Electronic Journal of Mathematical Analysis and Applications
Vol. 10(2) July 2022, pp. 61-74.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

BLOW UP FOR A VISCOELASTIC WAVE EQUATION WITH SPACE-TIME POTENTIAL IN $\mathbb{R}^n$

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Abstract. In this paper, we consider the following wave equation:

$$\begin{cases} u_{tt} - \big(\Delta u - \int_0^t g(t-s)\Delta u(s)ds\big) + b(t,x)u_t = f(x,u), & t>0,\ x \in \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

with space-time dependent potential, where the initial data have compact support. Under suitable assumptions on the nonlinear function f, the relaxation function g and the damping potential b, we obtain blow up results using the perturbed energy method.

1. Introduction

In this paper, we are concerned with blow-up result for solutions to nonlinear wave equations of the form

$$\begin{cases} u_{tt} - \big(\Delta u - \int_0^t g(t-s)\Delta u(s)ds\big) + b(t,x)u_t = f(x,u), & t>0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with space-time dependent potential b and relaxation function g, where the initial data $u_0(x)$, $u_1(x)$ belong to appropriate spaces and $u = u(t,x)$.

Recent works in the literature, see [9, 10, 27, 28] have shown that the behavior of the problem (1) is influenced by the dissipation produced by $b(t,x)u_t$ with $b(t,x) > 0$. An interesting question is whether or not this damping in unbounded domains could prevent blow-up. In the case of nonlinear damping in bounded domains, Georgiev and Todorova [5] considered the interaction between the source and damping term in the nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{\alpha-2}u_t = |u|^{p-2}u & x \in \Omega, \quad t>0 \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \qquad x \in \Omega & u(x,t)|_{\partial\Omega} = 0, \quad t>0, \end{cases} \tag{2}$$

when $\alpha > 2$ and showed that for $p > \alpha$, the solution with negative initial energy cannot be global and is global for $p \le \alpha$.

2010 *Mathematics Subject Classification.* 35A01, 35B45, 35L15, 35L70.
Key words and phrases. Blow up, Relaxation function, Viscoelastic, Damping potential.
Submitted Aug. 16, 2021.

There is an extensive literature on global existence and blow-up of weak solutions to nonlinear wave equations of the form (2) on bounded smooth domains $\Omega \subset \mathbb{R}^n$. More specifically, Yang in [29], obtained blow up of solutions to a nonlinear wave problem of the form

$$\begin{cases} u_{tt} - \Delta u_t - \operatorname{div}\Big[|\nabla u|^{\gamma}\nabla u + |\nabla u_t|^{r}\nabla u_t\Big] + |u_t|^m u_t = |u|^p u, & x \in \Omega,\ t > 0 \\ u(x,t)|_{\partial\Omega} = 0, \quad t > 0 \qquad u(x,0) = u_0, \qquad u_t(x,0) = u_1, & x \in \Omega, \end{cases} \tag{3}$$

under the condition $p > \max\{\gamma, m\}$ and where the blow up time depends on $|\Omega|$. In [15] Messaoudi and Said-Houari studied a class of nonlinear wave equations having the form (3) and extended the blow up result of Yang [29] to the case where the blow up result holds regardless of the size of Ω and for $p > \max\{\gamma, m\}$, $\gamma > r$. The case of (3) with positive initial energy solution and space dependent coefficients was considered by Ogbiyele in [20].

Messaoudi [14], considered the nonlinear wave equation

$$u_{tt} - \big(\Delta u - \int_0^t g(t-s)\Delta u(s)ds\big) + u_t|u_t|^{m-2} = u|u|^{p-2},$$

when $p > m$ and obtained blow up results for positive initial energy solution. For a review on recent results on global existence, energy decay and blow up of solutions to nonlinear wave equations in bounded domains and their extensions to variable exponents, see [1, 4, 16, 17] and for other relevant results on blow-up and global existence for nonlinear wave equations in bounded smooth domains, the reader is referred to [2, 6, 7, 11, 13, 18, 19, 21, 23, 26].

In the case of unbounded domains, Levine et al.[12] considered global existence and blow-up of weak solutions to the Cauchy problem

$$u_{tt} - \operatorname{div}(|\nabla u|^{\gamma-2}\nabla u) + b(t,x)|u_t|^{m-2}u_t = f(x,u), \tag{4}$$

with $\gamma = 2$. They showed that when m, p satisfy the condition $p < \min\{m, 2(n-1)/(n-2)\}$, the solutions are global. In addition to the condition $p > \{2, m\}$ they also gave the restriction $p < \max\{2n/(n-2), mn/(n+1-m)\}$ for which the solution blows up when the initial energy is merely less than zero. In a related work, G. Todorova [25], studied the Cauchy problem (1) where $f(x,u) = -\mu(x)u + u|u|^{p-2}$ and argued that for the case $\mu = 0$, the additional restriction $p < mn/(n-m+1)$ is method driven.

More recently, Ogbiyele and Arawomo [22] obtained blow up of weak solutions to the nonlinear Cauchy problem (4) and obtained blow up results under suitable conditions on the damping potential b and the nonlinear function f. For other blow up results of cauchy viscoelastic equation in unbounded domains, see [8, 31].

Motivated by the results in the literature, we consider the wave problem (1) under suitable conditions on the damping coefficient b and the relaxation function g and establish blow up results using a differential inequality similar to that in [22] where the initial data u_0, u_1 are assumed to have compact support in a ball $B(R)$ of radius R about the origin and the solution satisfies the finite speed of propagation property $\operatorname{supp} u(t) \in B(R+t)$ for $t \in (0, \infty)$.

2. Preliminaries

In this section, we state some basic assumptions used in this paper. First, we introduce the following notations. $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, the Lebesgue space with norm $\|\cdot\|_p$ and $W^{i,p}(\mathbb{R}^n)$ the Banach space of functions in $L^p(\mathbb{R}^n)$ with $i(i \in \mathbb{N})$ generalized derivatives and $H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$. We denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbb{R}^n)$.

Definition 1. *By a weak solution to (1), we mean a function $u(t,x)$ satisfying the following*

(i) $u \in L^\infty\big([0,T]; H^1(\mathbb{R}^n)\big) \cap L^\infty\big([0,T]; L^p(\mathbb{R}^n)\big)$, $\quad u_t \in L^\infty\big([0,T]; L^2(\mathbb{R}^n)\big)$, $u_{tt} \in L^2\big([0,T]; H^{-1}(\mathbb{R}^n)\big)$,

(ii) $\int_0^t \big[\langle u_{tt}, v\rangle + \langle[\nabla u - \int_0^t g(t-s)\nabla u(s)ds], \nabla v\rangle + \langle b(t,x)u_t, v\rangle - \langle f(x,u), v\rangle\big] ds = 0$,
for $v \in C_0^\infty([0,T]\times\mathbb{R}^n)$ and a.e. $t \in [0,T]$ such that $u(0) \in H^1(\mathbb{R}^n)$, $u_t(0) \in L^2(\mathbb{R}^n)$.

Lemma 1. *(Sobolev, Gagliardo, Nirenberg*[3]*) Suppose that $1 \le p < n$. If $u \in W^{1,p}(\mathbb{R}^n)$, then $u \in L^{p^*}(\mathbb{R}^n)$ with*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Moreover, there is a constant $k = k(n,p)$ such that

$$\|u\|_{p^*} \le k\|\nabla u\|_p \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Lemma 2. *Let u be the solution of (1) for $n \ge 3$, then there exists a positive constant K such that*

$$\int |u|^2 dx \le K^2(R+t)^2 \int |\nabla u|^2 dx. \tag{5}$$

Proof. This follows directly from Holder's inequality, Lemma (1) and the finite speed of propagation property, thus we have

$$\int_{\mathbb{R}^n} |u|^2 dx \le \Big[\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx\Big]^{\frac{n-2}{n}} \Big[\int_{B(R+t)} dx\Big]^{\frac{2}{n}} \le k^2[\omega_n(R+t)^n]^{2/n}\|\nabla u\|_2^2,$$

where $K = K(k, n, \omega_n)$. □

Lemma 3. *(Modified Gronwall inequality,*[24]*) Let $\phi(t)$ be a non-negative function on $[0,\infty)$ satisfying*

$$\phi(t) \le B_1 + B_2 \int_0^t \phi^\delta(s)ds,$$

where B_1, B_2 are positive constants, then $\phi(t)$ satisfy the inequality

$$\phi(t) \le B_1[1 - (\delta-1)B_2B_1^{\delta-1}t]^{\frac{-1}{\delta-1}} \qquad \text{for} \quad \delta > 1.$$

Lemma 4. [22] *Let $y(t)$ be a continuous non-negative C^1 function on $[0,\infty]$ which satisfies*

$$y'(t) \ge a(t)y(t) + c(t)y^r(t), \tag{6}$$

(i) *if $a(t) < 0$, $c(t) > 0$ and $r > 1$, then $y(t)$ satisfies the following inequality*

$$y^{1-r}(t) \le e^{(1-r)\int_0^t a(s)ds}\left[y_0^{-(r-1)} - (r-1)\int_0^t c(s)e^{(r-1)\int_0^s a(\tau)d\tau}ds\right] \tag{7}$$

(ii) *and if* $a(t) \geq 0$, $c(t) > 0$ *and* $r > 1$. *we have*

$$y^{1-r}(t) \leq y_0^{1-r} - (r-1)\int_0^t c(s)ds,$$

where $y_0 = y(0) > 0$.

For the nonlinear function f, the damping potential b and the relaxation function g, we have the following:

(A_1) The real valued function $f(x,u)$ with $f(x,0) = 0$ is continuous on $\mathbb{R}^n \times \mathbb{R}$ and such that $|f(x,u)| \leq c_1|u|^{p-1}$ for the constants $c_1 > 0$ and $p > 1$.

(A_2) The potential $b(t,x)$, satisfy $\int_{B(R+t)} |b(t,x)|^{\frac{-n}{2-(n-2)(p-2)}} dx \in L^\infty_{loc}(J_T)$ and $\int_{B(R+t)} |b(t,x)|^{\frac{n}{2}} dx \in L^\infty_{loc}(J_T)$ where $J_T = (0,\infty)$.

(A_3) g is a differentiable function satisfying

$$g(s) \geq 0, \qquad 1 - \int_0^\infty g(s)ds = \ell > 0 \quad \text{and} \quad g'(s) \leq 0 \qquad \text{for} \quad s \geq 0.$$

We now define the modified energy functional $E(t)$ associated to problem (1) by

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\Big[1 - \int_0^t g(s)ds\Big]\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u) - \int_{\mathbb{R}^n}\int_0^u f(\cdot,y)dydx \tag{8}$$

and use the following notation for easy representation

$$(g \circ \nabla u) := \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds.$$

For the functional $E(t)$, we state the following lemma.

Lemma 5. *Suppose that the assumptions* (A_1) *to* (A_3) *hold. Let* u *be a solution of the problem (1), then for* $t \geq 0$, *the energy functional* $E(t)$ *satisfies*

$$E'(t) \leq -\int_0^t b(t,x)|u_t|^2 dx - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}(g' \circ \nabla u). \tag{9}$$

Moreover, we have

$$E(t) \leq E(0). \tag{10}$$

Proof. By multiplying (1) by u_t and integrating over $\mathbb{R}^n$, we obtain the estimate (9) for any regular solution. Thus by using density arguments, we get the desired result. □

3. Local Existence

In this section, we consider the existence of a weak solution to (1) in the maximal interval $[0,T]$ for $T < \infty$, using the Galerkin approximation technique.

Theorem 1. *Suppose that the assumptions* (A_1) - (A_3) *hold, and let* $2 < p \leq \frac{2(n-1)}{n-2}$ *for* $n \geq 3$, *then there exist a unique solution*

$$u \in C([0,T]; H^1(\mathbb{R}^n)) \quad \text{and} \qquad u_t \in C([0,T]; L^2(\mathbb{R}^n)),$$

with initial data $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$ *having compact support for* T *small enough.*

Proof. First, we assume the sequence of functions $(w_j)_{j\in\mathbb{N}}$ to be a basis in $H^1(\mathbb{R}^n)$ which is also orthonormal in $L^2(\mathbb{R}^n)$ and consider weak solution of the form

$$u^n(t)=\sum_{j=1}^{n}a_{jn}(t)w_j, \tag{11}$$

which satisfies the following approximate problem corresponding to (1):

$$\langle u^n_{tt},w_j\rangle+\langle[\nabla u^n-\int_0^t g(t-s)\nabla u^n(s)ds],\nabla w_j\rangle+\langle b(t,\cdot)u^n_t,w_j\rangle-\langle f(\cdot,u^n),w_j\rangle=0, \tag{12}$$

for $w_j\in H^1(\mathbb{R}^n)$ with initial conditions

$$u^n(0)=u^n_0\equiv\sum_{j=1}^{n}d_{jn}w_j\to u_0 \quad \text{strongly in } H^1(\mathbb{R}^n) \text{ as } n\to\infty \tag{13}$$

and

$$u^n_t(0)=u^n_1\equiv\sum_{j=1}^{n}c_{jn}w_j\to u_1 \quad \text{strongly in } L^2(\mathbb{R}^n) \text{ as } n\to\infty, \tag{14}$$

where $a_{jn}(t)=\langle u^n(t),w_j\rangle$, $d_{jn}=\langle u^n_0,w_j\rangle$, and $c_{jn}=\langle u^n_1,w_j\rangle$. Since the coefficients are continuous, then there exist a solution $u^n(t)$ for the system (12) -(14) and for some interval $[0,t_n)$ where $0<t_n<T$. We use the a-priori estimates below to show that the solution is bounded on the whole interval $[0,T]$.

Set $w_j=u^n_t(t)$ in (12) and using assumption (A_1), the resulting equation is

$$\begin{aligned}\frac{d}{dt}\Big[\frac{1}{2}\|u^n_t\|^2+&\frac{1}{2}\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^n\|_2^2+\frac{1}{2}(g\circ\nabla u^n)\Big]+\int_{\mathbb{R}^n}b(t,\cdot)|u^n_t|^2dx\\ &\le c_1\int_{\mathbb{R}^n}|u^n|^{p-1}|u^n_t|dx-\frac{1}{2}g(t)\|\nabla u^n\|_2^2+\frac{1}{2}(g'\circ\nabla u^n).\end{aligned} \tag{15}$$

From assumption (A_3), the second term and third term on the right hand side of (15) are negative, and for the first term on the right hand side of (15), using Holder and Young's inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}|u^n|^{p-1}|u^n_t|dx\\ &\le\Big[\int_{\mathbb{R}^n}b(t,x)|u^n_t|^2dx\Big]^{\frac{1}{2}}\Big[\int_{\mathbb{R}^n}b(t,x)^{-1}|u^n|^{2(p-1)}dx\Big]^{\frac{1}{2}}\\ &\le\Big[\int_{\mathbb{R}^n}b(t,x)|u^n_t|^2dx\Big]^{\frac{1}{2}}\Big[\Big[\int_{\mathbb{R}^n}|u^n|^{\frac{2n}{n-2}}dx\Big]^{\frac{(p-1)(n-2)}{n}}\Big[\int_{B(R+t)}|b(t,x)|^{\frac{-n}{2-(n-2)(p-2)}}dx\Big]^{\frac{2-(n-2)(p-2)}{n}}\Big]^{\frac{1}{2}}\\ &\le\epsilon_1\int_{\mathbb{R}^n}b(t,x)|u^n_t|^2dx+C(\epsilon_1)k^2\|\nabla u\|_2^{2(p-1)}\Big[\int_{B(R+t)}|b(t,x)|^{\frac{-n}{2-(n-2)(p-2)}}dx\Big]^{\frac{2-(n-2)(p-2)}{n}},\end{aligned} \tag{16}$$

where from assumption (A_2), $\sup_{t\in[0,T]}\Big[\int_{B(R+t)}|b(t,x)|^{\frac{-n}{2-(n-2)(p-2)}}dx\Big]^{\frac{2-(n-2)(p-2)}{n}}\le K_0$. Therefore, employing the estimate (16) in (15), we obtain

$$\begin{aligned}\frac{d}{dt}\Big[\frac{1}{2}\|u^n_t\|^2+\frac{1}{2}\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^n\|_2^2+\frac{1}{2}(g\circ\nabla u^n)\Big]&\\ +(1-\epsilon_1)\int_{\mathbb{R}^n}b(t,\cdot)|u^n_t|^2dx\le c_1K_0C(\epsilon_1)\|\nabla u\|_2^{2(p-1)},&\end{aligned} \tag{17}$$

where we choose $\epsilon_1 < 1$. Furthermore, integrating (17) over t for $t \in [0,T]$ and setting

$$\begin{aligned}\mathcal{H}_n(t) =& \frac{1}{2}\|u_t^n\|^2 + \frac{1}{2}\Big[1 - \int_0^t g(s)ds\Big]\|\nabla u^n\|_2^2 + \frac{1}{2}(g \circ \nabla u^n) \\ &+ [1-\epsilon_1]\int_0^t \int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^2 dxds,\end{aligned} \tag{18}$$

we have that there exists a positive constant K_1 independent of $n \in \mathbb{N}$ such that (17) yields

$$\mathcal{H}_n(t) \le \mathcal{H}_n(0) + K_1 \int_0^t \mathcal{H}_n^{p-1}(s)ds, \tag{19}$$

for $t \in [0,T]$. And applying Lemma 3, we get

$$\mathcal{H}_n(t) \le \mathcal{H}_n(0)\Big[1-(p-2)K_1\mathcal{H}_n^{p-2}(0)t\Big]^{\frac{-1}{p-2}}, \qquad p > 2. \tag{20}$$

Therefore, from (18) and (20), there exist a positive constant K_2 independent of $n \in N$ such that the following estimates hold

$$\|u_t^n\|^2 \le K_2, \tag{21}$$

$$\|\nabla u^n\|_2^2 \le K_2, \tag{22}$$

and

$$\int_0^t \int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^2 dxds \le K_2 \tag{23}$$

Setting $v = w_j$ in (12), we have

$$|\langle u_{tt}^n, v\rangle| \le |\langle[\nabla u^n + \int_0^t g(t-s)\nabla u^n(s)ds], \nabla v\rangle| + |\langle b(t,\cdot)u_t^n, v\rangle| + |\langle f(\cdot,u^n), v\rangle|. \tag{24}$$

Now, for the last term on the right hand side of (24), using Holder's inequality and (22), we have

$$|\langle f(\cdot,u^n), v\rangle| \le \|f(\cdot,u^n(t))\|_{p'}\|v\|_p \le K_3\|f(\cdot,u^n(t))\|_{p'}\|v\|_{1,2} \tag{25}$$

and from (22) and assumption (A_1), we get

$$\|f(\cdot,u^n(t))\|_{p'} \le c_2\|u^n(t)\|_p^{p-1} \le K_4\|\nabla u^n(t)\|_2^{p-1} \le K_5 \quad \text{for } t \in [0,T]. \tag{26}$$

For the second term on the right hand side of (24), using Hölder and Sobolev inequalities, and assumption (A_2), we have the estimate

$$\begin{aligned}|\langle b(t,\cdot)u_t^n, v\rangle| &\le \Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^2 dx\Big]^{\frac{1}{2}}\Big[\int\limits_{B(R+t)} |b(t,x)|^{\frac{n}{2}} dx\Big]^{\frac{1}{n}}\|v\|_{\frac{2n}{n-2}} \\ &\le K_6\Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^2 dx\Big]^{\frac{1}{2}}\|v\|_{1,2}.\end{aligned} \tag{27}$$

Now, substituting the estimates (25), (27) in (24) and using Hölder's inequality for the first term on the right hand side of (24), we have the following estimate

$$\begin{aligned}|\langle u_{tt}^n, v\rangle| \le K_7\Big(&\|\nabla u^n(t)\|_2 + \int_0^t g(t-s)\|\nabla u^n(s)\|_2 ds + \|f(\cdot,u^n(t))\|_{p'} \\ &+ \Big[\int_{\mathbb{R}^n} b(t,\cdot)|u_t^n|^2 dx\Big]^{\frac{1}{2}}\Big)\|v\|_{1,2}\end{aligned}$$

and thus, using the estimates (22) and (26), we obtain

$$\|u^n_{tt}(t)\|_{-1,2} \le K_8\Big(\Big[\int_{\mathbb{R}^n} b(t,\cdot)|u^n_t|^2dx\Big]^{\frac{1}{2}} + \int_0^t g(t-s)\|\nabla u^n(s)\|_2 ds + 1\Big).$$

By applying Hölder's inequality, integrating the resulting estimate over t for $t \in [0,T]$ and employing the estimate (22), (23) and the assumption on g, we obtain

$$\int_0^t \|u^n_{tt}(t)\|^2_{-1,2}\, dt \le K_9 \int_0^t \Big(\int_{\mathbb{R}^n} b(t,\cdot)|u^n_t|^2dx + \|\nabla u^n(t)\|_2^2 + 1\Big)dt \le K_{10},$$

for $t \in [0,T]$. Therefore, for any $T > 0$ we have that the nonlinear terms are uniformly bounded on $[0,T]$ and it follows that the solution $u^n(t)$ of (12) exist on $[0,T]$ for each n.

Hence from the estimates above, we can obtain a subsequence u^k of u^n and pass the limit in the approximate problem to obtain a weak solution satisfying

(b_1) $u^k(t) \to u(t)$ weakly-star in $L^\infty\big([0,T];H^1(\mathbb{R}^n)\big)$
(b_2) $u^k_t(t) \to u_t(t)$ weakly in $L^\infty\big([0,T];L^2(\mathbb{R}^n)\big)$
(b_3) $u^k_{tt}(t) \to u_{tt}(t)$ weakly-star in $L^2\big([0,T];H^1(\mathbb{R}^n)\big)$
(b_4) $f(\cdot,u^k(t)) \to \phi(t)$ weakly-star in $L^\infty\big([0,T];L^{p'}(\mathbb{R}^n)\big)$
(b_5) $|b(t,\cdot)|^{\frac{1}{2}}u^k_t \to |b(t,\cdot)|^{\frac{1}{2}}u_t$ weakly in $L^2\big([0,T]\times\mathbb{R}^n\big)$,

Now, letting $n \to \infty$ in (12) and using (b_1) - (b_5), we obtain

$$\int_0^T \Big[\big(u_{tt},v\big) + \big([\nabla u - \int_0^t g(t-s)\nabla u(s)ds], \nabla v\big) + \big(b(t,\cdot)u_t, v\big) - \big(\phi, v\big)\Big]\,dt = 0,$$

for all $v \in L^2([0,T];H^1(\mathbb{R}^n))$. The proof for $f(\cdot,u^n) = \phi$ is the same as in [30], so we omit it. □

4. Blow up

In this section, we consider blow up of solution to (1) having negative initial energy. Our technique follows the one in [22]. however we employ Lemma 4 in obtaining the blow up estimate of solution to (1).

First, we define the function $H(t)$ by

$$H(t) := -E(t), \tag{28}$$

then from (8), we have

$$0 < H(0) \le H(t) \le \int_{\mathbb{R}^n}\int_0^u f(\cdot,y)dydx \le \frac{c_1}{p}\|u\|_p^p. \tag{29}$$

Moreover, from (9) the derivative $H'(t)$ satisfy

$$H'(t) = \int_0^t b(t,x)|u_t|^2dx + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\big(g' \circ \nabla u\big). \tag{30}$$

Furthermore, for the Cauchy problem (1), we define the function $L(t)$ by

$$L(t) := \lambda(t)H^{1-\varrho}(t) + \mu\beta(t)\int_{\mathbb{R}^n} uu_t dx, \tag{31}$$

for suitable choice of ϱ satisfying

$$0 < \varrho = \frac{p-2}{2p}, \tag{32}$$

where λ and β are positive functions depending on the support radius R and satisfying the following conditions:

l_1 : $\lambda'(t) \geq 0$

l_{2_1} : $\beta(t)\lambda'(t) - \lambda(t)\beta'(t) \geq 0$ and $\frac{\beta'(t)}{\beta(t)} < 0$

or

l_{2_2} : $\beta(t)\lambda'(t) - \lambda(t)\beta'(t) \geq 0$ and $\frac{\beta'(t)}{\beta(t)} \geq 0$

l_3 : $\lambda(t) \geq \delta_{_R}(t)\beta(t)$

where

$$\delta_{_R}(t) = \Big[\int\limits_{B(R+t)} |b(t,x)|^{\frac{p}{p-2}} dx \Big]^{\frac{p-2}{p}},$$

such that one of the following

l_{4_1} : $D(t) := \int_0^\infty \phi(s)^{-1} \big[\frac{\beta(s)}{\lambda(s)}\big]^{\frac{2p}{p+2}} ds = \infty$

or

l_{4_2} : $D(t) := \int_0^\infty \beta(s) \big[\phi(s)\lambda(s)^{\frac{2p}{p+2}}\big]^{-1} ds = \infty$

l_{5_1} : $D(t) := \int_0^\infty \phi(s)^{-1} \big[\frac{\beta(s)}{\lambda(s)}\big]^{\frac{2p}{p+2}} ds < \infty$

or

l_{5_2} : $D(t) := \int_0^\infty \beta(s) \big[\phi(s)\lambda(s)^{\frac{2p}{p+2}}\big]^{-1} ds < \infty$

is satisfied for $\phi(t) = \max\{1, \ \big[(R+t)^{\frac{n(p-2)}{2p}} \delta_{_R}^{-1}\big]^{\frac{1}{1-\varrho}}\}$.

The weighted functions $\lambda(t)$ and $\beta(t)$ are used here to compensate for the lack of $L^2 \hookrightarrow L^p$ injection arising as a result of the unboundedness of the domain for $0 \leq 2 < p$.

Theorem 2. *Let $u(t,x)$ be a solution of the problem (1) with compact support in the ball $B(R)$ and suppose that the assumptions (l_1), (l_{2_1}), (l_3) and (l_{4_1}) are satisfied. In addition, assume that $f(\cdot,u)$ satisfies*

(B_1) $\int_{\mathbb{R}^n} uf(\cdot,u)dx - q\int_{\mathbb{R}^n}\int_0^u f(\cdot,y)dydx \geq \rho_0\|u\|_p^p$,

(B_2) $\int_0^\infty g(s)ds \leq \frac{q-2}{q-1}$,

for positive constants ρ_0 and q with $q \in (2,p)$. Then no weak solution of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0u_1dx > 0$ can exist on the whole of $[0,\infty)$.

Proof. From (31), we have that the derivative of $L(t)$ yields

$$\begin{aligned} L'(t) =& \lambda'(t)H^{1-\varrho}(t) + \mu\beta'(t)\int_{\mathbb{R}^n} uu_tdx + \lambda(t)(1-\varrho)H^{-\varrho}(t)H'(t) \\ &+ \mu\beta(t)\|u_t\|^2 + \mu\beta(t)\int_{\mathbb{R}^n} uu_{tt} \end{aligned} \tag{33}$$

and using the equation (1), we obtain

$$\begin{aligned} L'(t) &= \lambda'(t)H^{1-\varrho}(t) + \mu\beta'(t)\int_{\mathbb{R}^n} uu_tdx + \lambda(t)(1-\varrho)H^{-\varrho}(t)H'(t) \\ &+ \mu\beta(t)\|u_t\|^2 - \mu\beta(t)\|\nabla u\|^2 + \mu\beta(t)\int_{\mathbb{R}^n} uf(\cdot,u)dx \\ &- \mu\beta(t)\int_{\mathbb{R}^n} b(t,x)u_tudx + \mu\beta(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} \nabla u(s)\nabla u(t)dxds. \end{aligned} \tag{34}$$

The second to the last term in (34), can be estimated using Holder's inequality to get

$$\begin{aligned}\int_{\mathbb{R}^n} b(t,x)u_t u dx \leq & \Big[\int_{B(R+t)} |b(t,x)|^{\frac{p}{p-2}} dx\Big]^{\frac{p-2}{2p}} \Big[\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx\Big]^{\frac{1}{2}} \|u\|_p \\ \leq & \Big[\delta_R(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx\Big]^{\frac{1}{2}} \|u\|_p^{\frac{p}{2}} \|u\|_p^{\frac{2-p}{2}}.\end{aligned}$$

Hence, using Young's inequality and (29), it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} b(t,x)u_t u dx \leq & C(\delta_1)K_{11}H^{-\varrho}(t)\delta_R(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx \\ & + \delta_1 K_{11} H^{-\varrho}(0)\|u\|_p^p,\end{aligned} \tag{35}$$

where $K_{11} = K_{11}(c_1, p, \varrho)$. For the last term on the right hand side of (34), using Cauchy-Schwarz inequality we obtain

$$\begin{aligned}&\int_0^t g(t-s)\int_{\mathbb{R}^n} \nabla u(s)\nabla u(t) dx ds \\ &= -\int_0^t g(t-s)\int_{\mathbb{R}^n} |\nabla u(t) - \nabla u(s)|\nabla u(t) dx ds + \int_0^t g(s)ds\int_{\mathbb{R}^n} |\nabla u|^2 dx \\ &\geq -\frac{1}{2}\int_0^t g(s)ds\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u) + \int_0^t g(s)ds\|\nabla u\|_2^2.\end{aligned} \tag{36}$$

Therefore, using the estimate (35) and (36) in (34), we obtain

$$\begin{aligned}L'(t) \geq & \lambda'(t)H^{1-\varrho}(t) + \mu\beta'(t)\int_{\mathbb{R}^n} u u_t dx + \lambda(t)(1-\varrho)H^{-\varrho}(t)H'(t) \\ & + \mu\beta(t)\|u_t\|^2 - \mu\beta(t)\Big(1 - \frac{1}{2}\int_0^t g(s)ds\Big)\|\nabla u\|_2^2 + \mu\beta(t)\int_{\mathbb{R}^n} u f(\cdot,u)dx \\ & - C(\delta_1)K_{11}H^{-\varrho}(t)\mu\beta(t)\delta_R(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx \\ & - \delta_1 K_{11}H^{-\varrho}(0)\mu\beta(t)\|u\|_p^p - \frac{\mu\beta(t)}{2}(g\circ\nabla u).\end{aligned} \tag{37}$$

From the energy identity,

$$q\int_{\mathbb{R}^n}\int_0^u f(\cdot,y)dy dx = \frac{q}{2}\|u_t\|^2 + \frac{q}{2}\Big[1 - \int_0^t g(s)ds\Big]\|\nabla u\|_2^2 + \frac{q}{2}(g\circ\nabla u) + qH(t),$$

for $2 < q < p$. Then, using assumption (B_1), we obtain

$$\int_{\mathbb{R}^n} u f(\cdot,u)dx \geq \frac{q}{2}\|u_t\|^2 + \frac{q}{2}\Big[1 - \int_0^t g(s)ds\Big]\|\nabla u\|_2^2 + \frac{q}{2}(g\circ\nabla u) + qH(t) + \rho_0\|u\|_p^p. \tag{38}$$

Therefore, we have

$$\begin{aligned} L'(t) \geq{}& \lambda'(t)H^{1-\varrho}(t) + \mu\beta'(t)\int_{\mathbb{R}^n} uu_t dx + \mu\beta(t)(1+\tfrac{q}{2})\|u_t\|^2 + q\mu\beta(t)H(t) \\ &+ \mu\beta(t)\big[\rho_0 - \delta_1 K_{11} H^{-\varrho}(0)\big]\|u\|_p^p + \frac{\mu\beta(t)}{2}\big[(q-2)-(q-1)\int_0^t g(s)ds\big]\|\nabla u\|_2^2 \\ &+ \big[\lambda(t)(1-\varrho) - C(\delta_1)K_{11}\mu\beta(t)\delta_{_R}(t)\big]H^{-\varrho}(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx \\ &+ \frac{\mu\beta(t)[q-1]}{2}(g\circ\nabla u). \end{aligned} \tag{39}$$

We choose μ small enough such that

$$\lambda(t)(1-\varrho) \geq \mu C(\delta_1)K_{11}\beta(t)\delta_{_R}(t). \tag{40}$$

Also, from the definition of $L(t)$ and assumption (l_{2_1}), we have that

$$\begin{aligned} &\mu\beta'(t)\int_{\mathbb{R}^n} uu_t dx + \lambda'(t)H^{1-\varrho}(t) \\ &\quad=\big[\frac{\beta'(t)}{\beta(t)}\big]L(t) + \beta(t)\big[\frac{\beta(t)\lambda'(t)-\lambda(t)\beta'(t)}{\beta^2(t)}\big]H^{1-\varrho}(t) \\ &\quad\geq\big[\frac{\beta'(t)}{\beta(t)}\big]L(t). \end{aligned} \tag{41}$$

Hence, using the estimate (40) - (41) and assumption (B_2) in (39), we obtain

$$\begin{aligned} L'(t) \geq{}& \big[\frac{\beta'(t)}{\beta(t)}\big]L(t) + \mu\beta(t)\Big[\rho_0 - \delta_1 K_{11}H^{-\varrho}(0)\Big]\|u\|_p^p + \frac{\mu\beta(t)[q-1]}{2}(g\circ\nabla u) \\ &+ \mu\beta(t)(1+\tfrac{q}{2})\|u_t\|^2 + q\mu\beta(t)H(t). \end{aligned} \tag{42}$$

Therefore, if we choose δ_1 small enough such that $\rho_0 \geq \delta_1 K_{11}H^{-\varrho}(0)$, then there exist a positive constant K_μ such that (42) satisfies

$$L'(t) - \big[\frac{\beta'(t)}{\beta(t)}\big]L(t) \geq K_\mu\beta(t)\big[\|u_t\|^2 + (g\circ\nabla u) + \|u\|_p^p + H(t)\big], \tag{43}$$

where $K_\mu := \mu\min\{q,\ [\rho_0 - \delta_1 K_{11}H^{-\varrho}(0)],\ (1+\frac{q}{2}),\ \frac{(q-1)}{2}\}$. Hence, since

$$L(0) = \lambda(0)H^{1-\varrho}(0) + \mu\beta(0)\int_{B(R)} u_0u_1 dx > 0,$$

then from (43), we have that $L(t)$ is an increasing function for $t \geq 0$, satisfying

$$L(t) \geq \frac{\beta(t)}{\beta(0)}L(0) > 0, \qquad \forall t \geq 0.$$

On the other hand

$$\begin{aligned} L^{\frac{1}{1-\varrho}}(t) ={}& \Big[\lambda(t)H^{1-\varrho}(t) + \mu\beta(t)\int_{\mathbb{R}^n} uu_t dx\Big]^{\frac{1}{1-\varrho}} \\ \leq{}& 2^{\frac{1}{1-\varrho}}\Big[\big[\lambda(t)\big]^{\frac{1}{1-\varrho}}H(t) + \big[\mu\beta(t)\big]^{\frac{1}{1-\varrho}}\Big[\int_{\mathbb{R}^n} uu_t dx\Big]^{\frac{1}{1-\varrho}}\Big] \end{aligned} \tag{44}$$

and using Hölder inequality, we get

$$\Big|\int_{\mathbb{R}^n} uu_t dx\Big| \leq [\omega_n(R+t)^n]^{\frac{(p-2)}{2p}}\|u\|_p\|u_t\|_2,$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Then by Young's inequality, we have

$$\Big[\|u\|_p\|u_t\|_2\Big]^{\frac{1}{1-\varrho}} \leq K_{12}\Big[\|u\|_p^{\frac{\varepsilon}{1-\varrho}} + \|u_t\|^{\frac{\theta}{1-\varrho}}\Big], \tag{45}$$

where $K_{12} = K_{12}(\varepsilon, \theta, \varrho)$ and $\frac{1}{\varepsilon} + \frac{1}{\theta} = 1$. Now choosing $\theta = 2(1-\varrho)$ and setting $\frac{\varepsilon}{1-\varrho} = \frac{2}{1-2\varrho} \leq p$, then (45) yields

$$\Big|\int_{\mathbb{R}^n} uu_t dx\Big|^{\frac{1}{1-\varrho}} \leq K_{13}(R+t)^{\frac{n(p-2)}{2p(1-\varrho)}}\Big[\|u\|_p^p + \|u_t\|^2\Big]. \tag{46}$$

Substituting (46) in (44), we have

$$L^{\frac{1}{1-\varrho}}(t) \leq 2^{\frac{1}{1-\varrho}}\Big[\big[\lambda(t)\big]^{\frac{1}{1-\varrho}}H(t) + K_{13}\big[(R+t)^{\frac{n(p-2)}{2p}}\mu\beta(t)\big]^{\frac{1}{1-\varrho}}\Big[\|u\|_p^p + \|u_t\|^2\Big]\Big] \tag{47}$$

and from (40), the estimate (47) yields

$$L^{\frac{1}{1-\varrho}}(t) \leq \big[2\lambda(t)\big]^{\frac{1}{1-\varrho}}\Big[H(t) + K_{14}\big[(R+t)^{\frac{n(p-2)}{2p}}\delta_R^{-1}\big]^{\frac{1}{1-\varrho}}\Big[\|u\|_p^p + \|u_t\|^2\Big]\Big], \tag{48}$$

where $K_{14} = K_{14}(K_{13},\ K_{11},\ \varrho,\ \mu,\ C(\delta_1),\ H(0))$.
Define $\phi(t) := \max\{1,\ K_{14}\big[(R+t)^{\frac{n(p-2)}{2p}}\delta_R^{-1}\big]^{\frac{1}{1-\varrho}}\}$, then we have from (48) that

$$L^{\frac{1}{1-\varrho}}(t) \leq \big[2\lambda(t)\big]^{\frac{1}{1-\varrho}}\phi(t)\Big[H(t) + \|u_t\|^2 + \|u\|_p^p\Big]. \tag{49}$$

Now combining (43) and (49), we have the following estimate

$$L'(t) - \beta'(t)[\beta(t)]^{-1}L(t) \geq K_\mu^*\beta(t)\big[\big[\lambda(t)\big]^{\frac{1}{1-\varrho}}\phi(t)\big]^{-1}L^{\frac{1}{1-\varrho}}(t), \tag{50}$$

where $K_\mu^* = 2^{-\frac{1}{1-\varrho}}K_\mu$. From Lemma 4, we have that (50) satisfies the following inequality

$$L(t) \geq \Big[\frac{\beta(t)}{\beta_0}\Big]\Bigg[L_0^{-\frac{\varrho}{1-\varrho}} - \Big[\frac{\varrho K_\mu^*\beta_0^{-\frac{\varrho}{1-\varrho}}}{1-\varrho}\Big]\int_0^t \phi(s)^{-1}\Big[\frac{\beta(s)}{\lambda(s)}\Big]^{\frac{1}{1-\varrho}}ds\Bigg]^{\frac{-(1-\varrho)}{\varrho}}, \tag{51}$$

with $\varrho = \frac{p-2}{2p}$. The desired result follows. □

Theorem 3. *Let $u(t,x)$ be a solution of the problem (1) and suppose that the assumptions (l_1), (l_{2_1}), (l_3) and (l_{5_1}) are satisfied. In addition, assume that $f(u)$ satisfies*

(B_1) $\int_{\mathbb{R}^n} uf(\cdot,u)dx - q\int_{\mathbb{R}^n}\int_0^u f(\cdot,y)dydx \geq \rho_0\|u\|_p^p$,
(B_2) $\int_0^\infty g(s)ds \leq \frac{q-2}{q-1}$,

for positive constants $q \in (2,p)$. Then, there exist a finite time T_* satisfying

$$D(T_*) \leq \frac{1-\varrho}{\varrho K_\mu^*}\Big[\beta_0/L_0\Big]^{\frac{\varrho}{1-\varrho}},$$

where $D(t)$ is the function defined in (l_{5_1}) and $\varrho = \frac{p-2}{2p}$ such that the solution u of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0u_1dx > 0$ blows up.

The proof follows from that of Theorem 2.

Theorem 4. *Let $u(t,x)$ be a solution of the problem (1) and suppose that the assumptions l_{2_1} and l_{4_1} in Theorem 2 are replaced by l_{2_2} and l_{4_2}, then no weak solution of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0u_1dx > 0$ can exist on the whole of $[0,\infty)$.*

In addition if the assumptions l_{2_1} and l_{5_1} in Theorem 3 are replaced by l_{2_2} and l_{5_2}, Then there exist a finite time T_* such that the solution of (1) with compact support and satisfying $E(0) < 0$ and $\int_{B(R)} u_0 u_1 dx > 0$ blows up.

The proof can be deduced from the proof of Theorem 2, where in this case, the estimate for the blow up time is given by

$$L(t) \geq \left[L_0^{\frac{-\varrho}{1-\varrho}} - \Big[\frac{\varrho K_\mu^*}{1-\varrho}\Big] \int_0^t \beta(s)\big[\phi(s)\lambda(s)^{\frac{1}{1-\varrho}}\big]^{-1} ds \right]^{\frac{-(1-\varrho)}{\varrho}}. \tag{52}$$

5. Applications

For $b(t,x) = (1+t)^k,\ k > 0$, we have that $\delta_R(t) = C(1+t)^k (R+t)^{\frac{n(p-2)}{p}}$. Then, from assumption (l_3), it follows that

$$\frac{\lambda(t)}{\beta(t)} \approx C(1+t)^k (R+t)^{\frac{n(p-2)}{p}} \tag{53}$$

and the assumption (l_1) is satisfied for $a + \frac{kp+n(p-2)}{p} > 1$ where $\beta(t) = (R+t)^a$. Moreover, for $k > -\frac{n(p-2)}{2p}$, $\max\big[1,\ K_{14}\big[(R+t)^{\frac{n(p-2)}{2p}} \delta_R^{-1}(t)\big]^{\frac{1}{1-\varrho}}\big] = 1$. Therefore we have that

$$\int_0^t \phi(s)^{-1} \big[\tfrac{\beta(s)}{\lambda(s)}\big]^{\frac{1}{1-\varrho}} ds = K_{15} \int_0^t (1+s)^{-\frac{2[kp+n(p-2)]}{p+2}} ds \tag{54}$$

The condition (l_{4_1}) holds for $p < \frac{2(2n+1)}{2(n+k)-1}$, and the blow up to Theorem 2 holds in the interval.

$$\frac{2n}{n+k+a} < p \leq \min\Big\{\frac{2(2n+1)}{2(n+k)-1}, \frac{2n}{n-2}\Big\} \tag{55}$$

In addition, the condition (l_{5_1}) holds for $\frac{2(2n+1)}{2(n+k)-1} < p$ and the blow up Theorem 3 is satisfied for

$$\max\Big\{\frac{2n}{n+k+a}, \frac{2(2n+1)}{2(n+k)-1}\Big\} < p \leq \frac{2n}{n-2} \tag{56}$$

For the case $k < -\frac{n(p-2)}{2p}$, $k < 0$ and K_{14} large enough, that is $K_{14} > 1$ say, we have that $\max\big[1,\ K_{14}\big[(R+t)^{\frac{n(p-2)}{2p}} \delta_R^{-1}(t)\big]^{\frac{1}{1-\varrho}}\big] = K_{14}\big[(R+t)^{\frac{n(p-2)}{2p}} \delta_R^{-1}(t)\big]^{\frac{1}{1-\varrho}} \approx K_{16}(1+t)^{\frac{-[n(p-2)+2kp]}{p+2}}$

Thus, we have that

$$\int_0^t \phi(s)^{-1} \big[\tfrac{\beta(s)}{\lambda(s)}\big]^{\frac{1}{1-\varrho}} ds = K_{17} \int_0^t (1+s)^{\frac{-n(p-2)]}{p+2}} ds \tag{57}$$

The condition (l_{4_1}) holds for $p < \frac{2(n+1)}{n-1}$, and the blow up to Theorem 2 holds in the interval.

$$\frac{2n}{n+k+a} < p \leq \min\Big\{\frac{2(n+1)}{n-1}, \frac{2n}{n-2}\Big\} \tag{58}$$

In addition, the condition (l_{5_1}) holds for $\frac{2(n+1)}{n-1} < p$ and the blow up Theorem 3 is satisfied for

$$\max\Big\{\frac{2n}{n+k+a}, \frac{2(n+1)}{n-1}\Big\} < p \leq \frac{2n}{n-2} \tag{59}$$

Note that for $\beta(t) = 1$, the estimates (55) - (59) hold with $a = 0$

For $b(t,x) = C(1+t)^k |x|^q$, we have that

$$\begin{aligned} \delta_{_R}(t) =& C(1+t)^k \Big[\int_{B(R+t)} |x|^{\frac{qp}{p-2}} dx\Big]^{\frac{p-2}{p}} \\ =& C(1+t)^{k+q}(R+t)^{\frac{n(p-2)}{p}} \end{aligned} \tag{60}$$

and the argument follows as before with k replaced by $k+q$.

References

[1] L. Alkhalifa, H. Dridi, K. Zennir et al. Blow-Up of Certain Solutions to Nonlinear Wave Equations in the Kirchhoff-Type Equation with Variable Exponents and Positive Initial Energy. *Journal of Function Spaces*, **2021**, (2021), 1-9.

[2] A. Braik, Y. Miloudi, K. Zennir, Exponential growth for a semi-linear viscoelastic heat equation with $L^p_\rho(\mathbb{R}^n)$-norm in Bi-Laplacian type, Bulletin of the transilvania University of Brasov **11**(60), (2018) Series III: Mathematics, Informatics, Physics, 77–88.

[3] H. Brezis, Functional analysis, Springer Science + Business Media, LLC 2011.

[4] H. Dridi, K. Zennir, New Class of Kirchhoff Type Equations with Kelvin-Voigt Damping and General Nonlinearity: Local Existence and Blow-up in Solutions. *J. Part. Diff. Eq.*, **34**(4),(2021), 313–347.

[5] V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term. *J. Differ. Equ.* **109**,(1994), 295–308 .

[6] F. Hebhoub, K. Zennir, T. Miyasita, M. Biomy. Blow up at well defined time for a coupled system of one spatial variable Emden-Fowler type in viscoelasticities with strong nonlinear sources. *AIMS Mathematics*, **6**(1), (2021), 442–455.

[7] J. Jeong, J. Park, Y. H. Kang, Global nonexistence of solutions for a quasilinear wave equation with acoustic boundary conditions. *Boundary Value Problems*. **42**,(2017), 1.

[8] M. Kafini, S. A. Messaoudi, A blow-up result in a Cauchy viscoelastic problem. *Applied Mathematics Letters*, **21**, (2008), 549–553.

[9] N. Lai, H. Takamura and K. Wakasa, Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent. *J. Differential Equations* **263**, (2017), 5377–5394.

[10] N. Lai and H. Takamura, Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity, *Nonlinear Analysis: Real World Applications* **45**, (2019), 83–96.

[11] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.* **5**, (1974), 138–146.

[12] H. A. Levine, S. R. Park and J. Serrin, Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinear damped wave equation, *J. Math. Anal. Appl.,* **228**, (1998), 181–205.

[13] W. Liu, M. Wang, Global nonexistence of solutions with positive initial energy for a class of wave equations, *Math. Meth. Appl. Sci.* **32** (2009), 594–605.

[14] S. A. Messaoudi, Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation, *J. Math. Anal. Appl.,* **320**(2), (2006), 902–915.

[15] S. A. Messaoudi, B. Said-Houari, Global nonexistence of solutions of a class of wave equations with nonlinear damping and source terms, *Math. Methods. Appl. Sci.* **27**, (2004), 1687–1696.

[16] S. A. Messaoudi, A. A. Talahmeh, A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities, Applicable Analysis, **96**(9), (2017), 1509-1515, DOI: 10.1080/00036811.2016.1276170

[17] S. A. Messaoudi, A . A. Talahmeh, On wave equation: review and recent results, *Arab. J. Math.* **7**, (2018), 113–145.

[18] T. Miyasita, K. Zennir, Finite time blow-up for a viscoelastic wave equation with weak-strong damping and power nonlinearity, *Osaka J. Math.*, **58**, (2021), 661–669.

[19] P. A. Ogbiyele, On global existence and nonexistence of solutions to a quasi-linear wave equation with memory, nonlinear damping and source terms, *Scientific African* (2021), doi: https://doi.org/10.1016/j.sciaf.2021.e00974.

[20] P. A. Ogbiyele, Global existence and blow up of positive initial energy solution of a quasilinear wave equation with nonlinear damping and source terms, *Int. J. Dynamical Systems and Differential Equations*, **10**(4), (2020), 299-320.

[21] P. A. Ogbiyele, On a quasilinear wave equation with memory and nonlinear source terms, *J. Nigerian Math. Soc.*, **37**, (2018), 89–109.

[22] P. A. Ogbiyele, P. O. Arawomo, Existence and Blow up Time Estimate for a Negative Initial Energy Solution of a Nonlinear Cauchy Problem, *Acta Appl Math.*, **170**, (2020), 443–458.

[23] E. Piskin, On the decay and blow up of solutions for a quasilinear hyperbolic equations with nonlinear damping and source terms, *Bound. Value Probl.* **127**,(2015), 1.

[24] T. Taniguchi, Existence and asymptotic behaviour of solutions to weakly damped wave equations of Kirchhoff type with nonlinear damping and source terms, *J. Math. Anal. Appl.* **361**(2), (2010), 566–578.

[25] G. Todorova, Cauchy problem for nonlinear wave equation with nonlinear damping and source terms, *Nonlinear Analysis,* **(41)**, (2000), 891–905.

[26] E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, *J. Differential Equations*, **186**, (2002), 259 –298.

[27] J. Wirth,Wave equations with time-dependent dissipation. I. Non-effective dissipation, *J. Differential Equations*, **222**, (2006), 487–514.

[28] J. Wirth, Wave equations with time-dependent dissipation. II. Effective dissipation, *J. Differential Equations*, **232**, (2007), 74–103.

[29] Z. Yang, Blowup of solutions for a class of nonlinear evolution equations with nonlinear damping and source terms, *Math. Methods Appl. Sci.* **25**, (2002), 825–833.

[30] Z. Yang, G. Chen, Global existence of solutions for quasi-linear wave equations with viscous damping, *J. Math. Anal. Appl.* **285**,(2003), 606 –620.

[31] Z. Yong, A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in $\mathbb{R}^n$. *Appl. Math. Lett.*, **18**, (2005), 281–286.

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