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Existence and Blow up Time Estimate for a Negative Initial Energy Solution of a Nonlinear Cauchy Problem

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Abstract In this paper, we consider nonlinear wave equations with dissipation having the form

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{\gamma-2}\nabla u\right) + b(t,x)|u_t|^{m-2}u_t = g(x,u)$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^n$. We obtain existence and blow up results under suitable assumptions on the positive function $b(t,x)$ and the nonlinear function $g(x,u)$. The existence result was obtained using the Galerkin approach while the blow up result was obtained via the perturbed energy method. Our result improves on the perturbed energy technique for unbounded domains.

Keywords Nonlinear wave equation · Global existence · Blow up · Finite speed of propagation

Mathematics Subject Classification (2010) 35A01 · 35B45 · 35L15 · 35L70

1 Introduction

In this paper, we consider existence and blow up of solution to a nonlinear wave equation

$$\begin{cases} u_{tt} - \operatorname{div}\left[|\nabla u|^{\gamma-2}\nabla u\right] + b(t,x)|u_t|^{m-2}u_t = g(x,u) & t \in [0,\infty), \quad x \in \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) & x \in \mathbb{R}^n \end{cases} \tag{1}$$

with space–time dependent dissipation. $u = u(t,x)$ is an unknown real valued function on $[0,\infty) \times \mathbb{R}^n$ and the initial data u_0, u_1 is assumed to have compact support in a ball $B(R)$

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of radius R about the origin, where R satisfies the condition $\operatorname{supp}\{u_0(x), u_1(x)\} \subset \{|x| \leq R\}$ and such that the solution satisfy the finite speed of propagation property

$$\operatorname{supp} u(t) \in B(R+t) \quad t \in [0, \infty)$$

Under these circumstances, the expectation is that the spread of the support could hinder finite-time blow-up of solution that is seen in the case of bounded domains, except for the case where the damping is absent or linear as blow-up can indeed occur.

In the case of bounded smooth domains $\Omega \subset \mathbb{R}^n$, there is an extensive literature on global existence and blow up of solutions of non-linear wave equations having negative initial energy and of the form

$$\begin{cases} u_{tt} - \Delta u_t - \operatorname{div}\Big[|\nabla u|^\gamma \nabla u + |\nabla u_t|^r \nabla u_t\Big] + |u_t|^m u_t = |u|^p u & x \in \Omega,\ t > 0 \\ u(x,t)|_{\partial\Omega} = 0, \quad t > 0 \qquad u(x,0) = u_0, \qquad u_t(x,0) = u_1, & x \in \Omega \end{cases} \tag{2}$$

that is, when the dissipation being considered, arises from an internal nonlinear damping term. see [1, 16, 18, 19] to mention but a few

Yang in [17], obtained blow up of solutions to (2) under the condition $p > \max\{\gamma, m\}$ and where the blow up time depends on $|\Omega|$.

In [10] Messaoudi and Said-Houari studied a class of nonlinear wave equations having the form (2) and obtained blow up result for $p > \max\{\gamma, m\}$ and $\gamma > r$, where the blow up result holds regardless of the size of Ω. Thus extending the result of Yang [17].

Liu and Wang [9] considered a class of wave equations of the form (2) and established blow up results for certain solutions with non-positive initial energy as well as positive initial energy. This further improves the results of Yang [17] and Messaoudi and Said-Houari [10].

In [13], Piskin investigated the energy decay of solutions for quasi-linear hyperbolic equations of the form (2) with nonlinear damping and source terms and obtained blow up result for the case $m = 0$, using the concavity method. More recently, Jeong, et al. [3] considered global nonexistence of solutions to a quasi-linear wave equation of the form (2), with acoustic boundary conditions and satisfying $p > \max\{\gamma, m\}$ and $\gamma > r$.

For a review on recent results on global existence, energy decay and blow up of solutions to nonlinear wave equations in bounded domains see [11], and for blow-up and global existence results for nonlinear wave equation of the form (2) with space dependent coefficients, see [12].

We note here that the interaction between source and damping terms was first studied in the work of Levine [6, 7], where they considered existence and asymptotic behaviour of solutions to (2) for the case $m = 0$. Their results were extended by Georgiev and Todorova [2] to the nonlinear case. In considering the relationship between m and p, for solutions that vanish on $\partial\Omega \times [0, \infty)$, they showed that for $m \geq p$, the solution with negative initial energy is global in time and for $p > m$ the solution cannot be global.

In the case of unbounded domains, due to lack of $L^m \hookrightarrow L^p$ injection, an important and challenging question is whether the nonlinear damping in (1) would be sufficient enough to prevent blow-up from occurring. Levine et al. [8] considered global existence and blow-up of weak solutions to the Cauchy problem (1) with $\gamma = 2$, they showed that when m, p satisfy the condition $p < \min\{m, 2(n-1)/(n-2)\}$, the solutions are global. In addition to the condition $p > \{2, m\}$ they also gave the restriction $p < \max\{2n/(n-2), mn/(n+1-m)\}$ for which the solution blows up when the initial energy is merely less than zero. In a related work, G. Todorova [15], studied the Cauchy problem (1) where $g(x,u) = -\mu(x)u + u|u|^{p-2}$

and argued that for the case $\mu = 0$, the additional restriction $p < mn/(n-m+1)$ is method driven.

In the semilinear case with scale invariant damping, that is

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu_0}{(1+t)^{\nu_0}} u_t = |u|^p \quad in \quad [0,\infty) \times \mathbb{R}^n \\ \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n \end{cases} \tag{3}$$

with $\mu_0 > 0$, $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ and $n \in \mathbb{N}$. Lai et al. in [4] obtained blow up and lifetime estimate result for solutions to (3) when $\nu_0 = 1$ and showed that blow up will occur for large exponent. He established similar result in [5] for combined nonlinearity in comparison to the scattering behavior of wave equations without damping.

In this paper, we investigate the existence and blow up time estimate for a negative initial energy solution to (1) in the energy space $u_0 \in W^{1,\gamma}(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ having compact support and for $t \in [0,\infty)$. For the blow up time estimate, we use a differential inequality of the form

$$L'(t) \geq k(t)L(t) + \mu h(t)L^\nu(t) \quad \text{for} \quad t > 0 \tag{4}$$

for $\nu > 1$, $\mu > 0$ and $h(t)$, $k(t)$ are functions of t to be determined later. In this case, the influence of the unboundedness of the domain on the growth behavior and the estimate of the upper bounds for the blow up time is taken into consideration. In Sect. 5, we give examples to various types of damping coefficients including the semilinear case (3).

2 Preliminaries

In this section, we state some basic assumptions used in this paper. For simplicity, we introduce the following notations.

p' Hölder conjugate of p where $p' = \frac{p}{p-1}$.
$\|\cdot\|_p$ the usual $L^p(\mathbb{R}^n)$ norm for $1 \leq p \leq \infty$.
$W^{k,p}(\mathbb{R}^n)$ is the Banach space of functions in L^p with $k (k \in \mathbb{N})$ generalized derivatives.
$J_T := [0,\infty)$.

For the nonlinear functions g and b, we have the following assumptions;

(A_1) $g \in C(\mathbb{R})$, $g(\cdot,s)s \geq 0$, such that

$$\rho_1|s|^{p-1} \leq |g(\cdot,s)| \leq \rho_2|s|^{p-1}, \quad s \in \mathbb{R} \tag{5}$$

where ρ_1 and ρ_2 are positive constants and $0 < p \leq \frac{n\gamma}{n-\gamma}$ when $n > \gamma$.

(A_2) $b(t,x) > 0$ is continuous and for $p \leq m$,

(i) $\sup_{x \in B(R+t)} b(t,x) \in L^\infty_{loc}(J_T)$
(ii) $\int_{B(R+t)} b(t,x)^{\frac{n\gamma}{n\gamma - m(n-\gamma)}} dx \in L^\infty_{loc}(J_T)$, whenever $0 < m \leq \frac{n\gamma}{n-\gamma}$ and $n > \gamma$

and for $p > m$, we assume that

(iii) $\int_{B(R+t)} b(t,x)^{\frac{p}{p-m}} dx \in L^\infty_{loc}(J_T)$

Definition 1 *We define a weak solution of* (1) *as a function* $u(t,x)$ *satisfying the following*

(i) $u \in L^\infty([0,T]; W^{1,\gamma}(\mathbb{R}^n)) \cap L^\infty([0,T]; L^p(\mathbb{R}^n))$
$u_t \in L^\infty([0,T]; L^2(\mathbb{R}^n)) \cap L^p([0,T] \times \mathbb{R}^n)$, $u_{tt} \in L^{m'}([0,T]; W^{-1,\gamma'}(\mathbb{R}^n))$

(ii) *we have*

$$\int_0^t \Big[\langle u_{tt}, v\rangle + \langle |\nabla u|^{\gamma-2}\nabla u, \nabla v\rangle + \langle b(t,x)|u_t|^{m-2}u_t, v\rangle - \langle g(x,u), u\rangle\Big] ds = 0$$

for $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$ *and a.e* $t \in [0,T]$ *and such that*

(iii)

$$u(0) = u_0 \in W^{1,\gamma}(\mathbb{R}^n), \quad u_t(0) = u_1 \in L^2(\mathbb{R}^n)$$

We define the energy function associated with (1) by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |u_t|^2 dx + \frac{1}{\gamma}\int_{\mathbb{R}^n} |\nabla u|^\gamma dx - \int_{\mathbb{R}^n}\int_0^u g(\cdot, y) dy dx \tag{6}$$

and for this energy function (6), we have the following lemma;

Lemma 1 *Assume that* (A_1) *and* (A_2) *hold. Let* u *be a solution of* (1), *then the energy function* $E(t)$ *of the problem* (1) *is defined by* (6). *In addition,* $E(t)$ *is non increasing and satisfies*

$$E'(t) = -\int_{\mathbb{R}^n} b(t,x)|u_t|^m dx \tag{7}$$

Moreover, we have

$$E(t) \le E(0) \tag{8}$$

Furthermore, we state the following lemmas for estimates as regard existence and blow up results.

Lemma 2 (*Modified Gronwall inequality*, [14]) *Let* $\phi(t)$ *be a non-negative function on* $[0,\infty)$ *satisfying*

$$\phi(t) \le B_1 + B_2 \int_0^t \phi^\delta(s) ds$$

where B_1, B_2 *are positive constants, then* $\phi(t)$ *satisfy the inequality*

$$\phi(t) \le B_1[1 - (\delta-1)B_2 B_1^{\delta-1} t]^{\frac{-1}{\delta-1}} \quad \textit{for } \delta > 1.$$

Lemma 3 *Let* $y(t)$ *be a continuous non-negative* C^1 *function on* $[0,\infty]$ *which satisfies*

$$y'(t) \ge a(t)y(t) + c(t)y^r(t) \tag{9}$$

(i) *if* $a(t) < 0$, $c(t) > 0$ *and* $r > 1$, *then* $y(t)$ *satisfies the following inequality*

$$y^{1-r}(t) \le e^{(1-r)\int_0^t a(s)ds}\left[y_0^{-(r-1)} - (r-1)\int_0^t c(s) e^{(r-1)\int_0^s a(\tau)d\tau} ds\right] \tag{10}$$

(ii) *and if* $a(t) \geq 0$, $c(t) > 0$ *and* $r > 1$. *we have*

$$y^{1-r}(t) \leq y_0^{1-r} - (r-1)\int_0^t c(s)ds$$

where $y_0 = y(0) > 0$.

Proof (Lemma 3)
Case (i)(if $a(t) < 0$, $c(t) > 0$ and $r > 1$):
Divide (9) by $y^r(t)$ and multiply the resulting inequality by its integrating factor to obtain

$$\frac{d}{dt}\left[y^{1-r}(t)e^{-(1-r)\int_0^t a(s)ds}\right] \leq (1-r)c(t)e^{-(1-r)\int_0^t a(s)ds}$$

Integrate this over $[0, t]$ and multiply both sides of the resulting inequality by $e^{(1-r)\int_0^t a(s)ds}$ to get the desired result.
Case (ii)(if $a(t) \geq 0$, $c(t) > 0$ and $r > 1$):
In this case, (9) reduces to

$$y'(t) \geq c(t)y^r(t) \tag{11}$$

Divide (11) by $y^r(t)$ and integrate the resulting inequality over $[0, t]$, this gives the desired result. □

Now we are set to present the first result.

3 Existence

In this section, using the Galerkin approximation technique, we shall discuss the existence of a weak solution to (1) in the maximal interval $[0, T]$ for $T < \infty$,

Theorem 1 *Suppose that the assumption* (A_1) *and* (A_2) *are satisfied. Then the problem* (1) *admits a unique weak solution* u *on* $[0, T]$ *such that*

$$u \in C\big([0, T]; W^{1,\gamma}(\mathbb{R}^n)\big), \qquad u_t \in C\big([0, T]; L^2(\mathbb{R}^n)\big) \cap L^p\big([0, T] \times \mathbb{R}^n\big)$$

Proof To prove the existence result, we make use of the Galerkin approximation technique. First, we assume the sequence of functions $(w_j)_{j\in\mathbb{N}}$ to be a basis in $W^{1,\gamma}(\mathbb{R}^n)$ which is also orthonormal in $L^2(\mathbb{R}^n)$ and we consider a weak solution of the form

$$u^n(t) = \sum_{j=1}^n a_{jn}(t)w_j \tag{12}$$

which satisfies the following approximate problem corresponding to (1)

$$\langle u_{tt}^n, w_j\rangle + \langle |\nabla u^n|^{\gamma-2}\nabla u^n, \nabla w_j\rangle + \langle b(t,\cdot)|u_t^n|^{m-2}u_t^n, w_j\rangle = \langle g(\cdot, u^n), w_j\rangle \tag{13}$$

for $w_j \in W^{1,\gamma}(\mathbb{R}^n)$ with initial conditions

$$u^n(0) = u_0^n \equiv \sum_{j=1}^n d_{jn}w_j \to u_0 \quad \text{strongly in } W^{1,\gamma}(\mathbb{R}^n) \text{ as } n \to \infty \tag{14}$$

and

$$u_t^n(0) = u_1^n \equiv \sum_{j=1}^{n} c_{jn} w_j \to u_1 \quad \text{strongly in } L^2(\mathbb{R}^n) \text{ as } n \to \infty \tag{15}$$

where $a_{jn}(t) = \langle u^n(t), w_j \rangle$, $d_{jn} = \langle u_0^n, w_j \rangle$, and $c_{jn} = \langle u_1^n, w_j \rangle$. Since the coefficients are continuous, then there exist a solution $u^n(t)$ for the system (13)-(15) and for some interval $[0, t_n)$ where $0 < t_n < T$. We will need the a-priori estimates below, to show that the solution is bounded on the whole interval $[0, T]$.

Set $w_j = u_t^n(t)$ in (13) and using assumption (A_1), the resulting equation is

$$\frac{d}{dt}\Big[\frac{1}{2}\|u_t^n\|^2 + \frac{1}{\gamma}\|\nabla u^n\|_\gamma^\gamma\Big] \le -\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx + \rho_2 \int_{\mathbb{R}^n} |u^n|^{p-1}|u_t^n| dx \tag{16}$$

For the last term on the right hand side of (16), using Holder and Young's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u^n|^{p-1}|u_t^n| dx \le \left[\int_{\mathbb{R}^n} |u_t^n|^p dx\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}^n} |u^n|^p dx\right]^{\frac{p-1}{p}} \\
&\le \epsilon_1 \left[\int_{\mathbb{R}^n} b(t,x)|u_t^n|^m dx\right]^{\frac{p}{m}} \left[\int_{B(R+t)} |b(t,x)|^{\frac{-p}{m-p}} dx\right]^{\frac{m-p}{m}} + C(\epsilon_1)\|u^n\|_p^p \\
&\le \epsilon_1\epsilon_2 \int_{\mathbb{R}^n} b(t,x)|u_t^n|^m dx + \epsilon_1 C(\epsilon_2) \int_{B(R+t)} |b(t,x)|^{\frac{-p}{m-p}} dx + C(\epsilon_1)\|u^n\|_p^p
\end{aligned} \tag{17}$$

and employing the estimate (17) in (16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\|u_t^n\|^2 + \frac{1}{\gamma}\|\nabla u^n\|_\gamma^\gamma\Big] + (1-\epsilon_1\epsilon_2\rho_2)\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx \\
&\le C(\epsilon_1)\rho_2\|u^n\|_p^p + \epsilon_1 C(\epsilon_2)\rho_2 \int_{B(R+t)} |b(t,x)|^{\frac{-p}{m-p}} dx
\end{aligned} \tag{18}$$

Now, for the first term on the right hand side of (18), we observe from Sobolev inequality that for $t \in [0, T]$ and $p < \frac{n\gamma}{n-\gamma}$, the following estimate holds

$$\begin{aligned}
\|u\|_p &\le \left[\int_{\mathbb{R}^n} |u|^{\frac{n\gamma}{n-\gamma}} dx\right]^{\frac{n-\gamma}{n\gamma}} \left[\int_{B(R+t)} dx\right]^{\frac{n\gamma-p(n-\gamma)}{np\gamma}} \\
&\le [\omega_n(R+t)^n]^{\frac{n\gamma-p(n-\gamma)}{np\gamma}} \|u\|_{\frac{n\gamma}{n-\gamma}} \le K[\omega_n(R+t)^n]^{\frac{n\gamma-p(n-\gamma)}{np\gamma}} \|\nabla u\|_\gamma
\end{aligned} \tag{19}$$

where $W^{1,\gamma}(\mathbb{R}^n)$ is embedded continuously in $L^{\frac{N\gamma}{N-\gamma}}(\mathbb{R}^n)$ with an embedding constant K and ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Hence (18) reduces to

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\|u_t^n\|^2 + \frac{1}{\gamma}\|\nabla u^n\|_\gamma^\gamma\Big] + (1-\epsilon_1\epsilon_2\rho_2)\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx \\
&\le C(\epsilon_1)K^p\rho_2[\omega_n(R+t)^n]^{\frac{n\gamma-p(n-\gamma)}{n\gamma}} \|\nabla u\|_\gamma^p + \epsilon_1 C(\epsilon_2)\rho_2 b_R(t)
\end{aligned} \tag{20}$$

where $b_R(t) = \int_{B(R+t)} |b(t,x)|^{\frac{-p}{m-p}} dx$ and $\epsilon_1\epsilon_2 < \frac{1}{\rho_2}$. Furthermore, setting

$$\mathcal{H}_n(t) = \frac{1}{2}\|u_t^n\|^2 + \frac{1}{\gamma}\|\nabla u^n\|_\gamma^\gamma + (1-\epsilon_1\epsilon_2\rho_2)\int_0^t \int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dxds \tag{21}$$

and integrating (20) over t for $t \in [0,T]$, we have from assumption (A_2) that there exist positive constants K_1 and K_2 such that (20) yields

$$\mathcal{H}_n(t) \leq \mathcal{H}_n(0) + K_1 + K_2 \int_0^t \mathcal{H}_n^{\frac{p}{\gamma}}(s)ds \tag{22}$$

for $t \in [0,T]$ where $\epsilon_1 C(\epsilon_2)\rho_2 \int_0^t b_R(s)ds \leq \epsilon_1 C(\epsilon_2)\rho_2 T \sup_{t\in[0,T]} b_R(t) \leq K_1$.

and $C(\epsilon_1)K^p\rho_2 \sup_{t\in[0,T]}[\omega_n(R+t)^n]^{\frac{n\gamma-p(n-\gamma)}{n\gamma}} \leq K_2$.

Applying Lemma 2, we get

$$\mathcal{H}_n(t) \leq K_1\left(1 - K_2 K_3^{\frac{p-\gamma}{\gamma}} t\right)^{\frac{-\gamma}{p-\gamma}} \tag{23}$$

for $t \in [0,T]$ and $K_3 = \mathcal{H}_n(0) + K_1$. Thus, there exist a positive constant K_4 independent of $n \in \mathbb{N}$ such that

$$\mathcal{H}_n(t) \leq K_4 \tag{24}$$

Therefore, from (21) and (24), we obtain the following estimates

$$\|u_t^n\|^2 \leq K_4 \tag{25}$$

$$\|\nabla u^n\|_\gamma^\gamma \leq K_4 \tag{26}$$

and

$$\int_0^t \int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dxds \leq K_4 \tag{27}$$

Furthermore, from (27), assumption (A_2) and Holder's inequality, we have

$$\int_{\mathbb{R}^n} |u_t^n|^p dx \leq \left[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx\right]^{\frac{p}{m}} \left[\int_{B(R+t)} |b(t,\cdot)|^{\frac{-p}{m-p}} dx\right]^{\frac{m-p}{m}} \leq K_5$$

where K_5 is a positive constant independent of n.

Setting $v = w_j$ in (13), we have

$$|\langle u_{tt}^n, v\rangle| \leq |\langle |\nabla u^n|^{\gamma-2}\nabla u^n, \nabla v\rangle| + |\langle b(t,\cdot)|u_t^n|^{m-2}u_t^n, v\rangle| + |\langle g(\cdot,u^n), v\rangle| \tag{28}$$

Now, for the last term on the right hand side of (28), using Holder's inequality and (19), we have

$$|\langle g(\cdot,u^n), v\rangle| \leq \|g(\cdot,u^n(t))\|_{p'}\|v\|_p \leq K_6\|g(\cdot,u^n(t))\|_{p'}\|v\|_{1,\gamma} \tag{29}$$

and from (26) and assumption (A_1), we get

$$\|g(\cdot,u^n(t))\|_{p'} \leq \rho_2\|u^n(t)\|_p^{p-1} \leq K_7 \quad \text{for } t \in [0,T] \tag{30}$$

For the second term on the right hand side of (28), using Hölder and Sobolev inequalities, and assumption (A_2), we have the estimate

$$\begin{aligned}&\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^{m-2}u_t^n v\,dx\\&\le \Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx\Big]^{\frac{m-1}{m}}\Big[\int_{B(R+t)} |b(t,x)|^{\frac{n\gamma}{n\gamma-m(n-\gamma)}}dx\Big]^{\frac{n\gamma-m(n-\gamma)}{n\gamma m}}\|v\|_{\frac{n\gamma}{n-\gamma}} \qquad (31)\\&\le K_8\Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx\Big]^{\frac{m-1}{m}}\|v\|_{1,\gamma}\end{aligned}$$

Now, substituting the estimates (29), (31) in (28) and using Hölder's inequality for the first term on the right hand side of (28), we have the following estimate

$$\begin{aligned}|\langle u_{tt}^n, v\rangle| \le K_9\Big(&\|\nabla u^n(t)\|_{-1,\gamma'} + \|g(\cdot,u^n(t))\|_{p'}\\&+\Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx\Big]^{\frac{m-1}{m}}\Big)\|v\|_{1,\gamma}\end{aligned}$$

and thus, using the estimates (26) and (30), we obtain

$$\|u_{tt}^n(t)\|_{-1,\gamma'} \le K_{10}\Big(\Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx\Big]^{\frac{m-1}{m}}+1\Big)$$

By applying Hölder's inequality, integrating the resulting estimate over t for $t\in[0,T]$ and employing the estimate (27), we obtain

$$\int_0^t \|u_{tt}^n(s)\|_{-1,\gamma'}^{m'}ds \le K_{11}\int_0^t\Big(\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^m dx+1\Big)ds \le K_{12}$$

for $t\in[0,T]$. Therefore, for any $T>0$ we have that the nonlinear terms are uniformly bounded on $[0,T]$ and it follows that the solution $u^n(t)$ of (13) exist on $[0,T]$ for each n.

Hence from the estimates above, we can obtain a subsequence u^k of u^n and pass the limit in the approximate problem to obtain a weak solution satisfying

b_1 $u^k(t)\to u(t)$ weakly-star in $L^\infty\big([0,T];W^{1,\gamma}(\mathbb{R}^n)\big)$
b_2 $u_t^k(t)\to u_t(t)$ weakly in $L^\infty\big([0,T];L^2(\mathbb{R}^n)\big)\cap L^p\big([0,T]\times\mathbb{R}^n\big)$
b_3 $u_{tt}^k(t)\to u_{tt}(t)$ weakly-star in $L^{m'}\big([0,T];W^{-1,\gamma'}(\mathbb{R}^n)\big)$
b_4 $g(\cdot,u^k(t))\to\phi(t)$ weakly-star in $L^\infty\big([0,T];L^{p'}(\mathbb{R}^n)\big)$
b_5 $|b(t,\cdot)|^{\frac{m-1}{m}}|u_t^k|^{m-2}u_t^k\to\varepsilon(t)$ weakly in $L^{m'}\big([0,T]\times\mathbb{R}^n\big)$.

Now, letting $n\to\infty$ in (13) and using (b_1)–(b_5), we obtain

$$\int_0^T\Big[\big(u_{tt},v\big)+\big(|\nabla u|^{\gamma-2}\nabla u,\nabla v\big)+\big(b(t,\cdot)|u_t|^{m-2}u_t,v\big)-\big(\phi,v\big)\Big]dt=0$$

for all $v\in L^m([0,T];W^{1,\gamma}(\mathbb{R}^n))$. The proof for $g(\cdot,u^n)=\phi$ is the same as in [19], so we omit it. □

4 Blow up

In this section, we consider the blow up property of the solution to (1) having negative initial energy. Our technique follows the one in [8], however we employ a differential inequality of the form (4) in obtaining the blow up estimate of solution to (1).

First, we define the function $H(t)$ by

$$H(t) := -E(t) \tag{32}$$

then from (6), we have

$$0 < H(0) \leq H(t) \leq \int_{\mathbb{R}^n} \int_0^u g(\cdot, y) dy dx \leq \frac{\rho_2}{p} \|u\|_p^p \tag{33}$$

Moreover, from (8) the derivative $H'(t)$ satisfy

$$H'(t) = \int_{\mathbb{R}^n} b(t,x) |u_t|^m dx \tag{34}$$

Furthermore, for the Cauchy problem (1), we define the function $L(t)$ by

$$L(t) := \lambda(t) H^{1-\varrho}(t) + \mu \beta(t) \int_{\mathbb{R}^n} u u_t dx \tag{35}$$

for suitable choice of ϱ satisfying

$$0 < \varrho = \frac{p-m}{mp} \tag{36}$$

where λ and β are positive functions depending on the support radius R and satisfying the following conditions

l_1: $\lambda'(t) \geq 0$

l_{2_1}: $\beta(t)\lambda'(t) - \lambda(t)\beta'(t) \geq 0$ and $\frac{\beta'(t)}{\beta(t)} < 0$
or
l_{2_2}: $\beta(t)\lambda'(t) - \lambda(t)\beta'(t) \geq 0$ and $\frac{\beta'(t)}{\beta(t)} \geq 0$
l_3: $\lambda(t) \geq \eta_R(t)\beta(t)$
where

$$\eta_R(t) = \left[\int_{B(R+t)} |b(t,x)|^{\frac{p}{p-m}} dx \right]^{\frac{p-m}{(m-1)p}}$$

such that one of the following
l_{4_1}: $D(t) := \int_0^\infty \phi(s)^{-1} \left[\frac{\beta(s)}{\lambda(s)}\right]^{\frac{mp}{p(m-1)+m}} ds = \infty$
or
l_{4_2}: $D(t) := \int_0^\infty \beta(s) \left[\phi(s)\lambda(s)^{\frac{mp}{p(m-1)+m}}\right]^{-1} ds = \infty$
l_{5_1}: $D(t) := \int_0^\infty \phi(s)^{-1} \left[\frac{\beta(s)}{\lambda(s)}\right]^{\frac{mp}{p(m-1)+m}} ds < \infty$
or
l_{5_2}: $D(t) := \int_0^\infty \beta(s) \left[\phi(s)\lambda(s)^{\frac{mp}{p(m-1)+m}}\right]^{-1} ds < \infty$
is satisfied for $\phi(t) = \max\{1, \left[(R+t)^{\frac{n(p-2)}{2p}} \eta_R^{-1}\right]^{\frac{1}{1-\varrho}}\}$.

The weighted functions $\lambda(t)$ and $\beta(t)$ are used here to compensate for the lack of $L^m \hookrightarrow L^p$ injection arising as a result of the unboundedness of the domain for $0 \leq m < p$. In [8] the function

$$L(t) := \lambda_0 \lambda(t) H^{1-\varrho}(t) + \beta(t) \int_{\mathbb{R}^n} u u_t dx$$

was used under the following assumptions

$$\eta_R(t) \leq \lambda(t)/\beta(t), \quad \lambda(t), \beta(t) > 0, \quad \lambda'(t) \geq 0$$

and $t^{n(p-2)/2p} \beta'(t) = o(\beta(t))$ as $t \to \infty$. Under these assumptions, they obtained an inequality of the form

$$L(t) \geq j(t) L^{\nu}(t), \quad \nu > 1 \tag{37}$$

In [15], the function (35) was used with $\lambda(t) = (R+t)^A$ and $\beta(t) = 1$ where A is a positive constant. As will be discussed later, in this case $\frac{\beta'(t)}{\beta(t)} = 0$ and the inequality (37) is expected. Notwithstanding for $\frac{\beta'(t)}{\beta(t)} < 0$, an inequality of the form (4) is expected.

Theorem 2 *Let $u(x,t)$ be a solution of the problem (1) with compact support in the ball $B(R)$ and suppose that the assumptions (l_1), (l_{2_1}), (l_3) and (l_{4_1}) are satisfied. In addition, assume that $g(\cdot, u)$ satisfies*

(B_1) $\int_{\mathbb{R}^n} u g(\cdot, u) dx - q \int_{\mathbb{R}^n} \int_0^u g(\cdot, y) dy dx \geq \rho_0 \|u\|_p^p$

for positive constants $q \in (\gamma, p)$. Then for $m < p$, no weak solution of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0 u_1 dx > 0$ can exist on the whole of $[0, \infty)$

Proof From (35), we have that the derivative of $L(t)$ yields

$$\begin{aligned} L'(t) = & \lambda'(t) H^{1-\varrho}(t) + \mu\beta'(t) \int_{\mathbb{R}^n} u u_t dx + \lambda(t)(1-\varrho) H^{-\varrho}(t) H'(t) \\ & + \mu\beta(t) \int_{\mathbb{R}^n} |u_t|^2 dx + \mu\beta(t) \int_{\mathbb{R}^n} u u_{tt} \end{aligned} \tag{38}$$

and using the equation (1), we obtain

$$\begin{aligned} L'(t) = & \lambda'(t) H^{1-\varrho}(t) + \mu\beta'(t) \int_{\mathbb{R}^n} u u_t dx + \lambda(t)(1-\varrho) H^{-\varrho}(t) H'(t) \\ & [] + \mu\beta(t) \int_{\mathbb{R}^n} |u_t|^2 dx - \mu\beta(t) \int_{\mathbb{R}^n} |\nabla u|^{\gamma} dx + \mu\beta(t) \int_{\mathbb{R}^n} u g(\cdot, u) dx \\ & - \mu\beta(t) \int_{\mathbb{R}^n} b(t,x) |u_t|^{m-2} u_t u dx \end{aligned} \tag{39}$$

The second to the last term in (39), can be estimated using Holder's inequality to get

$$\begin{aligned} \int_{\mathbb{R}^n} b(t,x)|u_t|^{m-2} u_t u dx & \leq \left[\int_{B(R+t)} |b(t,x)|^{\frac{p}{p-m}} dx \right]^{\frac{p-m}{mp}} \left[\int_{\mathbb{R}^n} b(t,x) |u_t|^m dx \right]^{\frac{m-1}{m}} \|u\|_p \\ & \leq \left[\eta_R(t) \int_{\mathbb{R}^n} b(t,x)|u_t|^m dx \right]^{\frac{m-1}{m}} \|u\|_p^{\frac{p}{m}} \|u\|_p^{\frac{m-p}{m}} \end{aligned}$$

Hence, using Young's inequality and (33), it follows that

$$\int_{\mathbb{R}^n} b(t,x)|u_t|^{m-2}u_t u dx \leq C(\delta_1)H^{\varrho-\varrho_1}(0)H^{-\varrho}(t)\eta_R(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^m dx + \delta_1 H^{-\varrho}(0)\|u\|_p^p \tag{40}$$

where $\varrho_1 = \frac{p-m}{mp}$. Therefore, using the estimate (40) in (39), we obtain

$$\begin{aligned} L'(t) \geq\; & \lambda'(t)H^{1-\varrho}(t) + \mu\beta'(t)\int_{\mathbb{R}^n} uu_t dx + \lambda(t)(1-\varrho)H^{-\varrho}(t)H'(t) \\ & + \mu\beta(t)\|u_t\|^2 - \mu\beta(t)\|\nabla u\|_\gamma^\gamma + \mu\beta(t)\int_{\mathbb{R}^n} ug(\cdot,u)dx \\ & - C(\delta_1)H^{\varrho-\varrho_1}(0)H^{-\varrho}(t)\mu\beta(t)\eta_R(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^m dx \\ & - \delta_1 H^{-\varrho}(0)\mu\beta(t)\|u\|_p^p \end{aligned} \tag{41}$$

From the energy identity,

$$q\int_{\mathbb{R}^n}\int_0^u g(\cdot,y)dydx = \frac{q}{2}\|u_t\|^2 + \frac{q}{\gamma}\|\nabla u\|_\gamma^\gamma + qH(t)$$

for $\gamma < q < p$. Then, using assumption (B_1), we obtain

$$\int_{\mathbb{R}^n} ug(\cdot,u)dx \geq \frac{q}{2}\|u_t\|^2 + \frac{q}{\gamma}\|\nabla u\|_\gamma^\gamma + qH(t) + \rho_0\|u\|_p^p \tag{42}$$

Therefore, we have

$$\begin{aligned} L'(t) \geq\; & \lambda'(t)H^{1-\varrho}(t) + \mu\beta'(t)\int_{\mathbb{R}^n} uu_t dx + \mu\beta(t)(1+\frac{q}{2})\|u_t\|^2 \\ & + \mu\beta(t)\big[\rho_0 - \delta_1 H^{-\varrho}(0)\big]\|u\|_p^p + \mu\beta(t)\Big[\frac{(q-\gamma)}{\gamma}\Big]\|\nabla u\|_\gamma^\gamma + q\mu\beta(t)H(t) \\ & + \big[\lambda(t)(1-\varrho) - C(\delta_1)H^{\varrho-\varrho_1}(0)\mu\beta(t)\eta_R(t)\big]H^{-\varrho}(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^m dx \end{aligned} \tag{43}$$

We choose μ small enough such that

$$\lambda(t)(1-\varrho) \geq C(\delta_1)H^{\varrho-\varrho_1}(0)\mu\beta(t)\eta_R(t) \tag{44}$$

Also, from the definition of $L(t)$ and assumption (l_{2_1}), we have that

$$\begin{aligned} & \mu\beta'(t)\int_{\mathbb{R}^n} uu_t dx + \lambda'(t)H^{1-\varrho}(t) \\ & = \Big[\frac{\beta'(t)}{\beta(t)}\Big]L(t) + \beta(t)\Big[\frac{\beta(t)\lambda'(t) - \lambda(t)\beta'(t)}{\beta^2(t)}\Big]H^{1-\varrho}(t) \\ & \geq \Big[\frac{\beta'(t)}{\beta(t)}\Big]L(t) \end{aligned} \tag{45}$$

Hence, using the estimate (44) and (45) in (43), we obtain

$$\begin{aligned} L'(t) \geq& [\frac{\beta'(t)}{\beta(t)}]L(t) + \mu\beta(t)\Big[\rho_0 - \delta_1 H^{-\varrho}(0)\Big]\|u\|_p^p \\ &+ \mu\beta(t)(1+\frac{q}{2})\|u_t\|^2 + \mu\beta(t)[\frac{(q-\gamma)}{\gamma}]\|\nabla u\|_\gamma^\gamma + q\mu\beta(t)H(t) \end{aligned} \tag{46}$$

Therefore, if we choose δ_1 small enough such that $\rho_0 \geq \delta_1 H^{-\varrho}(0)$, then there exist a positive constant C_μ such that (46) satisfies

$$L'(t) - [\frac{\beta'(t)}{\beta(t)}]L(t) \geq C_\mu \beta(t)\Big[\|u\|_p^p + \|u_t\|^2 + \|\nabla u\|_\gamma^\gamma + H(t)\Big] \tag{47}$$

where $C_\mu := \mu \min\{q,\ \frac{q-\gamma}{\gamma},\ [\rho_0 - \delta_1 H^{-\varrho}(0)],\ (1+\frac{q}{2})\}$. Hence, since

$$L(0) = \lambda(0)H^{1-\varrho}(0) + \mu\beta(0)\int_{B(R)} u_0 u_1 dx > 0$$

then from (47), we have that $L(t)$ is an increasing function for $t \geq 0$, satisfying

$$L(t) \geq \frac{\beta(t)}{\beta(0)} L(0) > 0 \qquad \forall t \geq 0.$$

On the other hand

$$\begin{aligned} L^{\frac{1}{1-\varrho}}(t) =& \Big[\lambda(t)H^{1-\varrho}(t) + \mu\beta(t)\int_{\mathbb{R}^n} u u_t dx\Big]^{\frac{1}{1-\varrho}} \\ \leq& 2^{\frac{1}{1-\varrho}}\Big[\big[\lambda(t)\big]^{\frac{1}{1-\varrho}} H(t) + \big[\mu\beta(t)\big]^{\frac{1}{1-\varrho}}\Big[\int_{\mathbb{R}^n} u u_t dx\Big]^{\frac{1}{1-\varrho}}\Big] \end{aligned} \tag{48}$$

Now, using Hölder inequality, we get

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} u u_t dx\Big| &\leq [\omega_n (R+t)^n]^{\frac{(p-2)}{2p}} \|u\|_p \|u_t\|_2 \\ &\leq [\omega_n (R+t)^n]^{\frac{(p-2)}{2p}} \|u\|_p \|u_t\|_2 \end{aligned}$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Then by Young's inequality, we have

$$\Big[\|u\|_p \|u_t\|_2\Big]^{\frac{1}{1-\varrho}} \leq C_8 \Big[\|u\|_p^{\frac{\epsilon}{1-\varrho}} + \|u_t\|^{\frac{\theta}{1-\varrho}}\Big] \tag{49}$$

where $C_8 = C_8(\epsilon, \theta, \varrho)$ and $\frac{1}{\epsilon} + \frac{1}{\theta} = 1$. Now choosing $\theta = 2(1-\varrho)$ and setting $\frac{\epsilon}{1-\varrho} = \frac{2}{1-2\varrho} \leq p$, so that $\varrho \leq \varrho_2$, where $\varrho_2 = \frac{p-2}{2p}$, then (49) yields

$$\Big|\int_{\mathbb{R}^n} u u_t dx\Big|^{\frac{1}{1-\varrho}} \leq C_9 (R+t)^{\frac{n(p-2)}{2p(1-\varrho)}} \Big[\|u\|_p^p + \|u_t\|^2\Big] \tag{50}$$

Combining the choice of ϱ in (50), with the previous choices, we observe that $\varrho_1 = \min\{\varrho_1,\ \varrho_2\}$, and therefore we have

$$L^{\frac{1}{1-\varrho}}(t) \leq 2^{\frac{1}{1-\varrho}}\Big[\big[\lambda(t)\big]^{\frac{1}{1-\varrho}} H(t) + C_9\big[(R+t)^{\frac{n(p-2)}{2p}} \mu\beta(t)\big]^{\frac{1}{1-\varrho}}\Big[\|u\|_p^p + \|u_t\|^2\Big]\Big] \tag{51}$$

and from (44), the estimate (51) yields

$$L^{\frac{1}{1-\varrho}}(t) \le \left[2\lambda(t)\right]^{\frac{1}{1-\varrho}}\left[H(t) + C_{10}\left[(R+t)^{\frac{n(p-2)}{2p}}\eta_R^{-1}\right]^{\frac{1}{1-\varrho}}\left[\|u\|_p^p + \|u_t\|^2\right]\right] \tag{52}$$

where $C_{10} = C_{10}(C_9,\ \varrho,\ C(\delta_1),\ H(0))$.

Define $\phi(t) := \max\{1,\ C_{10}\left[(R+t)^{\frac{n(p-2)}{2p}}\eta_R^{-1}\right]^{\frac{1}{1-\varrho}}\}$, then we have from (52) that

$$L^{\frac{1}{1-\varrho}}(t) \le \left[2\lambda(t)\right]^{\frac{1}{1-\varrho}}\phi(t)\left[H(t) + \|u\|_p^p + \|u_t\|^2\right] \tag{53}$$

Now combining (47) and (53), we have the following estimate

$$L'(t) - \beta'(t)[\beta(t)]^{-1}L(t) \ge C_\mu^*\beta(t)\left[\left[\lambda(t)\right]^{\frac{1}{1-\varrho}}\phi(t)\right]^{-1}L^{\frac{1}{1-\varrho}}(t) \tag{54}$$

where $C_\mu^* = 2^{-\frac{1}{1-\varrho}}C_\mu$. From Lemma 3, we have that (54) satisfies the following inequality

$$L(t) \ge \left[\frac{\beta(t)}{\beta_0}\right]\left[L_0^{-\frac{\varrho}{1-\varrho}} - \left[\frac{\varrho C_\mu^*\beta_0^{-\frac{\varrho}{1-\varrho}}}{1-\varrho}\right]\int_0^t \phi(s)^{-1}\left[\frac{\beta(s)}{\lambda(s)}\right]^{\frac{1}{1-\varrho}}ds\right]^{\frac{-(1-\varrho)}{\varrho}} \tag{55}$$

with $\varrho = \frac{p-m}{mp}$. The desired result follows. $\square$

Theorem 3 *Let $u(x,t)$ be a solution of the problem (1) and suppose that the assumptions (l_1), (l_{2_1}), (l_3) and (l_{5_1}) are satisfied. In addition, assume that $g(u)$ satisfies*

(B_1) $\int_{\mathbb{R}^n} ug(u)dx - q\int_{\mathbb{R}^n}\int_0^u g(y)dydx \ge \rho_0\|u\|_p^p$

for positive constants $q \in (\gamma, p)$ and $m < p$. Then, there exist a finite time T_* satisfying

$$D(T_*) \le \frac{1-\varrho}{\varrho C_\mu^*}\left[\beta_o/L_0\right]^{\frac{\varrho}{1-\varrho}}$$

where $D(t)$ is the function defined in (l_{5_1}) and $\varrho = \frac{p-m}{mp}$ such that the solution u of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0u_1dx > 0$ blows up.

The proof follows from that of Theorem 2.

Theorem 4 *Let $u(x,t)$ be a solution of the problem (1) and suppose that the assumptions l_{2_1} and l_{4_1} in Theorem 2 are replaced by l_{2_2} and l_{4_2}, then no weak solution of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0u_1dx > 0$ can exist on the whole of $[0,\infty)$.*

In addition if the assumptions l_{2_1} and l_{5_1} in Theorem 3 are replaced by l_{2_2} and l_{5_2}, Then there exist a finite time T_* such that the solution of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0u_1dx > 0$ blows up.

The proof can be deduced from the proof of Theorem 2, where in this case, the estimate for the blow up time is given by

$$L(t) \ge \left[L_0^{\frac{-\varrho}{1-\varrho}} - \left[\frac{\varrho C_\mu^*}{1-\varrho}\right]\int_0^t \beta(s)\left[\phi(s)\lambda(s)^{\frac{1}{1-\varrho}}\right]^{-1}ds\right]^{\frac{-(1-\varrho)}{\varrho}} \tag{56}$$

5 Applications

For $b(t,x)=(1+t)^k$, we have that

$$\eta_R(t)=C(1+t)^{\frac{k}{m-1}}(R+t)^{\frac{n(p-m)}{p(m-1)}} \tag{57}$$

and for $\beta(t)=(R+t)^a$. Then, from assumption (l_3), it follows that $\lambda(t)$ takes the form $(1+t)^{a+\frac{kp+n(p-m)}{p(m-1)}}$ and the assumption (l_1) is satisfied for $\frac{nm}{n+k+a(m-1)}<p$.

The case

$$\max\Big[1,\ C_{10}\big[(R+t)^{\frac{n(p-2)}{2p}}\eta_R^{-1}(t)\big]^{\frac{1}{1-\varrho}}\Big]=1$$

This holds for $2\le m<3$. Moreover, we have that

$$\int_0^t \phi(s)^{-1}\left[\frac{\beta(s)}{\lambda(s)}\right]^{\frac{1}{1-\sigma}}ds=\int_0^t(1+s)^{-\frac{m[kp+n(p-m)]}{(m-1)[p(m-1)+m]}}ds \tag{58}$$

In addition, the condition (l_{4_1}) holds for $\frac{m(nm+m-1)}{m(n+k)-(m-1)^2}<p$ and the blow up Theorem 2 is satisfied for

$$\max\left\{\frac{nm}{n+k+a(m-1)},\frac{m(nm+m-1)}{m(n+k)-(m-1)^2}\right\}<p\le\frac{n\gamma}{n-\gamma} \tag{59}$$

The condition (l_{5_1}) holds for $p<\frac{m(nm+m-1)}{m(n+k)-(m-1)^2}$, and the blow up to Theorem 3 holds in the interval.

$$\frac{nm}{n+k+a(m-1)}<p\le\min\left\{\frac{m(nm+m-1)}{m(n+k)-(m-1)^2},\frac{n\gamma}{n-\gamma}\right\} \tag{60}$$

Note that for $\beta(t)=1$, the estimates (59) and (60) hold with $a=0$

For $b(t,x)=C(1+t)^k|x|^q$, we have that

$$\begin{aligned}\eta_R(t)&=C(1+t)^{\frac{k}{m-1}}\left[\int_{B(R+t)}|x|^{\frac{qp}{p-m}}dx\right]^{\frac{p-m}{p(m-1)}}\\&=C(1+t)^{\frac{k+q}{m-1}}(R+t)^{\frac{n(p-m)}{p(m-1)}}\end{aligned} \tag{61}$$

and the argument follows as before with k replaced by $k+q$.

A good example in this direction is the semilinear wave equation (3). In this case $k=-\nu_0=-1$, $m=2$, $\gamma=2$. Choosing $\beta(t)=1$, we have that the blow up Theorem 2 holds for

$$\max\left\{\frac{2n}{n-1},\frac{2(2n+1)}{2n-3}\right\}<p\le\frac{2n}{n-2}$$

and for $n=3$, $\frac{14}{3}<p\le 6$. Also, the blow up Theorem 3 holds for

$$\frac{2n}{n-1}<p\le\min\left\{\frac{2(2n+1)}{2n-3},\frac{2n}{n-2}\right\}$$

and for $n=3$, we have $3<p\le\frac{14}{3}$.

The case

$$\max\Big[1,\ C_{10}\big[(R+t)^{\frac{n(p-2)}{2p}}\eta_R^{-1}(t)\big]^{\frac{1}{1-\varrho}}\Big]=C_{10}\big[(R+t)^{\frac{n(p-2)}{2p}}\eta_R^{-1}(t)\big]^{\frac{1}{1-\varrho}}$$

which satisfies the case $m \geq 3$.

For $b(t,x) = (1+t)^k$ and $\beta(t) = (R+t)^a$ as defined before, we have that $\lambda(t)$ takes the form $(1+t)^{a+\frac{kp+n(p-m)}{p(m-1)}}$. In this case the integral is given by

$$\int_0^t \phi(s)^{-1}\left[\frac{\beta(s)}{\lambda(s)}\right]^{\frac{1}{1-\sigma}} ds = C_{10}\int_0^t (1+s)^{-[\frac{n(p-2)}{2p}][\frac{mp}{p(m-1)+m}]} ds$$

Furthermore, condition (l_{4_1}) is satisfied for $\frac{2m(n+1)}{m(n-2)+2} < p$ and the blow up Theorem 2 holds for

$$\max\left\{\frac{nm}{n+k+a(m-1)}, \frac{2m(n+1)}{m(n-2)+2}\right\} < p \leq \frac{n\gamma}{n-\gamma}$$

Condition (l_{5_1}) is satisfied for $p < \frac{2m(n+1)}{m(n-2)+2}$ and the blow up Theorem 3 holds for

$$\frac{nm}{n+k+a(m-1)} < p \leq \min\left\{\frac{2m(n+1)}{m(n-2)+2}, \frac{n\gamma}{n-\gamma}\right\}$$

Finally, the case $b(t,x) = 1$ can be deduced from the previous examples by setting $k = q = 0$.

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