u'(t)\epsilon}{h(u'(t))} \tag{3.20}$$

Using (i) and (ii), we get

$$\frac{dR(u'(t))}{dt} + \phi(t)\frac{dG(u(t))}{dt} \leq \frac{u'(t)\epsilon}{h(u'(t))} \text{ for all } \;\; t \geq t_0 \tag{3.21}$$

Integrating by part from t_0 to t

$$R(u'(t)) - \phi(t)G(u(t)) + \int_{t_0}^{t} \phi'(s)G(u(s))ds \leq \epsilon \int_{t_0}^{t} \frac{u'(s)}{h(u'(s))}ds$$

Taking the absolute value of both sides, we have

$$|R(u'(t)) - \phi(t)G(u(t))| \leq \epsilon \int_{t_0}^{t} \frac{|u'(s)|}{h(|u'(s)|)}ds + \int_{t_0}^{t} \phi'(s)G(|u(s)|)ds \tag{3.22}$$

Setting

$$|R(u'(t)) - \phi(t)G(u(t))| \geq \alpha(t)|u(t)||u'(t)| \tag{3.23}$$

for $\alpha(t) \in C(\mathbf{I}, \mathbf{R}_+)$ So by (3.8), equation (3.7) becomes

$$\alpha(t)|u(t)||u'(t)| \leq \epsilon \int_{t_0}^{t} \frac{|u'(s)|}{h(|u'(s)|)}ds + \int_{t_0}^{t} |\phi'(s)|G(|u(s)|)ds \tag{3.24}$$

Using condition (iv), we get

$$\delta|u(t)||u'(t)| \leq \epsilon \int_{t_0}^{t} \frac{|u'(t)|}{h(|u'(s)|)} ds + \int_{t_0}^{t} |\phi'(s)|G(|u(s)|)ds \tag{3.25}$$

It follows that

$$|u(t)| \leq \frac{\epsilon}{\delta|u'(t)|} \int_{t_0}^{t} \frac{|u'(t)|}{|h(u'(s)|)} ds + \frac{1}{\delta|u'(t)|} \int_{t_0}^{t} |\phi'(s)|G(|u(s)|)ds \tag{3.26}$$

Using condition(v) and setting $|u'(t)| \leq \lambda$, for $\lambda \geq 0$, we have

$$|(u(t))| \leq \frac{\epsilon}{\delta\lambda} P + \frac{1}{\delta\lambda} \int_{t_0}^{t} |\phi'(s)|G(|u(s)|)ds$$

From the fact that $G \in S$ for

$$G(|u(t)|) = \omega(|u(t)|)$$

it follows that

$$|(u(t))| \leq \frac{\epsilon}{\delta\lambda} P + \frac{1}{\delta\lambda} \int_{t_0}^{t} |\phi'(s)|\omega(|u(s)|)ds \tag{3.27}$$

Setting

$$L = \frac{\epsilon}{\delta\lambda} P \tag{3.28}$$

We have

$$\frac{|(u(t))|}{L} \leq 1 + \frac{1}{\delta\lambda} \int_{t_0}^{t} |\phi'(s)|\omega(\frac{|u(s)|}{L})ds \tag{3.29}$$

using lemma 2.2 and equation(2.3) we obtain

$$\frac{|u(t)|}{L} \leq \Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} \int_{t_0}^{t} |\phi'(s)|ds \right)$$

By condition (iii), we have

$$\frac{|u(t)|}{L} \leq \Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} M \right)$$

$$|u(t)| \leq L\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} M \right) \tag{3.30}$$

Substituting for L from (3.28), equation (3.30) becomes

$$|u(t)| \leq \frac{\epsilon}{\delta\lambda} P\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} M \right)$$

Since

$$|u(t) - u(t_0)| \leq |u(t)|$$

we have

$$|u(t) - u_0(t)| \leq \frac{\epsilon}{\delta\lambda} P\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda}M\right)$$

Hence, equation(3.16) is Hyers-Ulam stable with Hyers-Ulam constant K given as

$$K = \frac{1}{\delta\lambda} P\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda}M\right).$$

Example. To investigate Hyers-Ulam stability of the second order nonlinear differential equation of the form

$$u''(t) + t^{-3}u^2(t)u'^4(t) = 0,$$

where $-3\int_{t_0}^{t} \frac{ds}{s^4} < \infty$ and $h(u'(t)) = u'^4(t))$, $u(t_0) = u'(t_0) = 0$.

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