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ON THE BEHAVIOUR OF SOLUTIONS FOR A CLASS OF THIRD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

A. T. ADEMOLA[1], A. M. MAHMOUD[2], P. O. ARAWOMO[3]

Abstract. In this paper, a new class of third order nonlinear neutral delay differential equations is discussed. By reducing the third order nonlinear neutral delay differential equations to systems of first order, the second method of Lyapunov is engaged by constructing a complete Lyapunov functional and used to establish criteria that guarantee uniform asymptotic stability of the trivial solution and uniform ultimate boundedness of solutions. The obtained results are not only new but also include many outstanding results in the literature. Finally, the correctness and effectiveness of the obtained results are justified with examples.

1. Introduction

The problem of asymptotic stability, boundedness, integrability, existence and uniqueness of periodic solutions for differential equations with or without delay has received considerable attention of authors over the years, see for example Arino *et al.* [14], Burton [15, 16], Driver [18], Lakshmikantham *et al.* [25], Yoshizawa [47, 48] which contains background knowledge. Other outstanding results include the papers of Ademola and Arawomo [1]–[6], Ademola *et al.* [7]–[9], Ademola and Ogundiran [10], Afuwape and Omeike [13], Chukwu [17], Graef *et al.* [19, 20, 21], Graef and Tunç [22], Gui [23], Omeike [27, 28], Remili and Oudjedi [30]–[31], Remili *et al.* [32], Sadek [33], Tejumola and Tchegnani [34], Tunç [36]–[45], Yao and Wang [46], Zhu [49] and the references cited therein.

It is well known that delay differential equations (DDEs) are mostly utilized to model many of the physical processes emanating from engineering and various branches of science such as atomic energy, information theory, control theory, chemistry, physics, biology and ecological system. As we all known that stability, boundedness, existence and uniqueness of solutions are the most important problems in the study of qualitative behaviour of solutions of FDEs. In fact, time delays occur most often in many physical and ecology systems, because the future state of the systems depend on both the present and past states. It is widely known that time delays often lead to instability of a stable system. Therefore, the study of FDEs has become the subject of many investigations.

In 1992 Zhu [49], developed sufficient conditions to guarantee the stability, boundedness and ultimate boundedness of solutions for the following third order nonlinear delay differential equations

$$x''' + ax'' + bx' + f(x(t-\tau)) = p(t),$$

and

$$x''' + ax'' + \phi(x'(t-\tau)) + f(x) = p(t).$$

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In 2003, Sadek [33] considered the stability and boundedness of solutions for the third order delay differential equation

$$x''' + ax'' + g(x'(t-\tau(t))) + f(x(t-\tau(t))) = p(t).$$

In 2006, Liu and Huang [26] used the coincidence degree theory to discussed the existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equations

$$(x(t) + Bx(t-\delta))' = g_1(t, x(t)) + g_2(t, x(t-\tau)) + p(t),$$

where $p : \mathbb{R} \to \mathbb{R}$ and $g_1, g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, τ, B and δ are constants, p is $T-$periodic, g_1 and g_2 are $T-$periodic in the first argument, $|B| \neq 1$ and $T > 0$. In 2007 the author in [45] studied stability and boundedness of solutions of nonlinear third order delay differential equations

$$x''' + a_1x'' + f_2(x(t-\tau(t))) + a_3x(t) = p(t, x, x', x(t-\tau(t)), x'(t-\tau(t)), x'')$$

In 2010, Omeike [27], Tunç [40] and [42] respectively considered new results on the stability of solution of some non autonomous third order delay differential equations

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-\tau)) = 0,$$

the stability and boundedness of solutions of nonlinear third order delay differential equations

$$x''' + g(x, x')x'' + f(x(t-\tau), x'(t-\tau)) + h(x(t-\tau)) = p(t, x, x', x(t-\tau), x'(t-\tau)), x'')$$

and some stability and boundedness conditions for non autonomous differential equations with deviating arguments

$$x''' + a(t)x'' + b(t)g_1(x'(t-\tau)) + g_2(x') + h(x(t-\tau)) = p(t, x, x', x(t-\tau), x'(t-\tau)), x'').$$

In 2013, Ademola and Arawomo [6] developed criteria which guarantee uniform asymptotic stability and boundedness of solutions for the third order nonlinear differential equation

$$x''' + f(x, x', x'')x'' + g(x(t-\tau(t)), x'(t-\tau(t)) + h(x(t-\tau(t))) = p(t, x, x', x''),$$

where f, g, h and p are continuous functions depending only on the arguments displayed.

In another interesting paper, Graef *et al.* [19] discussed sufficient conditions that guarantee the square integrability of all solutions and the asymptotic stability of the zero solution of a non-autonomous third order neutral delay differential equation

$$[x(t) + \beta x(t-\tau)]''' + a(t)\Big(Q(x(t))x'(t)\Big)' + b(t)\Big(R(x(t))x'(t)\Big) + c(t)f(x(t-r)) = h(t),$$

where β and r are constants with $0 \leq \beta \leq 1$ and $r \geq 0$, the functions $a, b, c : [0, \infty) \to [0, \infty)$, $Q, R : \mathbb{R} \to [0, \infty)$, $h : [0, \infty) \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$, are continuous, and $xf(x) > 0$ for $x \neq 0$.

Recently, in 2019 Oudjedi *et al.* [29] gave sufficient conditions for every solution to be converges to zero, bounded and square integrable for a class of third order neutral delay differential equations

$$[x(t) + \beta x(t-\tau)]''' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = p(t).$$

where, β and τ are constants with $0 \leq \beta \leq 1$ and $\tau \geq 0$, $h(t)$ and $f(x)$ continuous functions depending only on the arguments shown and $f'(x)$ exist and is continuous for all x.

The aim of this paper is to obtain conditions for uniform asymptotic stability of the zero solution, uniform ultimate boundedness and the existence of a unique periodic solution for the nonlinear non autonomous DDE

$$[x(t) + \phi x(t-\tau)]''' + a(t)x''(t) + b(t)g(x'(t-\tau)) + c(t)h(x(t-\tau)) = p(t). \tag{1.1}$$

Setting $x'(t) = y(t)$ and $x''(t) = z(t)$, equation (1.1) is equivalent to the system of first order differential equations

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= z(t) \\ Z'(t) &= -a(t)z(t) - b(t)g(y(t)) - c(t)h(x(t)) + p(t) \\ &\quad + \int_{t-\tau}^{t} [b(t)g'(y(s))z(s) + c(t)h'(x(s))y(s)]ds \end{aligned} \tag{1.2}$$

where $Z(t) = z(t) + \phi z(t-\tau)$, $\tau > 0$ is a constant delay, ϕ is a constant satisfying $0 \leq \phi \leq 1$, the functions $a(t), b(t), c(t), g(y), h(x)$ are continuous in their respective arguments on $\mathbb{R}^+, \mathbb{R}^+, \mathbb{R}^+, \mathbb{R}, \mathbb{R}$ respectively with $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. Besides, it is supposed that the derivatives $g'(y)$ and $h'(x)$ exist and are continuous for all x, y and $h(0) = 0$. Motivation for this paper comes from the works in [6, 19, 26, 27, 29, 33, 40, 42, 45, 49]. Results of this paper are not only new but extend some well known results on third order delay differential equations in the literature. An equally interesting problem is the second order delay differential equations of type (1.1). This has already been considered and the results arising in this direction will be published through another outlet. The main results are stated and proved in Sections 2 and 3 while in the last section, examples are given to illustrate and authenticate the obtained results.

2. Uniform Asymptotic Stability of the Trivial Solution

Let $X(t) = x(t) + \phi x(t-\tau)$, $Y(t) = y(t) + \phi y(t-\tau)$ and $Z(t) = z(t) + \phi z(t-\tau)$. When $p(t) \equiv 0$, the delay differential equations (1.1) and (1.2) respectively become

$$[x(t) + \phi x(t-\tau)]''' + a(t)x''(t) + b(t)g(x'(t-\tau)) + c(t)h(x(t-\tau)) = 0 \tag{2.1}$$

and

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= z(t) \\ Z'(t) &= -a(t)z(t) - b(t)g(y(t)) - c(t)h(x(t)) \\ &\quad + \int_{t-\tau}^{t} [b(t)g'(y(s))z(s) + c(t)h'(x(s))y(s)]ds, \end{aligned} \tag{2.2}$$

where the functions g and h are defined in Section 1. Let (x_t, y_t, z_t) be any solution of system (2.2), a continuously differentiable functional $V = V(t, x_t, y_t, z_t)$ employed in this work is

$$V = \sum_{i=0}^{2} V_i + \int_{t-\tau}^{t} \Big[\mu_1 y^2(s) + \mu_2 z^2(s)\Big]ds + \int_{-\tau}^{0}\int_{t+s}^{t} \Big[\mu_3 y^2(\tau) + \mu_4 z^2(\tau)\Big]d\tau ds \tag{2.3}$$

where

$$V_0 := \frac{1}{2}Z^2 + (\alpha + a)yZ + \frac{1}{2}a(t)(\alpha + a)y^2;$$

$$V_1 := 2c(t)(\alpha + a)\int_0^x h(s)ds + 2c(t)h(x)Y + \frac{1}{2}bb(t)Y^2;$$

$$V_2 := 2b(t)\int_0^y g(s)ds + \frac{1}{2}\beta bx^2 + a\beta xy + \beta xZ + \frac{1}{2}Z^2 + \frac{1}{2}a(\alpha + a)y^2 + (\alpha + a)yZ;$$

and a, b, α, β are positive constants with α, β satisfying the inequalities

$$\min\left\{\frac{c}{2b} - a, \frac{2c}{b} - a\right\} < \alpha < a, \qquad \beta < b. \tag{2.4}$$

Theorem 2.1. *Suppose that* $a_0, a_1, a, b_1, b, c_0, c, B, h_0, h_1, k, \alpha, \beta, \phi, \tau$ *are positive constants and that*

(i) $a_0 \leq a(t) \leq a_1$ *for all* $t \geq 0$;

(ii) $c_0 \leq c(t) \leq b(t) \leq b_1$, $b'(t) \leq c'(t) \leq 0$ *for all* $t \geq 0$;

(iii) $h(0) = 0$, $h_0 \leq \dfrac{h(x)}{x} \leq h_1$ *for all* $x \neq 0$, *and* $h'(x) \leq |h'(x)| \leq c$ *for all* x;

(iv) $b \leq \dfrac{g(y)}{y} \leq B$ *for all* $y \neq 0$, $|g'(y)| \leq k$ *for all* y, *with* $c < ab$, *and* $a_0 = 2a$,

$$\tau < \min\left\{\frac{c_0 h_0}{b_1(k+c)}, \frac{2(\alpha+a)b - c - A_1}{A_2}, \frac{a - \alpha - A_3}{A_4}\right\}, \tag{2.5}$$

where

$$A_1 := \frac{1}{2}b_1[(b\phi + 2h_1)(1+\phi) + b];$$

$$A_2 := \frac{1}{2}b_1 c[\beta + 2(1+\alpha+a+\phi) + (\alpha+a)(k+c)b_1];$$

$$A_3 := \frac{1}{2}[bb_1(1+\phi)+\beta] + 3\phi^2(a-\alpha) + \frac{3c_0^2\phi^2}{2[2(\alpha+a)b-c]}\left(\frac{g(y)}{y} - b\right)^2 + \frac{1}{2}\phi(\beta + \phi b b_1);$$

$$A_4 := b_1\left[(1 + \frac{1}{2}\beta + \phi)k + (\alpha+a)c + (1+\phi)(k+c)\right].$$

Then the trivial solution of system (2.2) *is uniformly asymptotically stable.*

Corollary 2.2. *If the nonlinear delay functions* $g(y'(t-\tau))$ *and* $h(x(t-\tau))$ *are replaced by functions* $g(y'(t))$ *and* $h(x(t))$ *respectively in* (2.2), *then the trivial solution of the new system of ordinary differential equations is uniformly asymptotically stable.*

Remark 2.1. We note the following:

(i) If $\phi = 0$, $a(t) = a$, $b(t) = b$, $c(t) = c$, where a, b, c are positive constants, $g(x'(t-\tau)) = x'(t)$ and $h(x(t-\tau)) = x(t)$ then equation (2.1) reduces to third order linear ordinary differential equation

$$x'''(t) + ax''(t) + bx'(t) + cx(t) = 0. \tag{2.6}$$

Furthermore, hypotheses (i) to (v) of Theorem 2.1 reduce to Routh-Hurwitz conditions $a > 0, ab > c$ and $c > 0$ for asymptotic stability of the trivial solution of the linear third order differential equation (2.6);

(ii) If $\phi = 0$, equation (1.1) specializes to some of the delay equations discussed in [17, 27, 33, 40, 46, 49];

(iii) Whenever $\phi = 0$ and $\tau = 0$ then equation (1.1) reduces to third order nonlinear ordinary differential equations that had been discussed by authors in the literature, some of these authors include but not limited to [1, 2, 3, 4, 8, 11, 12];

(iv) Observation from relevant literature shows that there are no results on second and third order delay differential equations of the type (1.1), except in [19, 26, 29], where existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equations was discussed;

(v) The results of this paper do not only new but extend some outstanding results in the literature such as in [6, 26, 27, 29, 33, 40, 42, 45, 49] and the references cited therein; and

(vi) Note that the solution (x_t, y_t, z_t) can also be written in the form $(x_t, y_t, \theta z_t)$ where $\theta = 1 + \phi$.

Next, we shall state and proofs two major lemmas that are prominent to the proof of Theorem 2.1 and subsequent results

Lemma 2.3. *Under the hypotheses of Theorem 2.1 there exist positive constants D_0, D_1, D_2 and D_3 such that*

$$D_0(x^2(t)+y^2(t)+z^2(t)) \leq V(t,x_t,y_t,z_t) \leq D_1(x^2(t)+y^2(t)+z^2(t))+Q(t,y,z), \quad (2.7)$$

for all $t \geq 0, x, y, z$ and $z(t-\tau)$ where

$$Q(t,y,z) := D_2\int_{t-\tau}^{t}(y^2(s)+z^2(s))ds + D_3\int_{-\tau}^{0}\int_{t+s}^{t}(y^2(\tau)+z^2(\tau))d\tau ds$$

Proof. Let (x_t, y_t, z_t) be any solution of system (2.2), the functional V_i, $(i=0,1,2)$ can be rewritten in the following forms

$$V_0 = \frac{1}{4}(Z+(\alpha+a)y)^2+\frac{1}{4}(\alpha+a)a(t)\left(y+\frac{Z}{a(t)}\right)^2+\frac{1}{4}(\alpha+a)[a(t)-(\alpha+a)]y^2+\frac{1}{4a(t)}[a(t)-(\alpha+a)]Z^2;$$

since $h(0)=0$, we find that

$$V_1 = \frac{1}{2}bb(t)\left[Y+\frac{2c(t)}{bb(t)}h(x)\right]^2+\frac{2c(t)}{b}\int_0^x\left[(\alpha+a)b-\frac{2c(t)}{b(t)}h'(s)\right]h(s)ds;$$

and

$$V_2 = 2b(t)\int_0^y g(s)ds+\frac{1}{2}\beta(b-\beta)x^2+\frac{1}{2}(\beta x+ay+Z)^2+\frac{1}{2}\alpha a\left(y+\frac{Z}{a}\right)^2+\frac{1}{2a}(a-\alpha)Z^2.$$

Since $a_0 \leq a(t) \leq a_1$ for all $t \geq 0$, with the fact that $(Z+(\alpha+a)y)^2 \geq 0$ and $\left(y+\frac{Z}{a(t)}\right)^2 \geq 0$ for all y, Z with $a_0 = 2a$, we find that

$$V_0 \geq \frac{1}{4}(a-\alpha)\left[(\alpha+a)y^2+\frac{1}{a_1}Z^2\right] \quad (2.8)$$

for all y and Z. Furthermore, $c_0 \leq c(t) \leq b(t) \leq b_1$ for all $t \geq 0$, $h'(x) \leq c$ for all x, $\frac{h(x)}{x} \geq h_0$ for all $x \neq 0$, and $[Y+\frac{2c(t)}{bb(t)}h(x)]^2 \geq 0$ for all $t \geq 0, x$ and Y, it follows that

$$V_1 \geq \frac{2}{b}\left[(\alpha+a)b-2c\right]c_0h_0x^2, \quad (2.9)$$

for all x. In addition, since $b \leq \frac{g(y)}{y}$ for all $y \neq 0$, $(\beta x+ay+Z)^2 \geq 0$ for all x, y, Z and $\left(y+\frac{Z}{a}\right)^2 \geq 0$ for all y, Z we have

$$V_2 \geq \frac{1}{2}\beta(b-\beta)x^2+c_0by^2+\frac{1}{2a}(a-\alpha)Z^2. \quad (2.10)$$

Substituting in equation (2.3), estimates (2.8),(2.9) and (2.10), noting that the integrals are nonnegative and $a_0 = 2a$, there exists a positive constant $\delta_0 > 0$ such that

$$V \geq \delta_0(x^2+y^2+Z^2) \quad (2.11)$$

for all $t \geq 0, x, y$ and Z, where

$$\delta_0 := \min\left\{\frac{1}{b}[(\alpha+a)b-2c]c_0h_0+\frac{1}{2}\beta(b-\beta), \frac{1}{4}(\alpha+a)(a-\alpha)+bc_0, \frac{1}{4}(a-\alpha)\left(\frac{2}{a}+\frac{1}{a_1}\right)\right\}.$$

Now, from inequality (2.11), we find that, if $z(t) = -\phi z(t-\tau)$,

$$V(t, x_t, y_t, z_t) = 0 \text{ if and only if } x^2 + y^2 + Z^2 = 0 \tag{2.12a}$$

and

$$V(t, x_t, y_t, z_t) > 0 \text{ if and only if } x^2 + y^2 + Z^2 \neq 0 \tag{2.12b}$$

$$V(t, x_t, y_t, z_t) \to +\infty \text{ as } x^2 + y^2 + Z^2 \to \infty. \tag{2.12c}$$

Furthermore, since $\frac{h(x)}{x} \leq h_1$ for all $x \neq 0$, $\frac{g(y)}{y} \leq B$ for all $y \neq 0$, and the fact that $2|q_1 q_2| \leq q_1^2 + q_2^2$, there exist positive constants δ_1, δ_2 and δ_3 such that

$$V \leq \delta_1(x^2 + y^2 + z^2) + \delta_2 \int_{t-\tau}^{t} [y^2(s) + z^2(s)]ds + \delta_3 \int_{-\tau}^{0}\int_{t+s}^{t} [y^2(\tau) + z^2(\tau)]d\tau ds, \tag{2.13}$$

where

$$\delta_1 := \frac{1}{2}\max\{\beta(1+b+b)+2b_1h_1(1+\alpha+a), (\alpha+a)(1+a+a_1)+b_1(b+2h_1)(1+\phi)^2 a\beta+2b_1B, 1+\alpha+a\};$$

$$\delta_2 := \max\{\mu_1, \mu_2\};$$

and

$$\delta_3 := \max\{\mu_3, \mu_4\}.$$

Combining estimates (2.11) and (2.13), the inequality (2.7) of Lemma 2.3 is satisfied with $\delta_0, \delta_1, \delta_2, \delta_3$ equivalent to D_0, D_1, D_2, D_3 respectively. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Under the assumptions of Theorem 2.1 there exists a positive constant D_4 such that along the solution path of system (2.2) we have*

$$V'_{(2.2)}(t, x_t, y_t, Z_t) \leq -D_4(x^2(t) + y^2(t) + z^2(t)) \tag{2.14}$$

for all $t \geq 0, x, y, z$

Proof. Let (x_t, y_t, z_t) be any solution of (2.2). The derivative of the functional V with respect to independent variable t along the solution of (2.2) is

$$\begin{aligned} V'_{(2.2)} = &-\sum_{i=3}^{4} V_i + \sum_{i=5}^{7} V_i + \mu_1[y^2(t) - y^2(t-\tau)] + \mu_2[z^2(t) - z^2(t-\tau)] \\ &+ \tau[\mu_3 y^2 + \mu_4 z^2] - \int_{t-\tau}^{t} [\mu_3 y^2(s) + \mu_4 z^2(s)]ds + a\beta y^2, \end{aligned} \tag{2.15}$$

where

$$V_3 := \frac{1}{2}\beta c(t)\frac{h(x)}{x}x^2 + \left[(\alpha + a)b(t)\frac{g(y)}{y} - c(t)h'(x)\right]y^2 + [a(t) - (\alpha + a)]z^2;$$

$$V_4 := \sum_{j=1}^{5} V_{4j};$$

$$V_{41} := \frac{1}{4}\beta c(t)\frac{h(x)}{x}x^2 + \beta\left[b(t)\frac{g(y)}{y} - b\right]xy + \frac{1}{3}\left[(\alpha + a)b(t)\frac{g(y)}{y} - c(t)h'(x)\right]y^2;$$

$$V_{42} := \frac{1}{4}\beta c(t)\frac{h(x)}{x}x^2 + \beta[a(t) - a]xz + \frac{1}{3}[a(t) - (\alpha + a)]z^2;$$

$$V_{43} := \frac{1}{3}\left[(\alpha + a)b(t)\frac{g(y)}{y} - c(t)h'(x)\right]y^2 + (\alpha + a)[a(t) - a]yz + \frac{1}{3}[a(t) - (\alpha + a)]z^2;$$

$$V_{44} :=[a(t)-(\alpha+a)]\left[\frac{1}{3}z^2+2\phi zz(t-\tau)\right];$$
$$V_{45} :=\frac{1}{3}\left[(\alpha+a)b(t)\frac{g(y)}{y}-c(t)h'(x)\right]y^2+\phi b(t)\left[\frac{g(y)}{y}-b\right]yz(t-\tau);$$
$$V_5 :=(bb(t)+\beta)yz+2\phi c(t)h'(x)yy(t-\tau)+\beta\phi yz(t-\tau+\phi bb(t)y(t-\tau)z$$
$$+\phi^2 bb(t)y(t-\tau)z(t-\tau);$$
$$V_6 :=2(\alpha+a)c'(t)\int_0^x h(s)ds+2c'(t)h(x)Y+\frac{1}{2}bb'(t)Y^2+2b'(t)\int_0^y g(s)ds$$

and

$$V_7 := [\beta x+2(\alpha+a)y+2z+2\phi z(t-\tau)]\int_{t-\tau}^{t}\left[b(t)g'(y(s))z(s)+c(t)h'(x(s))y(s)\right]ds.$$

Now since $c(t)\geq c_0$ for all $t\geq 0$, $\dfrac{h(x)}{x}\geq h_0$ for all $x\neq 0$, $\dfrac{g(y)}{y}\geq b$ for all $y\neq 0$, $h'(x)\leq c$ for all x, $a(t)\geq a_0$ and $c(t)\leq b(t)$ for all $t\geq 0$, it is not difficult to show that

$$V_3\geq \frac{1}{2}\beta c_0h_0x^2+\frac{1}{2}[(2\alpha b-c)+(2ab-c)]y^2+[a-\alpha]z^2, \tag{2.16}$$

for all $t\geq 0, x, y$ and z. Also, applying the following inequalities

$$\beta^2\left(b(t)\frac{g(y)}{y}-b\right)^2<\frac{1}{6}\beta c_0h_0[2(\alpha+a)b-c],$$
$$\beta^2(a(t)-a)^2<\frac{1}{3}\beta c_0h_0[a-\alpha],$$

and

$$(\alpha+a)^2(a(t)-a)^2<\frac{2}{9}[2(\alpha+a)b-c][a-\alpha],$$

in V_{41}, V_{42} and V_{43} respectively, to obtain the following inequalities

$$V_{41}\geq\left[\frac{1}{2}\sqrt{\beta c_0h_0}|x|-\sqrt{\frac{1}{6}[2(\alpha+a)b-c]}|y|\right]^2\geq 0,\ \forall\ x,y;$$
$$V_{42}\geq\left[\frac{1}{2}\sqrt{\beta c_0h_0}|x|-\sqrt{\frac{1}{3}(a-\alpha)}|z|\right]^2\geq 0,\ \forall\ x,z;$$

and

$$V_{43}\geq\left[\sqrt{\frac{1}{6}[2(\alpha+a)b-c]}|y|-\sqrt{\frac{1}{3}(a-\alpha)}|z|\right]^2\geq 0,\ \forall\ y,z.$$

Furthermore, since $a_1>\alpha+a, c_0>0$, $2\alpha b>c$ and $2ab>c$ it follows that

$$[z+3\phi z(t-\tau)]^2\geq 0$$

for all $z, z(t-\tau)$, and

$$[y+\frac{3c_0\phi}{2[2(\alpha+a)b-c]}\left(\frac{g(y)}{y}-b\right)z(t-\tau)]^2\geq 0$$

for all $y, z(t-\tau)$, so that

$$V_{44}\geq -3\phi^2(a-\alpha)z^2(t-\tau)$$

and

$$V_{45}\geq -\frac{3c_0^2\phi^2}{2[2(\alpha+a)b-c]}\left(\frac{g(y)}{y}-b\right)^2z^2(t-\tau).$$

Combining $V_{4i}, (i = 1, 2, \cdots, 5)$, to get

$$V_4 \geq -3\phi^2 \left[a - \alpha + \frac{c_0^2}{2[2(\alpha + a)b - c]} \left(\frac{g(y)}{y} - b \right)^2 \right] z^2(t - \tau). \tag{2.17}$$

Next,

$$\begin{aligned} V_5 \leq & \frac{1}{2} \Big[[b_1(b + 2\phi h_1)] y^2 + [bb_1(1 + \phi) + \beta] z^2 + b_1[\phi b(1 + \phi) + 2h_1] y^2(t - \tau) \\ & + \phi[\beta + \phi bb_1] z(t - \tau) \Big]. \end{aligned} \tag{2.18}$$

Moreover, V_6 can be rewritten in the form

$$V_6 = b'(t) V_{61}$$

where

$$V_{61} := 2(\alpha + a) \frac{c'(t)}{b'(t)} \int_0^x h(s) ds + 2 \frac{c'(t)}{b'(t)} h(x) Y + \frac{1}{2} b Y^2 + 2 \int_0^y g(s) ds.$$

Since $b'(t) \leq c'(t) \leq 0$ for all $t \geq 0$, $b > 0$, $[Y + 2b^{-1}h(x)]^2 \geq 0$ for all x and Y, there exists a positive constant δ_* such that

$$V_{61} \geq \delta_*(x^2 + y^2) \geq 0 \tag{2.19}$$

for all x, y, where

$$\delta_* := \min\{b^{-1}[2(\alpha + a)b - c]h_0, b\}.$$

From estimate (2.19) and the assumption that $b'(t) \leq c'(t) \leq 0$ for all $t \geq 0$, it follows that

$$V_6 = b'(t) V_{61} \leq 0, \ \forall \ t \geq 0, x, y. \tag{2.20}$$

Since $|g'(y)| \leq k$ for all y and $|h'(x)| \leq c$ for all x, it follows that

$$\begin{aligned} V_7 \leq & \frac{1}{2} \beta b_1 (k + c) \tau x^2 + (\alpha + a) b_1 (k + c) \tau y^2 + b_1[(1 + \phi(k + c))] \tau z^2 \\ & + \frac{1}{2} b_1 c [2(1 + \alpha + a + \phi) + \beta] \int_{t-\tau}^t y^2(s) ds \\ & + b_1[(1 + \phi + \frac{1}{2}\beta) k + (\alpha + a) c] \int_{t-\tau}^t z^2(s) ds. \end{aligned} \tag{2.21}$$

Substituting estimates (2.16), (2.17), (2.18), (2.20) and (2.21) in equation (2.15), to obtain

$$\begin{aligned}
V'_{(2.2)} =& -\frac{1}{2}\beta\left[c_0h_0 - b_1(k+c)\tau\right]x^2 - \left[\frac{1}{2}[2(\alpha+a)b-c]\right.\\
&-\left[\frac{1}{2}b_1(b+2\phi h_1)+a\beta+\mu_1\right]-\left.[\mu_3+(\alpha+a)(k+c)b_1]\tau\right]y^2\\
&-\left[(a-\alpha)-[\frac{1}{2}(bb_1(1+\phi)+\beta)+\mu_2]-[\mu_4+b_1(1+\phi)(k+c)]\tau\right]z^2\\
&-\left[\mu_1-\frac{1}{2}b_1[b\phi(1+\phi)+2h_1]\right]y^2(t-\tau) \qquad (2.22)\\
&-\left[\mu_2-\left[3\phi^2(a-\alpha)+\frac{3c_0^2\phi^2}{2[2(\alpha+a)b-c]}\left(\frac{g(y)}{y}-b\right)^2+\frac{1}{2}\phi(\beta+\phi bb_1)\right]\right]\times\\
&z^2(t-\tau)-\left[\mu_3-\frac{1}{2}b_1c[\beta+2(1+\alpha+a+\phi)]\right]\int_{t-\tau}^{t}y^2(s)ds\\
&-\left[\mu_4-b_1\left[(1+\frac{1}{2}\beta+\phi)k+(\alpha+a)c\right]\right]\int_{t-\tau}^{t}z^2(s)ds.
\end{aligned}$$

Choose

$$\mu_1=\frac{1}{2}b_1[b\phi(1+\phi)+2h_1],\quad \mu_2=3\phi^2(a-\alpha)+\frac{3c_0^2\phi^2}{2[2(\alpha+a)b-c]}\left(\frac{g(y)}{y}-b\right)^2+\frac{1}{2}\phi(\beta+\phi bb_1),$$

$$\mu_3=\frac{1}{2}b_1c[\beta+2(1+\alpha+a+\phi)],\quad \text{and}\quad \mu_4=b_1\left[(1+\frac{1}{2}\beta+\phi)k+(\alpha+a)c\right],$$

estimate (2.22) becomes

$$\begin{aligned}
V'_{(2.2)} =& -\frac{1}{2}\beta\left[c_0h_0-b_1(k+c)\tau\right]x^2\\
&-\left\{\frac{1}{2}[2(\alpha+a)b-c]-\left[\frac{1}{2}b_1(b+2\phi h_1)+a\beta+\frac{1}{2}b_1[b\phi(1+\phi)+2h_1]\right]\right.\\
&-\left.\left[\frac{1}{2}b_1c[\beta+2(1+\alpha+a+\phi)]+(\alpha+a)(k+c)b_1\right]\tau\right\}y^2-\left\{(a-\alpha)\right. \qquad (2.23)\\
&-\left[\frac{1}{2}(bb_1(1+\phi)+\beta)+\left[3\phi^2(a-\alpha)+\frac{3c_0^2\phi^2}{2[2(\alpha+a)b-c]}\left(\frac{g(y)}{y}-b\right)^2\right.\right.\\
&+\left.\left.\frac{1}{2}\phi(\beta+\phi bb_1)\right]\right]-b_1\left.\left[(1+\frac{1}{2}\beta+\phi)k+(\alpha+a)c+(1+\phi)(k+c)\right]\tau\right\}z^2.
\end{aligned}$$

Thus in view of estimates (2.23) and (2.5) there exists positive constant δ_4 such that

$$V'_{(2.2)} \le -\delta_4(x^2+y^2+z^2) \qquad (2.24)$$

for all $t \ge 0, x, y, z$. Thus, inequality (2.24) establishes inequality (2.14) of Lemma 2.4, this completes the proof of Lemma 2.4 with $\delta_4 \equiv D_4$. □

Proof of Theorem 2.1. Let (x_t, y_t, z_t) be any solution of equation (2.2), in view of inequalities (2.11), (2.13) and (2.24), the trivial solution $X_t \equiv 0$ of system (2.2) is uniformly stable and uniformly asymptotically stable. This completes the proof of Theorem 2.1. □

3. Uniform Ultimate Boundedness of Solutions

In this section, uniform boundedness and uniform ultimate boundedness of solutions of system (1.2) when $p(t) \neq 0$, will be discussed.

Theorem 3.1. *If assumptions (i) to (vi) of Theorem 2.1 hold and in addition*

$$|p(t)| \leq M, \ 0 < M < \infty, \ \forall \ t \in \mathbb{R}^+,$$

then the solution (x_t, y_t, z_t) *of system* (1.2) *is uniformly bounded and uniformly ultimately bounded provided that the inequality* (2.5) *is satisfied.*

Proof. Let (x_t, y_t, z_t) be any solution of system (1.2). The derivative of the functional V defined by equation (2.3), with respect to the independent variable t along the solution path of system (1.2) is

$$V'_{(1.2)} = V'_{(2.2)} + [\beta x + 2(\alpha + a)y + 2z + 2\phi z(t-\tau)]p(t). \tag{3.1}$$

From the inequality (2.24) there exists a positive constant δ_5 such that

$$V'_{(1.2)} \leq -\delta_4(x^2 + y^2 + z^2) + \delta_5(|x| + |y| + |z|)|p(t)|, \tag{3.2}$$

where

$$\delta_5 := \max\{\beta, 2(\alpha + a), 2(1 + \phi)\}.$$

Since $|p(t)| \leq M$, $0 < M < \infty$ for all $t \geq 0$ and the fact that $|x| < 1 + x^2$ for all $x \in \mathbb{R}$, choose $M > 0$ sufficiently small such that $\delta_4 - \delta_5 M > 0$, there exist positive constants δ_6 and δ_7 such that

$$V'_{(1.2)} \leq -\delta_6(x^2 + y^2 + z^2) + \delta_7, \tag{3.3}$$

for all $t \geq 0, x, y$ and z, where

$$\delta_6 := \delta_4 - \delta_5 M \text{ and } \delta_7 := 3\delta_5 M.$$

From estimates (2.11), (2.13) and (3.3) the solutions (x_t, y_t, z_t) of system (1.2) are uniformly bounded and uniformly ultimately bounded. This completes the proof of Theorem 3.1. □

Corollary 3.2. *If the nonlinear delay functions* $g(y'(t-\tau))$ *and* $h(x(t-\tau))$ *are replaced by functions* $g(y'(t))$ *and* $h(x(t))$ *respectively in* (1.2)*, and*

$$|p(t)| \leq M, \ 0 < M < \infty, \ \forall \ t \in \mathbb{R}^+,$$

then the solution (x_t, y_t, z_t) *of the new system of ordinary differential equations is uniformly bounded and uniformly ultimately bounded.*

Theorem 3.3. *If all hypotheses of Theorem 3.1 hold true and in addition the functions* $a(t), b(t), c(t)$ *and* $p(t)$ *are* $\omega-$*periodic functions of* t*, then there exists a unique periodic solution for system* (1.2) *of period* ω*, provided that the inequality* (2.5) *holds.*

Proof. Let (x_t, y_t, z_t) be any solution of system (1.2). From inequality (3.2), since

$$(|x| + |y| + |z|)^2 \leq 3(x^2 + y^2 + z^2)$$

for all x, y, z, it follows that

$$V'_{(1.2)} \leq -\delta_8(x^2 + y^2 + z^2) \leq 0, \tag{3.4}$$

for all $t \geq 0, x, y, z$, where

$$\delta_8 := \delta_4 - 3\delta_5 M > 0$$

for sufficiently small $M > 0$. From estimates (2.11), (2.12), (2.13) and (3.4), a unique, $\omega-$periodic solution of equation (1.2) exists. This completes the proof of Theorem 3.3. □

4. Example

In this section two examples will be given to illustrate the correctness of the obtained results of Sections 2 and 3.

Example 4.1. Consider the third order delay differential equation

$$\begin{aligned}&[x(t)+\phi x(t-\tau)]''' + \left(8+\frac{\sin(t/2)}{t/2}\right)x''(t) + \Big(2x'(t-\tau)+3\sin(x'(t-\tau)/3)\Big)\times\\ &\left(\frac{1}{2}+\frac{1}{3+4t^2}\right)+\left(\frac{1}{4}+\frac{1}{4+4t^2}\right)\left(\frac{1}{7}x(t-\tau)+\frac{x(t-\tau)\sin(x(t-\tau)/4)}{1+x^2(t-\tau)}\right)=0.\end{aligned} \tag{4.1}$$

As system of first order functioal differential equations, equation (4.1) becomes

$$\begin{aligned}&x' = y\\ &y' = z\\ &Z' = -\left(8\frac{\sin(t/2)}{t/2}\right)z-\left(\frac{1}{2}+\frac{1}{3+4t^2}\right)(2y+3\sin(y/3))-\left(\frac{x}{7}+\frac{x\sin(x/4)}{1+x^2}\right)\times\\ &\left(\frac{1}{4}+\frac{1}{4+4t^2}\right)+\left(\frac{1}{2}+\frac{1}{3+4t^2}\right)\int_{t-\tau}^{t}\Big(2+3\cos(y(s)/3)\Big)z(s)ds+\left(\frac{1}{4}+\frac{1}{4+4t^2}\right)\times\\ &\int_{t-\tau}^{t}\left(\frac{1}{7}+\frac{\sin(x(s)/4)}{1+x^2(s)}+\frac{x(s)\cos(x(s)/4)}{4(1+x^2(s))}-\frac{2x^2(s)\sin(x(s)/4)}{(1+x^2(s))^2}\right)y(s)ds.\end{aligned} \tag{4.2}$$

Now, comparing equations (2.2) and (4.2), the following functions are defined:

(i) The function

$$a(t) := 8 + H_1(t)$$

where

$$H_1(t) = \frac{\sin(t/2)}{t/2}.$$

From Figure 1 we notice that $-0.22 \le H_1(t) \le 1$ for all $t \in \mathbb{R}$, it follows that

$$7.8 = a_0 \le a(t) \le a_1 = 9.0 \tag{4.3}$$

for all $t \in \mathbb{R}$. The path of $H_1(t), a(t)$ and their bounds are depicted in Figure 1.

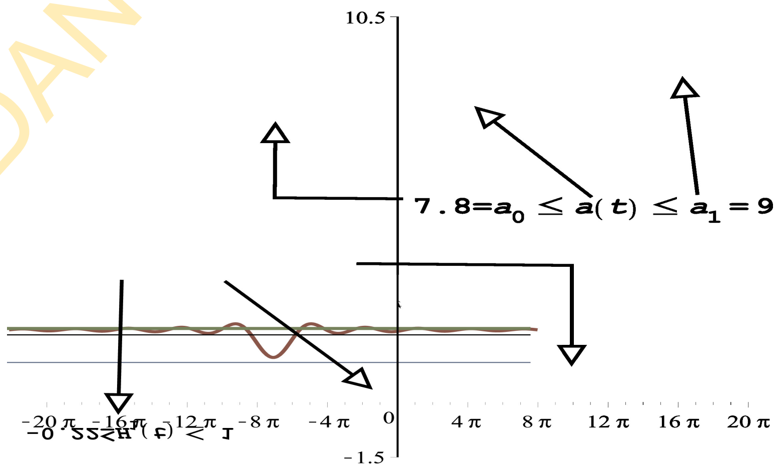

FIGURE 1.

Paths of the functions H_1, a and their respective bounds $t \in [-20\pi, 20\pi]$

From inequalities (4.3) since $a_0 = 2a$, it follows that

$$a = 3.9. \tag{4.4}$$

(ii) The functions

$$b(t) := \frac{1}{2} + H_2(t) \text{ and } c(t) := \frac{1}{4} + H_3(t)$$

where

$$H_2(t) = \frac{1}{3+4t^2} \text{ and } H_3(t) = \frac{1}{4+4t^2}.$$

Since $4 + 4t^2 > 3 + 4t^2$ for all $t \in \mathbb{R}$, then it follows that

$$H_3(t) < H_2(t) \tag{4.5}$$

for all $t \in \mathbb{R}$. Also, since

$$\lim_{t\to\infty} H_2(t) = 0 = \lim_{t\to\infty} H_3(t) \tag{4.6}$$

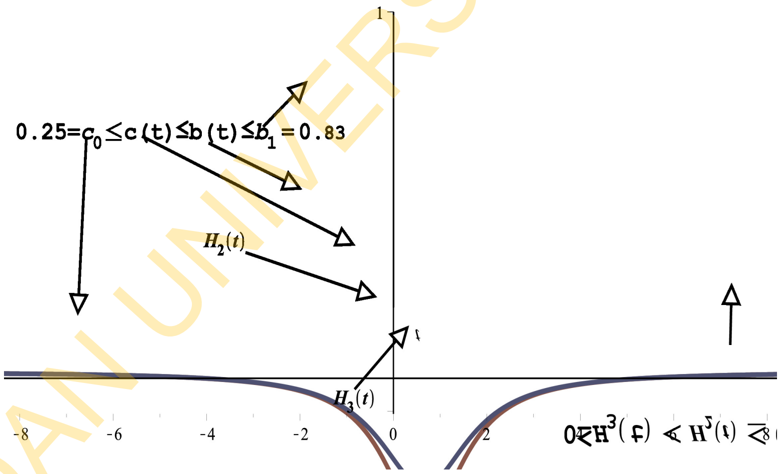

FIGURE 2.

The behaviour of the functions H_2, H_3, b and c, $t \in [-8, 8]$

it follows from estimates (4.5), (4.6), Figure 2 and the fact that the H_2 and H_3 are decreasing functions, the following inequalities hold

$$0 \le H_3(t) < H_2(t) \le 0.33 \tag{4.7}$$

for all $t \in \mathbb{R}$. From inequalities (4.7), it is easy to see that

$$0.25 = c_0 \le c(t) \le b(t) \le b_1 = 0.83$$

for all $t \in \mathbb{R}$, (see Figure 2).

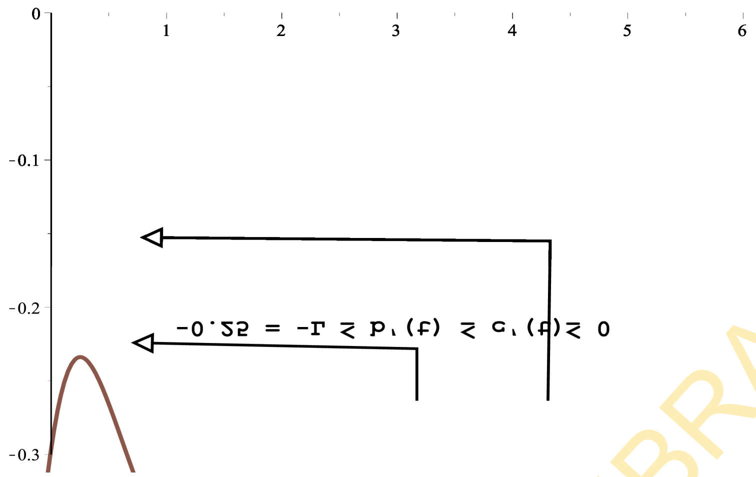

FIGURE 3.
The behaviour of the functions b' and c', $t \in [0, 6]$

Moreover, the derivatives of the functions b and c with respect to the independent variable t are

$$b'(t) = -\frac{8t}{(3+4t^2)^2} \text{ and } c'(t) = -\frac{8t}{(4+4t^2)^2}.$$

Noting that

$$b'(t) = -\frac{8t}{(3+4t^2)^2} < -\frac{8t}{(4+4t^2)^2} = c'(t) \tag{4.8}$$

for all $t \in \mathbb{R}$. Also,

$$\lim_{t\to\infty} b'(t) = 0 = \lim_{t\to\infty} c'(t) \tag{4.9}$$

Thus, from equations (4.9) and Figure 3, the inequality

$$b'(t) \le c'(t) \le 0$$

follows for all $t \in \mathbb{R}^+$.

(iii) The function

$$h(x) := \frac{1}{7}x + \frac{x\sin(x/4)}{1+x^2},$$

clearly $h(0) = 0$, and that

$$\frac{h(x)}{x} = \frac{1}{7} + H_4(x)$$

where

$$H_4(x) = \frac{\sin(x/4)}{1+x^2}.$$

From Figure 4, since

$$-0.12 \le H_4(x) \le 0.12$$

for all $x \in \mathbb{R}$, it follows that

$$0.02 = h_0 \le \frac{h(x)}{x} \le h_1 = 0.27, \ \forall\, 0 \neq x \in \mathbb{R}.$$

Furthermore, the derivative of the function h with respect to x is

$$h'(x) = \frac{1}{7} + H_5(x)$$

where

$$H_5(x) = \frac{\sin(x/4)}{1+x^2} + \frac{x\cos(x/4)}{4(1+x^2)} - \frac{2x^2\sin(x/4)}{(1+x^2)^2}.$$

Now since $H_5(x) \leq 0.16$ for all $x \in \mathbb{R}$, it follows that

$$h'(x) \leq c = 0.3,$$

for all $x \in \mathbb{R}$. The paths of functions H_5 and h' are shown in Figure 5.

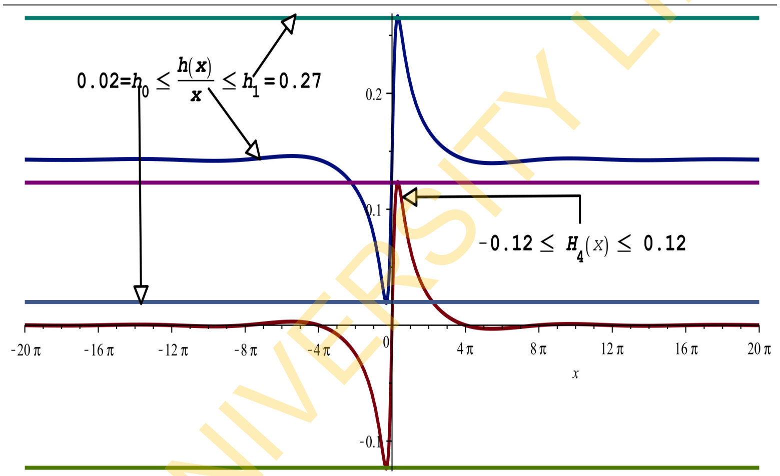

FIGURE 4.
The behaviour of the functions H_4 and $\frac{h(x)}{x}$, $x \in [-20\pi, 20\pi]$

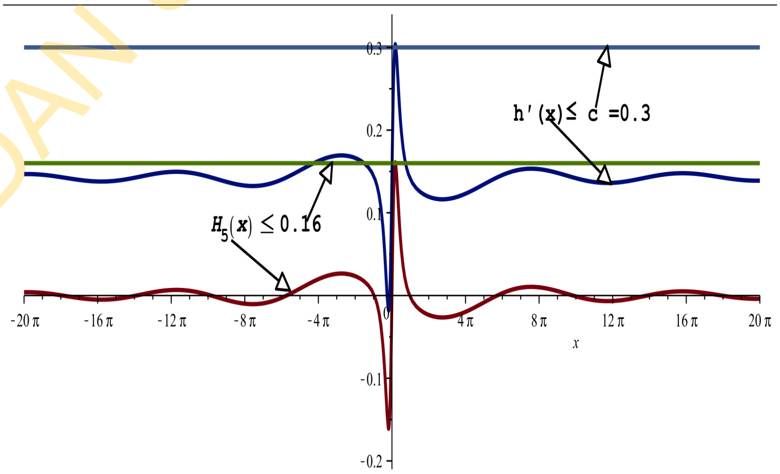

FIGURE 5.
The behaviour of the functions H_5 and $h'(x)$, $x \in [-20\pi, 20\pi]$

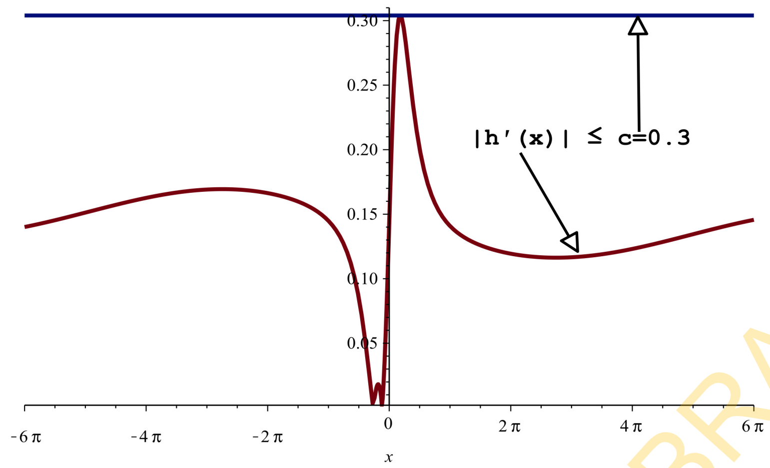

FIGURE 6.

The behaviour of $|h'(x)|$, $x \in [-6\pi, 6\pi]$

What is more, from Figure 6 it is easy to see that

$$|h'(x)| \leq c = 0.3$$

for all $x \in \mathbb{R}$.

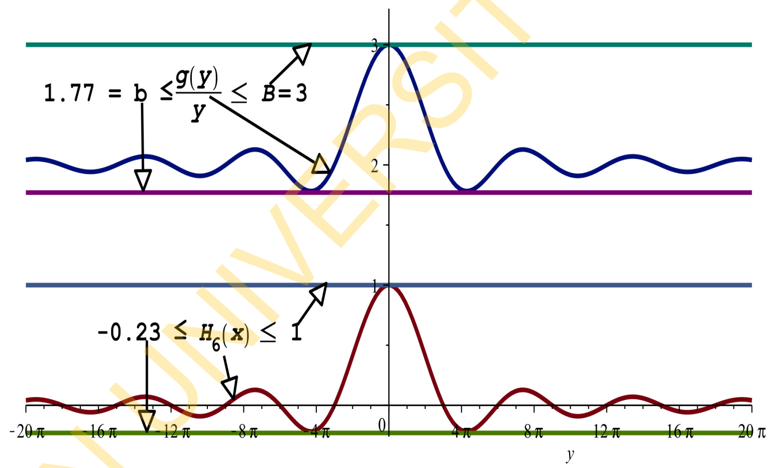

FIGURE 7.

The paths of H_6 and $\frac{g(y)}{y}$, $y \in [-20\pi, 20\pi]$

(iv) The function

$$g(y) := 2y + \frac{y\sin(y/3)}{y/3} \text{ and } \frac{g(y)}{y} = 2 + H_6(y)$$

where

$$H_6(y) = \frac{\sin(y/3)}{y/3}.$$

It can be seen, from Figure 7, that

$$-0.23 \leq H_6(y) \leq 1$$

for all $y \in \mathbb{R}$. It is not difficult to see from Figure 7 and the last inequality that

$$1.77 = b \leq \frac{g(y)}{y} \leq B = 3, \ \forall\ 0 \neq y \in \mathbb{R}. \tag{4.10}$$

Furthermore, the derivative of the function g with respect to variable y is

$$g'(y) = 2 + H_7(y)$$

where

$$H_7(y) = \cos(y/3).$$

Noting that $-1 \leq H_7(y) \leq 1$ for all $y \in \mathbb{R}$, it follows that

$$|g'(y)| \leq k = 3.0$$

for all $y \in \mathbb{R}$. For the behaviour of the functions H_7 and $|g'|$ see Figure 8. The aftermath of Theorem 2.1 gives the following summary

Summary 4.2. If the following constants hold $a_0 = 7.8, a_1 = 9.0, a = 3.9, b_1 = 0.83, b = 1.77, B = 3.0, c_0 = 0.25, h_0 = 0.02, h_1 = 0.27, L = 0.25, \alpha = 0.3, \beta = 0.8, \phi = 0.5$ and for all $t \geq 0$

(i) $7.8 \leq a(t) \leq 9.0$;
(ii) $0.25 \leq c(t) \leq b(t) \leq 0.83$, $-0.25 \leq b'(t) \leq c'(t) \leq 0$;
(iii) $h(0) = 0$, $0.02 \leq \frac{h(x)}{x} \leq 0.27$ for all $x \neq 0$, and $h'(x) = |h'(x)| \leq 0.3$ for all x;
(iv) $1.77 \leq \frac{g(y)}{y} \leq 3.0$ for all $y \neq 0$, $|g'(y)| \leq 3.0$ for all y;

$$\tau < \min\{0.002, 85.402, 1.165\} = 0.002, \tag{4.11}$$

where $A_1 = 3.935, A_2 = 0.125, A_3 = 1.667$ and $A_4 = 1.66$. The value 0.001 is chosen for τ.

Then the trivial solution of (4.2) is uniformly asymptotically stable provided that inequality (4.11).

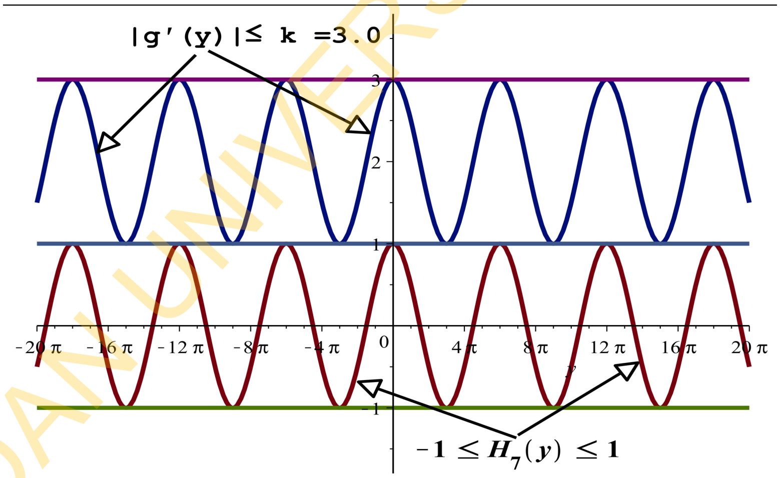

FIGURE 8. The paths of functions H_7 and $|g'|$, $y \in [-20\pi, 20\pi]$

Example 4.3. Consider the third order delay differential equation

$$[x(t) + \phi x(t-\tau)]''' + \left(8 + \frac{\sin(t/2)}{t/2}\right)x''(t) + \left(2x'(t-\tau) + 3\sin(x'(t-\tau)/3)\right) \times$$
$$\left(\frac{1}{2} + \frac{1}{3+4t^2}\right) + \left(\frac{1}{4} + \frac{1}{4+4t^2}\right)\left(\frac{1}{7}x(t-\tau) + \frac{x(t-\tau)\sin(x(t-\tau)/4)}{1+x^2(t-\tau)}\right) \tag{4.12}$$
$$= \frac{1}{10} + \frac{1}{1+t}.$$

As system of first order delay differential equations, equation (4.12) becomes

$$
\begin{aligned}
x' &= y\\
y' &= z\\
Z' &= -\left(8\frac{\sin(t/2)}{t/2}\right)z - \left(\frac{1}{2} + \frac{1}{3+4t^2}\right)(2y + 3\sin(y/3)) - \left(\frac{x}{7} + \frac{x\sin(x/4)}{1+x^2}\right)\times\\
&\left(\frac{1}{4} + \frac{1}{4+4t^2}\right) + \left(\frac{1}{2} + \frac{1}{3+4t^2}\right)\int_{t-\tau}^{t}\left(2 + 3\cos(y(s)/3)\right)z(s)ds \qquad (4.13)\\
&+\left(\frac{1}{4} + \frac{1}{4+4t^2}\right)\int_{t-\tau}^{t}\left(\frac{1}{7} + \frac{\sin(x(s)/4)}{1+x^2(s)} + \frac{x(s)\cos(x(s)/4)}{4(1+x^2(s))}\right.\\
&\left.-\frac{2x^2(s)\sin(x(s)/4)}{(1+x^2(s))^2}\right)y(s)ds + \frac{1}{10} + \frac{1}{1+t}.
\end{aligned}
$$

Comparing (1.2) and (4.13), items (i) to (iv) of Example 4.1 hold true for functions a, b, c, g and h. In addition, the function

$$p(t) := \frac{1}{10} + H_8(t)$$

where

$$H_8(t) = \frac{1}{1+t}.$$

Analysis shows that the function $H_8(t)$ decreases as t increases and that

$$\lim_{t\to\infty} H_8(t) = \lim_{t\to\infty}\left(\frac{1}{1+t}\right) = 0.$$

By completeness axiom, function H_8 has an upper bound i.e.,

$$H_8(t) \le 1$$

for $t = 0$. The paths and behaviour of functions H_8 and p are shown in Figure 9.

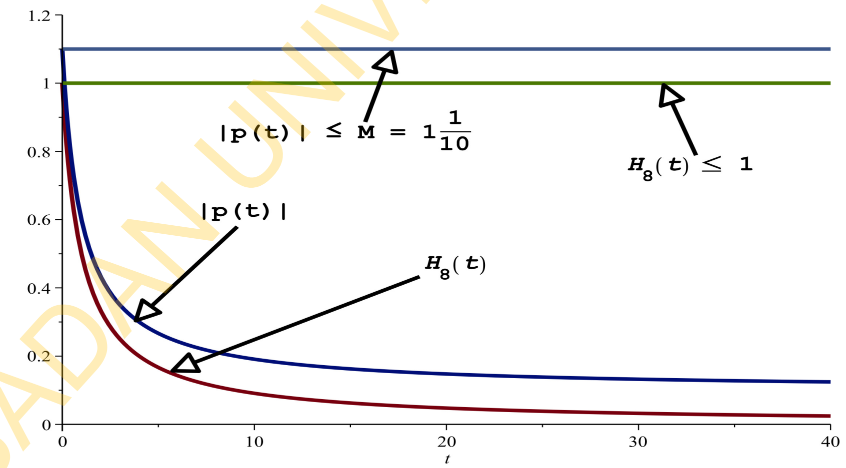

FIGURE 9.

The paths of functions H_8 and p, $t \in [0, 40]$.

Summary 4.4. If hypotheses (i) to (iv) of Summary 4.2 hold true, and in addition

$$|p(t)| \le M = 1\frac{1}{10}$$

for all $t \ge 0$, then the solutions of (4.13) is uniformly bounded and uniformly ultimately bounded provided the inequality (4.11) holds.

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[1] Department of Mathematics, Obafemi Awolowo University, Post Code 220005 Ile-Ife, Nigeria
[2] Department of Mathematics, Faculty of Science, New Valley Branch, Assiut University. New Valley, El-Khargah 72111, Egypt
[3] Department of Mathematics, University of Ibadan, Ibadan Nigeria
E-mail address: [1] atademola@oauife.edu.ng, [2] math_ayman27@yahoo.com, [3] womopeter@gmail.com